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On doubled 3-manifolds and minimal handle presentations for 4-manifolds

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Abstract. We extend our earlier work on free reduction problems for 2-complexes K in 4-manifolds N (i.e., the problem of effecting, by a geometric deformation of K in N, the free reduction of the relator words in the presentation associated with K). Here, the problem is recast, with new results, in terms of 2-handle presentations of 4-manifolds.

Let M_* be the complement of the interior of a closed 3-ball in the 3-manifold M, and let $2M_*$ be the connected sum of two copies M, via a boundary identification allowing the identification of $2M_*$ with the boundary of $M_* \times [-1, 1]$.

We show that algebraic handle cancellation associated with a 2-handle presentation of a 4-manifold with boundary $2M_*$ can be turned into geometric handle cancellation for handle presentations of possibly different 4-manifolds having the same boundary provided that certain obstruction conditions are satisfied. These conditions are identified as surgery equivalence classes of framed links in $\mathrm{Bd}(M_* \times [-1, 1])$. These links, without the framing information, were considered in previous work by the author.

The following is one of the main results here: Let M be a 3-manifold that is a rational homology sphere, and suppose that $M_* \times [-1, 1]$ has a handle presentation $\mathcal{H}$ with no handles of index greater than 2. Suppose $\mathcal{H}$ is a normal, algebraically minimal handle presentation. If the obstruction conditions are satisfied, then there is a 4-manifold N bounded by $2M_*$ that has a minimal handle presentation.

Another theorem states, independent of the Poincaré Conjecture, conditions for a homotopy 3-sphere to be S^3 in terms of minimal handle presentations and the triviality of the defined obstruction conditions.

Contents

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1. Introduction

An earlier version of this paper was written in the 1990's, before Perelman's solution of the Poincaré Conjecture. See [GP02], [GP03a], [GP03b], [CZ06], [CZER] and [MT06]. At the time, we were interested in using 2-handle presentations of 4-manifolds and formal 3-deformations of 2-complexes naturally associated to these handle presentations to cast some light on the Boileau–Zieschang Seifert fibered 3-manifolds, and see if homotopy 3-cells could be simplified using 3-deformations of cells in the presentation so that when a presentation involving a single cell was encountered, meaning a presentation arrived at by formal or abstract 3-deformations, involving a single cell, a 0-cell, (corresponding to the satisfaction of the Andrews–Curtis Conjecture), then we could recognize the fake cube as the 3-cube.

Reader take note. We are trying to develop tools for an alternate understanding of the Poincaré Conjecture, especially the case where 2-spines of homotopy 3-spheres satisfy the Andrews–Curtis Conjecture. Because of that we will avoid using the truth of the Poincaré Conjecture in the main theorems. Theorem 3 is the only theorem that is affected. We will discuss the matter at the end of the proof of Theorem 3.

For a closed 3-manifold M, let $M_*(k)$ denote the 3-manifold that results by removing the interiors of k disjoint 3-balls in M. Let $2M_*(k)$ denote the *double* of $M_*(k)$ obtained by identifying, in the obvious way, two copies of $M_*(k)$, one with orientation reversed, along their boundaries. When $k = 1$, we will use the more familiar notation $2M$ for the double. There have been some efforts to show that for the case $k = 1$, the 4-manifold $N = M_*(1) \times [-1, 1]$ has a 2-handle presentation (2-spine) with few 1-handles where few is defined in terms of the formal 3-deformation properties of $M_*(1)$. See [RC89], [RC93], [JM86]. Such handle presentations will be called *minimal* here and minimal will be defined in the next paragraph. From [RC89] and [RC93] we know that the existence of minimal 2 handle presentations can, in favorable circumstances, be reduced to some very difficult free reduction obstruction problems for attaching words for 2-handles. The existence of minimal handle presentations for the 4-manifolds described above would offer an interesting geometric invariant bounded above by Heegaard genus but not always equal to Heegaard genus. Also a minimal presentation would offer the chance for an alternative proof of the Poincaré conjecture by reducing it to a weak version of the Andrews–Curtis Conjecture. In this paper we attempt to avoid the free reduction problems in finding minimal handle presentations by emphasizing the trading of $M_* \times [-1,1]$ for other 4-manifolds with the same boundary.

Counting and index. In the notation here k–handles means handles of index k, and k handles means a collection of handles of cardinality k. Consider one of the manifolds $M_*(k)$. Define the *extended Nielsen genus* $\mathrm{en}(M_*(k))$ of $M_*(k)$ to be the minimum number of 1-cells taken over all 2-complexes to which $M_*(k)$ formally 3-deforms. Note that $M_*(k)$ 3-deforms to a wedge of a 2-spine of $M_*(1)$ with $(k-1)$ 2-spheres. From the characterization of equivalence under extended Nielsen operations, (see [RB93], [RC79a], [RC79b] [HM93], [PW75], [SY76]), it follows that we can define $\mathrm{en}(M_*(k))$ in the following alternate way. Take a 2-spine K of $M_*(1)$. Let $\mathcal{P}_K = \{x_1, \dots, x_n \mid r_1, \dots, r_n\}$ be the associated group presentation. Let $(*)_{k-1}$ denote $k-1$ copies of the trivial relator. Then for the presentation $\mathcal{P} = \{x_1, \dots, x_n \mid r_1, \dots, r_n, (*)_{k-1}\}$ minimize, by using extended Nielsen transformations, the number of generators x_i in $\mathcal{P}$. This minimum number is $\mathrm{en}(M_*(k))$, and for $k = 1$ we will write $\mathrm{en}(M)$ for $\mathrm{en}(M_*(1))$ sometimes. Notice that the 4-manifold $M_*(k) \times [-1,1]$ has, as boundary, the double $2M_*(k)$ and this boundary is equivalent to $2M\#(k-1)S^2 \times S^1$. The 4-manifold $M_*(k) \times [-1,1]$ collapses to $M_*(k)$. Let N be a 4-manifold bounded by $2M_*(k)$. A handle presentation $\mathcal{H}$ for N is said to be *minimal* (relative to the boundary $2M_*(k)$) provided that it has no handles of index exceeding 2 and the following conditions hold:

1. The number of 1-handles in $\mathcal{H}$ is equal to $\mathrm{en}(M_*(k))$.
2. If $K_{\mathcal{H}}$ is one of the 2-complexes naturally associated with $\mathcal{H}$, then $M_*(k) \diagup\!\!\overset{3}{\searrow} K_{\mathcal{H}}$.

Let $\mathrm{hg}(M)$ denote the Heegaard genus of a closed 3-manifold M. From the definitions of Heegaard genus and extended Nielsen genus, we know that $\mathrm{rank}(\pi_1(M)) \leq \mathrm{en}(M,k) \leq \mathrm{hg}(M_*(k))$. There are only a couple of classes of 3-manifolds for which strict inequality $\mathrm{rank}(\pi_1(M)) < \mathrm{hg}(M)$ is known to hold: One of these is a family of the Seifert fibered spaces identified by Boileau and Zieschang [BZ84]. (See also the graph manifolds of Moriah–Schultens and Weidmann [SW07], Schultens–Weidmann, and Weidmann [RW03]). For the Boileau–Zieschang examples, the rank is 2 and the Heegaard genus is 3. Moreover, by the work of Montesinos on the Boileau–Zieschang manifolds [JM86], $\mathrm{en}(M_*(1)) = 2$ for at least one of these examples. We attempt to prove here that for every closed 3-manifold M, and for every positive integer k, there is a 4-manifold N bounded by $2M_*(k)$

and a minimal handle presentation for N. For a 3-manifold M and a positive integer k, we will identify two obstructions to getting minimal handle presentations for 4-manifolds bounded by $2M_*(k)$. For $k = 1$, when the two obstructions vanish, then minimal handle presentations exist. The first obstruction will be an obstruction to the method used here. All known examples of this first obstruction, including some highly complicated ones, either vanish or can be made to vanish by a modification of the obstruction definition. When the first obstruction vanishes, the second obstruction becomes, in certain important cases, a topological obstruction. In the case of homotopy 3-spheres M for which $M_*(1)$ 3-deforms to a point, when the first obstruction vanishes, the second obstruction vanishes if and only if M is a 3-sphere. By Perelman's solution of the Poincaré Conjecture, this is thus always the case and we have the possibility of putting together an independent reduction of the Poincaré Conjecture to the Andrews–Curtis Conjecture.

Kapitza, (see [PK01],[PK11a], and [PK11b]), attempts to get a minimal handle presentation for a 4-manifold bounded by the double of one of the Boileau–Zieschang examples, specifically one that is identified by Montesinos as having extended Nielsen genus 2. He uses a Heegaard decomposition of the Boileau–Zieschang example of genus 95 to find an instance where the first obstruction to minimality vanishes. Notice that if there is a minimal handle presentation in this case, then $\mathrm{en}(M)$ is an integral invariant measuring geometric simplicity of 3-manifolds that is sometimes less than the Heegaard genus.

2. Definitions, examples, and implications

Presentations and reduced presentations. A *presentation* for a group G is an expression $\mathcal{P} = \{x_1, \dots, x_n \mid r_1, \dots, r_p\}$. The x_i's are the *generators* and the r_j's are the *relators*. The relators are words on the alphabet $\{x_i^{\pm}\}$. The relators are not necessarily freely reduced. We denote by $\overline{\mathcal{P}}$ the corresponding reduced presentation $\{x_1, \dots, x_n \mid \bar{r}_1, \dots, \bar{r}_p\}$ where $\bar{r}_i$ denotes the free reduction of the word r_i.

We will be working with 2-handle presentations for 4-manifolds and with longitudinal surgery presentations for 3-manifolds. All the work here is in PL. The work is done visually on diagrams. We can take freely from diagrams in DIFF and convert them to PL. The basic facts from references such as [JS68] and [RS82] will be assumed. For a handle presentation $\mathcal{H}$ and an integer ℓ, let $N(\mathcal{H}, \ell)$ or $N(\ell)$ for short, denote the union of all handles of index less than or equal to ℓ. For $N(\mathcal{H}, 1)$, we will always assume that this manifold has the form $J \times [-1, 1]$ where J is a 3-dimensional cube with handles. We will denote the boundary of this manifold by $\Sigma(n)$, a sphere with n handles.

Split 4-manifolds. Getting minimal handle presentations is regarded at the start as a problem of converting algebraic handle cancellation to geometric handle cancellation. We restrict attention to a special class of 4-manifolds where this problem can be treated as an obstruction problem. We say that a 4-manifold N is a *split* 4*-manifold* provided that there is a 3-dimensional cube with handles J and disjoint unions of disks D_- and D_+ together with homeomorphisms $h_\pm : \mathrm{Bd}(D_\pm) \times [0,1] \to \mathrm{Bd}(J)$ so that N is represented as the identification space,

$$\begin{aligned} D_- \times [0,1] \times [-1,-1/2] \coprod_{(x,s,t)=(h_-(x,s),t))} & J \times [-1,1] \\ \coprod_{(h_+(x,s),t))=(x,s,t)} & D_+ \times [0,1] \times [1/2,1]. \end{aligned}$$

If the images of h_- and h_+ are disjoint, then N can be reparameterized as an *ordinary product manifold* $M_* \times [-1,1]$. We do not know whether all 4-manifolds with boundary that have 2-handle presentations admit split structures; however we do know that every 3-deformation type of 2-complex is the natural spine for a 4-manifold with a split handle presentation in the sense of normal handle presentations defined below. See [CC84] and [RCip].

Complex associated with a handle presentation. With any handle presentation $\mathcal{H}$ for N, a manifold with boundary, where the index of the handles is at most n, there is an associated n–complex $K_{\mathcal{H}}$ that is well defined up to formal $(n+1)$–deformation. This is built up by deforming attaching cells for the successive handles. See [RS82] for example. For the 2-handle presentations here, the associated 2-complexes are well defined up to 3-deformation; thus invariants of extended Nielsen type are associated with them.

Normal handle presentations, algebraic handle cancellation. Consider a split 4-manifold with handle presentation $\mathcal{H}$ where the union $N(1)$ of the 0- and 1-handles is $J \times [-1,1]$. We will say that the presentation is a *normal handle presentation* provided that the following two conditions hold:

(1) The 2-handles may be taken to be the products

$$D_{-,j} \times [0,1] \times [-1,-1/2] \quad \text{and} \quad D_{+,j} \times [0,1] \times [1/2,1]$$

where $D_{\pm,j}$ are the components of $D_\pm$.

(2) The attaching spheres for the 2-handles are the curves

$$h_\pm((\mathrm{Bd}(D_\pm) \times \pm 1)).$$

For any normal handle presentation, $\mathcal{H}$, we say that p 1-handles can be *cancelled algebraically* provided that there is a basis $\{x_1, \ldots, x_n\}$ for the fundamental group of J corresponding to a complete system of meridian disks in J such that the attaching words for p of the 2-cells (read by the attaching maps) when projected to the fundamental group of J, read, after free reduction, p distinct basis elements $x_i (1 \le i \le p)$. For later reference we identify two copies of J:

$$\begin{aligned} J_+ &= J \times 1 \cup (\mathrm{Bd}(J)) \times [0.75, 1], \\ J_- &= J \times -1 \cup (\mathrm{Bd}(J)) \times [-1, -0.75]. \end{aligned}$$

Finally let us say that a normal handle presentation $\mathcal{H}$ for a 4-manifold N with boundary $2M_*$ is *algebraically minimal* (relative to the boundary $2M_*$) provided that following conditions hold:

(1) The 2-complex $K_{\mathcal{H}}$ associated with $\mathcal{H}$ formally 3-deforms to M_*.
(2) The number p of the 1-handles can be cancelled algebraically where $\mathcal{H}$ has n 1-handles and $n - p = \mathrm{en}(M_*)$.

Algebraic handle cancellation can be turned into actual geometric handle cancellation and the sought for minimal handle presentations if attaching spheres reflecting algebraic handle cancellation can be adjusted so that the attaching words are freely reduced. This approach is summarized in the statement of the two results below. The second result is well known, and the first one is effectively done by Kapitza [PK11a].

Theorem 1 (Christ [CC84])**.** *For each* 3*-manifold* M *and each integer* $k \geq 1$*, the* 4*-manifold* $M_*(k) \times [-1,1]$ *has a normal handle presentation in which all but* $\mathrm{en}(M_*)$ 1*-handles can be algebraically cancelled.*

Proposition 1. *For any one of the handle presentations described in Theorem* 1*, suppose that there is an isotopy of* $\mathrm{Bd}((J) \times [-1,1])$ *that causes, for the* 2*-cells that contribute to algebraic handle cancellation, the attaching words for the* 2*-cells to be freely reduced.*

Then $M_*(k) \times [-1,1]$ *has a minimal handle presentation.*

The work by Christ cited above is a Diplom Arbeit at the University of Frankfurt. Much sharper results here will come when we allow the actual 4-manifold $M_*(k) \times [-1,1]$ to change. We will employ, in §7, a variation on the usual surgery calculus for framed longitudinal surgeries on 3-manifolds [TW64a] and [TW64b], [RK78], and [GS99]. In this calculus, we will distinguish *equivalence* $\approx$ and *strong equivalence* $\overset{s}{\approx}$. For any 3-manifold M and corresponding $M_*(k)$, and for any normal handle presentation $\mathcal{H}$ for $M_*(k) \times [-1,1]$ that admits algebraic cancellation of all but $\mathrm{en}(M_*(k))$ 1-handles, we will associate three framed links $\mathcal{R}$, $\mathcal{L}$, and $\mathcal{T}$, involving disjoint links. The surgery equivalence classes of these framed links will measure obstructions to converting the algebraic handle cancellation in $\mathcal{H}$ to geometric handle cancellation for a handle presentation of a possibly different 4-manifold that has the same boundary $2M_*(k)$ as before. We will say that the obstruction conditions are *satisfied*, or say that the obstructions are *trivial*, if $\mathcal{L} \overset{s}{\approx} \emptyset$ and $\mathcal{T} \cup \mathcal{L} \approx \mathcal{T}$.

3. Statement of some of the main results, further implications

Theorem 2. *Let* M *be a closed compact* 3*-manifold that is a rational homology sphere, and let* $k = 1$*. Suppose that* $\mathcal{H}$ *is a normal, algebraically minimal handle presentation of* $M_*(1) \times [-1,1]$ *with* n 1*-handles. Let* $\mathcal{R}$*,* $\mathcal{L}$*,* $\mathcal{T}$ *be the associated framed links.*

If $\mathcal{L} \overset{s}{\approx} \emptyset$ and $\mathcal{T} \cup \mathcal{L} \approx \mathcal{T}$, then there is a 4-manifold N bounded by $2M_*(1)$ that has a minimal handle presentation.

Theorem 3. *Let M be a homotopy 3-sphere, let $\mathrm{en}(M_*(k)) = 0$ for $k = 1$ or 2, and suppose that $\mathcal{L} \overset{s}{\approx} \emptyset$ for some algebraically minimal normal handle presentation.*

Then there exists a minimal handle presentation for a 4-manifold bounded by $2M_*(k)$ if and only if M is a 3-sphere. For $k = 1$, M is a 3-sphere if and only if $\mathcal{T} \cup \mathcal{L} \approx \mathcal{T}$.

Consider a homotopy 3-sphere M. For any $k \geq 1$, $M_*(k)$ collapses to a bouquet of $(k-1)$ 2-spheres wedged with a 2-spine of the homotopy cube $M_*(1)$. It is well known that there is an integer $k \geq 1$ such that $\mathrm{en}(M_*(k)) = 0$. In this case, a minimal handle presentation for a 4-manifold bounded by $2M_*(k)$ would consist of a 0-handle and $(k-1)$ 2-handles. Thus the homotopy sphere with $(k-1)$ handles, $2M_*(k)$, would result from the 3-sphere by longitudinal surgery on a link of $k-1$ components. There is a long outstanding conjecture, the Generalized Property R Conjecture, that the framed link is equivalent, without stabilization by either of the two kinds, to the 0-framed unlink; hence in this case surgery returns a sphere with $(k-1)$ handles. See [RK97]. Thus, if the Generalized Property R Conjecture holds, then $2M$ and hence M would be a 3-sphere. For $k = 1$ or 2 special considerations apply. If $k = 1$, and a minimal handle presentation exists for some N, then N has just a 0-handle and $2M$ is the boundary of a 4-ball; so $2M$ is 3-sphere and hence M is a 3-sphere. If $k = 2$ and a minimal handle presentation exists, then $2M_*(2)$ is the connected sum of $2M$ and $S^2 \times S^1$. Gabai's solution to the Property R Conjecture [DG87] (see also Gordon–Luecke [GL89]) confirms, in this case, the Property R Conjecture, and shows that M is again a 3-sphere.

Effect of the Poincaré Conjecture on Theorem 3. First of all, we know that M is a 3-sphere and that the equivalence $\mathcal{T} \cup \mathcal{L} \approx \mathcal{T}$ always holds. This allows us to sketch out the reduction of the Poincaré Conjecture to the Andrews–Curtis Conjecture if we do not use the conjecture in carrying out these steps:

(1) Establish that $\mathcal{T} \cup \mathcal{L} \approx \mathcal{T}$ holds.
(2) Show that for some normal, algebraically minimal handle presentation for $M_* \times [-1, 1]$, the triviality $\mathcal{L} \overset{s}{\approx} \emptyset$ holds.

The first of the two tasks should be made easier by the fact that we know that the equivalence is true.

If the Andrews–Curtis Conjecture were false for 2-spines of homotopy 3-spheres, then $\mathrm{en}(M) \geq 2$ for some homotopy 3-sphere M. Now, by a result of Haken's [WH68], Heegaard genus is additive for connected sums, $\mathrm{hg}(M\#\dots\#M) = \mathrm{hg}(M) + \cdots + \mathrm{hg}(M)$, but extended Nielsen genus would not be additive for a homotopy 3-sphere M if $\mathrm{en}(M) > 0$. In particular we would have $\mathrm{en}(M\#\ldots\#M) \leq \mathrm{en}(M)$ (see [RC75]); so if minimal handle presentations always exist, then there would be, in this case, handle presentations for 4-manifolds bounded by doubled homotopy 3-spheres in which the number of 1-handles would be smaller than that promised by Heegaard genus. In fact, the difference $\mathrm{hg}(M) - \mathrm{en}(M)$ could be made arbitrarily large.

4. Cancellation segments; factorings

A disjoint system of simple closed curves $\{S_i\}$ on a compact, closed, orientable surface Q is *complete* if the curves do not separate Q and no other simple closed curve disjoint from the S_i can be added to the collection so as to preserve the nonseparation property. Similarly, a disjoint system of meridian disks in a cube with handles J is *complete* if the union does not separate, but any larger disjoint collection of meridian disks does separate. Let us say that a system of disjoint simple closed curves on a surface Q is *saturated* if it contains a complete subsystem. Similarly call a system of disjoint, properly embedded disks in a cube with handles *saturated* if it contains a complete system of meridian disks. Given a cube with handles J and a complete system of meridian disks $\{E_\ell\}$ on J, there is a standard way of reading elements from the fundamental group of J. One assigns positive and negative sides to the disks E_ℓ. One can regard $J \backslash \bigcup E_\ell$ as the basepoint of J. Then given any loop in J one reads the letters x_ℓ^- if the loop passes through E_ℓ from the positive to negative side and x_ℓ^+ if the loop passes through E_ℓ from the negative to the positive side. With a saturated system of disks we can do the same thing except the readings correspond to elements of the fundamental group only after intersections with certain disks E_ℓ are ignored. But even in this case, for homotopically trivial loops in J, the associated words cancel down to the empty word regardless of whether the system of disks includes more than a complete system.

Cancellation segments. Let J be a cube with n handles and $E = \bigcup E_\ell$ a saturated system of disks in J. Let $R = \bigcup R_i$ be a union of disjoint, oriented, simple closed curves $\bigcup R_i$ on $\mathrm{Bd}(J)$ intersecting E transversally. Let each R_i be provided a base point $*_i$ missing E. By a *cancellation segment* on one of the curves R_i we mean an arc A on $R_i \backslash *_i$ that begins and ends on some E_ℓ, abutting on the same side at the two ends, and is homotopic relative to its end points to an arc in E_ℓ. Denote the initial and terminal points of the cancellation segment A by $\iota(A)$ and $\tau(A)$. A cancellation segment A is *irreducible* if no proper initial subsegment or terminal subsegment is a cancellation segment. See Figure 1 for an example of a reducible cancellation segment. An equivalent, and useful, condition for irreducibility is the following: Lift A and E_ℓ to $\tilde{A}$ and $\tilde{E}_\ell$ in the universal cover $\tilde{J}$ of J so that $\tilde{A}$ begins and ends on $\tilde{E}_\ell$. Then A is irreducible if and only if $\tilde{A} \cap \tilde{E}_\ell = \mathrm{Bd}(\tilde{A})$.

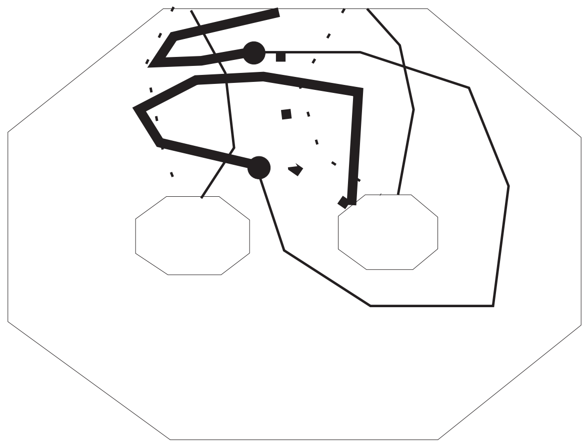

Figure 1.

Factoring, normal form. Let $\{A_i\}$ be a collection of irreducible cancellation segments in R defined by a union of disks E. Assume that whenever two of these segments intersect, one of them is contained in the other. By a *factoring* of the segments $\{A_i\}$ we mean a collection of cancellation segments $\{B_j\}$ containing $\{A_i\}$ where each B_j is contained in some A_i and the collection $\{B_j\}$ is maximal with respect to satisfying the irreducibility and containment conditions just mentioned. A particular factoring is described in the next paragraph. Factoring here corresponds to tree factoring in [RC89] and [RC93].

Initial normal form. For the least possible R_i, choose the least point of $R_i \cap E$ that is an initial point for a cancellation segment. Then among all cancellation segments beginning at this initial point, choose the one that has the least terminal point. Add this segment to the collection $\{A_i\}$ and apply this same construction repeatedly until a collection of cancellation segments B_j is created that accounts for every point of every $R_i \cap E$. We define this collection of irreducible cancellation segments to be an *initial normal form.*

5. A review of previous link obstructions

The framed link obstructions here derive from the free reduction approach to getting minimal handle presentations [RC89] and [RC93]. In the second reference a family of 1-dimensional links (without framing) measures the obstruction to geometrically realizing, by an isotopy, free reduction of attaching words for 2-handles. A natural framing on the links above will enable us to regard the link obstructions above as surgery obstructions.

The most elementary example of an obstruction to free reduction, the Hopf Link, will turn out to be trivial as a surgery obstruction. We review here the obstruction links of [RC93]. Consider first the case of an ordinary handle presentation $\mathcal{H}$ with attaching spheres $R = \bigcup R_i$. Let $R(0) \subset R$ be a subunion of attaching spheres. Let there be given a complete system of meridian disks in J giving rise to attaching words and an irreducible factoring of the cancellation segments. By taking thin regular neighborhoods of the disks E_ℓ that are small with respect to all items previously mentioned, we can get very close parallel copies $E_{\ell\pm}$ of the disks E_ℓ that enjoy the same transverse intersection properties with the curves in R as the disks E_ℓ do. For each arc A_{ij} from the factoring, extend A_{ij} slightly in R_i so that its initial and terminal points are shifted from some E_ℓ to a corresponding $E_{\ell\epsilon}$. Then push the extended arcs to one side of $R(0)$ using the fact that $R(0)$ is 2-sided in $\mathrm{Bd}(J)$. From Lemma 3.1 of [RC93] it follows that on each $E_{\ell\epsilon}$ containing end points of pushed arcs, the pairs of end points can be connected by disjoint arcs, called *lips*, properly embedded in that $E_{\ell\epsilon}$. Adding the lips to the adjusted cancellation segments creates a link L, the *obstruction link*, that is disjoint from R. For normal handle presentations, there is an immediate extension obtained by forming two links $L_+ \subset J_+$ and $L_- \subset J_-$ for the handles attached to $\mathrm{Bd}(J_+)$ and $\mathrm{Bd}(J_-)$ respectively. The union of the two links $L = L_+ \cup L_-$ is the *obstruction link*.

After a change of language from 2-complexes to 2-handles, we get the following main result of [RC93].

Theorem 4. *Let $\mathcal{H}$ be a normal handle presentation. Let $R(0) \subset R$ be a subunion of the attaching spheres for the 2-handles, and consider the family of all obstruction links L corresponding to the subunion $R(0)$.*

Then a necessary and sufficient condition in order that there be an isotopic deformation of $\mathrm{Bd}((J) \times [-1,1])$ that causes the attaching words for the 2-handles associated with $R(0)$ to be freely reduced is that one of the links $L = L_+ \cup L_-$ in the family be trivial in $J_+ \cup J_-$ in the sense that the components of L bound disjoint disks in $J_+ \cup J_-$.

6. Framed links and surgery

Given a 3-manifold M possibly with boundary, a *framed link* is a pair (S, A) $(S \subset M)$ where $S = \bigcup S_i$ is a link of simple closed curves S_i in M and $A = \bigcup A_i$ is a union of embedded annuli in M, called *framing annuli*, containing S in its boundary and defining trivializations $\mu : S \times D^2 \to M$ of tubular neighborhoods of the curves S_i in M' where M' is some 3-manifold containing M in its interior. (If $x \in S$ then $(x, 0) \to x$.) (For technical reasons it is sometimes necessary for us to allow curves S_i to reside totally or partially in $\mathrm{Bd}(M)$.) Two trivializations μ and μ' are *equivalent* if there is an ambient isotopy of M' fixing S and converting μ to μ'. If $x \in \mathrm{Bd}(D^2)$ then when a tubular neighborhood of S is removed from M', new solid tori are to be sewn in with each $\mu(S_i \times x)$ becoming a meridian in the resewn torus.

The connection with the framing annuli is that for one of the equivalence classes of trivializations μ, we have $A_i = \mu(S_i \times B)$ where B is an arc in D^2 running from 0 to $\mathrm{Bd}(D^2)$. When S is homologously trivial, the framing can be described by specifying the linking number of the two components of the boundary of the framing annulus where the two boundary components are oriented in parallel. When this linking number is 0, the framing is called the 0-*framing*. The framing is connected with surgery in the following way: The trivialization $\mu : S \times D^2 \to M'$ of the tubular neighborhood of the link S, defines a union of tori $\mu(S_i \times D^2)$ with distinguished boundary curves $\mu(S_i \times *)$ where $*$ is a point of $\mathrm{Bd}(D^2)$. Removing the union of tori from M' and resewing them so that meridian disks are attached to the curves $\mu(S_i \times *)$ yields a new 3-manifold that results from M' by *surgery on the framed link* (S, A).

The natural framing. Let J be a cube with handles and D a finite union of disjoint properly embedded disks in J. Let S be a simple closed curve in J. Assume that the following two conditions hold:

(1) S is either contained in $\mathrm{Bd}(J)$ or it is a union of proper arcs in D together with arcs in $\mathrm{Bd}(J)$ whose interiors intersect $\mathrm{Bd}(D)$ transversally in $\mathrm{Bd}(J)$.
(2) For each arc component A of $S \cap D$, the ends of the arc component of $S \backslash A$ abut on the same side of D in J.

Under these conditions, a preferred framing of S exists that we refer to as the *natural framing*. If S is contained in $\mathrm{Bd}(J)$, then S is 2-sided in $\mathrm{Bd}(J)$ and so there is an annulus A for S that is contained in $\mathrm{Bd}(J)$. We take this to define the natural framing. In the case where S has pieces in $\mathrm{Int}(J)$, the framing annulus is defined by a product embedding $A = \mu(S \times [0, 1])$ where $\mu(x, 0) = x$, and $A \subset \mathrm{Bd}(J) \cup D$. Here $\mu(x \times [0, 1])$ intersects D if and only if $x \in D$ in which case $\mu(x \times [0, 1]) \subset D$. To get the annulus, start laying out at some point of $S \backslash D$ following the rules given above. Given the start, there is always a unique side of S on which to continue. The question is, is it possible that we end up on the opposite side of S when we try to complete the layout? If that were to occur, we could complete another cycle about S defining a Moebius band. Using the abutment condition we could push this Moebius band to an immersion in $\mathrm{Bd}(J)$ exhibiting a nonorientable curve in the orientable $\mathrm{Bd}(J)$.

7. Operations on framed links

Consider a 3-manifold M possibly with boundary and a framed link $\mathcal{S} = (S, A)$. Consider the following operations on $\mathcal{S}$:

(1) Shrink the whole of M by pushing in along the fibers of a collar on $\mathrm{Bd}(M)$ so that M is pushed onto the closure of the complement of a collar on $\mathrm{Bd}(M)$. Apply the same push to the framing annuli. This

operation insures that all curves in a surgery presentation can be modified to have solid tori for tubular neighborhoods.
(2) Inverse of the previous operation.
(3) Replace any annulus A_i by an annulus A'_i that defines the same equivalence class of framing.
(4) Replace S and A by $H_1(S)$ and $H_1(A)$ where H_t is an ambient isotopy of M.
(5) Modify any S_k and A_k by sliding S_k over some $S_j (j \neq k)$.
(6) Add or delete a 0-framed Hopf Link that is contained in the interior of a 3-ball in M where the 3-ball does not intersect any other part of the surgery curves.
(7) Add or delete a 0-framed unknotted curve that is contained in the interior of a 3-ball in M where the 3-ball does not intersect any other part of the surgery curves.

Define two framed links in M to be *equivalent* if one can be converted to the other by a finite sequence of the operations above. The chief differences with the usual calculi for framed links (Wall [TW64a] and [TW64b], Kirby [RK78], Gompf & Stipsicz [GS99]) are two: Adding the 0-framed unknot is not considered in the two theories just cited, and the Kirby move involving a ± 1 framed unknot, is not considered here. The following two propositions are well known (for the second one see 5.11 of [GS99] or [TW64b]).

Proposition 2. *If $\mathcal{S}$ and $\mathcal{S}'$ are equivalent framed links in a 3-manifold M, then up to connected sum with copies of $S^2 \times S^1$, the two corresponding surgeries present the same 3-manifold.*

Proposition 3. *Let $\mathcal{S} = (S, A)$ be a framed link and $\{S_0, S_1\}$ a pair of homotopically trivial components of S such that:*
(1) *S_0 is evenly framed.*
(2) *S_1 has the 0-framing and bounds a disk intersecting S_1 transversally in a single point and not intersecting the remaining curves of S.*

Then deleting $S_0 \cup S_1$ from the surgery $\mathcal{S}$ results in an equivalent presentation.

Our interest is in framed links in Σ with all the curves located in $J_\pm$; however the transformations will be applied sometimes with $M = J_+ \cup J_-$ and sometimes with $M = \Sigma = \mathrm{Bd}(J \times [-1, 1])$.

8. Invariants for framed links

Here we look at properties that are invariant under the operations on the framed links. All of our framed links are associated with the sphere with n handles $\Sigma(n)$; although we look in some cases at the subspaces $J_\pm$ to consider equivalence. The source of invariants is the linking form defined on any framed link $\mathcal{S}$ consisting of homologously trivial curves. See for example Chapter 4 of [GS99]. Given a framed link $\mathcal{S}$ in $\Sigma(n)$ based on curves $S_1, \dots, S_p$ all homologously trivial in $\Sigma(n)$, the linking form is an integral symmetric bilinear form with matrix $B^{p\times p}$ defined as follows: Give each S_i an orientation. The entry b_{ij} is the linking number in $\Sigma(n)$ of the pair (S_i, S_j). The entry b_{ii} is obtained by taking the linking number of the two components of the boundary of a framing annulus A_i where both boundary components are oriented in the same direction. Two linking forms, represented by symmetric, integral matrices B and B' are *equivalent* if the corresponding matrices B and B' are congruent, that is, if there is a unimodular, integral symmetric matrix C such that $B' = CBC^T$. Notice that changing the orientation of a component results in an equivalent linking form. Invariants of the form are *signature* and *type*, *even* or *odd* accordingly as all the diagonal entries of the linking matrix are even or not. One can also refer to the individual curves and say that the framing on one of the curves is *even* or *odd* accordingly as the linking number of the boundary components of the framing annulus is even or odd.

Except for the stabilizing operations, the other surgery moves correspond to congruence of matrices. The stabilizing operations correspond to replacing a matrix by a direct sum with $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ or $(\,0\,)$. For any positive integer n, let O_n denote the direct sum of n copies of the 1×1 zero matrix (0).

Proposition 4. *Let $\mathcal{S}$ and $\mathcal{S}'$ be equivalent framed links with corresponding linking forms B and B'.*

(1) *After direct sum with suitable numbers of copies of $(\,0\,)$ and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, the matrices B and B' become congruent.*
(2) *If B and B' have the forms $B_0 \bigoplus O_p$ and $B'_0 \bigoplus O_q$ where B_0 and B'_0 have nonzero determinants, then* $\det(B'_0) = \pm\det(B_0)$.

Proof. The first condition is obvious. For the second, there is no loss in supposing that B and B' are actually congruent. Now by invariance of rank, the dimensions of B_0 and B'_0 are the same. By taking sums with $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and $(\,0\,)$, at the expense of changing the sign of the determinants of B_0 and B'_0, we may assume that we have the congruence situation above with $p = q$ and the dimensions of B_0 and B'_0 equal to p. But now for some $2p\times 2p$ unimodular integral matrix,

$$\left(\begin{smallmatrix}C&D\\E&F\end{smallmatrix}\right)$$

we have

$$\left(\begin{smallmatrix}C&D\\E&F\end{smallmatrix}\right)\begin{pmatrix}B_0&O_p\\O_p&O_p\end{pmatrix}\left(\begin{smallmatrix}C&D\\E&F\end{smallmatrix}\right)^T = \begin{pmatrix}B'_0&O_p\\O_p&O_p\end{pmatrix}.$$

This shows that $B'_0 = CB_0C^T$ and so the determinant of B_0 is a divisor of the determinant of B'_0. The same argument in reverse shows that the determinant of B'_0 is a divisor of the determinant of B_0. Thus the two determinants are the same up to sign. $\square$

Remark. We are not saying that decompositions as in (2) always exist, but only that when they do exist, the nonzero determinant is, up to sign change, an invariant.

9. Evenness of framing

We show below that the natural framing on the obstruction link $\mathcal{L}$ is an even framing. Here the condition of irreducibility on the cancellation segments is crucial. Figure 1 shows an example of an odd framing associated with a reducible cancellation segment.

Proposition 5. *Let S be a component of an obstruction link L. Then the natural framing on S is an even framing.*

Proof. Let S be based on an arc A and lip C in a disk D. By the irreducibility criterion, it follows that if we lift S and C to the universal cover $\tilde{J}$ of J so that A is lifted to $\tilde{A}$ that begins and ends on $\tilde{D}$, then $\mathrm{Int}(\tilde{A})$ fails to intersect $\tilde{D}$. From the Dehn Lemma [CP57] and loop theorem [SW58] and [JS60], $\tilde{S}$ bounds a disk E in $\tilde{J}$ with $\mathrm{Int}(E) \subset \mathrm{Int}(\tilde{J})$. In fact, $\tilde{D}$ separates $\tilde{J}$ into two pieces, and the disk E can be chosen to be in the closure of one of the two pieces. From this it is easy to see in fact that the framing of $\tilde{S}$ in $\tilde{J}$ by an annulus in E is equivalent to the natural framing and so projects to the natural framing of S.

Applying the projection map to E we get an immersion of E in J, and by some cut and paste together with piping, we can reduce the singularities of the projection map to clasp double arcs that begin and end on the lip. So we have a clasp disk bounded by S with an annular neighborhood of S mapping nonsingularly to a framing annulus equivalent to the natural framing on S.

But whenever an annulus in a clasp disk is used to frame the boundary, the framing is even; that is, the linking number of the boundary components of the annulus is an even number: The boundary curve meets the interior of the clasp disk transversally in an even number of points. $\square$

Proposition 6. *Let $S \subset L$ be an unknotted component of an obstruction link in J. Then the natural framing on S is the* 0*-framing.*

Proof. We know from the previous proposition that the framing on the curves is even. Let S be a component of the obstruction link L and let S bound a disk E in J. There is no loss in supposing that S and E have been pushed slightly into the interior of J. We know from Proposition 5 that S is the boundary of a clasp disk in J and that this clasp disk can be chosen so that when lifted to the universal cover $\tilde{J}$ of J, the clasp disk becomes a nonsingular disk E. Furthermore, this clasp disk E may be used to define the natural framing on S by means of an annulus A with lifting $\tilde{A}$. Since the projection map is nonsingular on $\tilde{S}$, we can calculate the linking number of the two components of $\mathrm{Bd}(A)$ by calculating linking in $\tilde{J}$. But since $\tilde{E}$ is embedded in $\tilde{J}$ the linking number of the two boundary components of $\tilde{A}$ must be 0. This shows that the natural framing of the components of L is the 0-framing. $\square$

10. Three framed obstruction links, reduced attaching curves

We describe three approaches to the framed obstruction link $\mathcal{L}$. We begin with ordinary handle presentations. We suppose that we are given a union of attaching curves R and a subunion $R(0)$ whose attaching words we want to see freely reduced. We assume that for some basis for the fundamental group of J, the curves in $R(0)$ are precisely the ones that read distinct free generators. We will use ι and τ to denote the initial and terminal points of oriented arc segments.

The Type 1 link. Let L be the obstruction link corresponding to the subset $R(0)$. For the obstruction link $\mathcal{L}$, we take the framed link (L, A) where A gives the natural framing on L in J. We define this to be the *framed obstruction link* of Type 1 associated with $R(0)$. The trefoil example of Figure 2 in [RC93] shows easily that this framed link is not always trivial in the calculus here. For a suitable orientation, the matrix for the bilinear intersection form is the 1×1–matrix $[2]$. If the surgery were trivial, then by Proposition 4, the determinant would have to be ± 1 instead of 2. The next two forms of the obstruction link are modifications intended to help make the obstruction links trivial.

The Type 2 link. Extend the set R minimally to a union of disjoint simple closed curves, R' in BdJ that meets M transversally and contains the union $R'(0)$ of a complete system of curves in BdJ where $R(0) \subset R'(0)$. If J has genus n, then this means that the extension contains n curves whose union does not separate BdJ. There are some arguments for preferring some extensions over others; however if Conjecture 2 of §13 is true, all extensions will have the desired effect of making the obstruction link trivial. Form the Type 1 framed obstruction link associated with $R'(0)$. We say that this is a *framed obstruction link of Type* 2.

The Type 3 link. Let an irreducible factoring of the cancellation segments of $R(0)$ into irreducible subsegments be given, for example the initial normal form. This factoring corresponds to a tree factoring in the language of [RC93]. We will form the obstruction link as before except that we will include components for all the subsegments in the irreducible factoring.

For each E take parallel disks as before. Use natural framing annuli on the curves R_i to push the arcs of the irreducible factoring into disjoint arcs as indicated in Figure 2. The procedure works like this: Push the original arcs half way across the framing annuli. Then find the maximal proper subarcs and push them three fourths of the way across the framing annuli. Then locate the maximal proper subarcs of these and push them seven eighths of of the way across. Continue in this way until all the subarcs are pushed off. As in the case of the Type 1 link, irreducibility allows us to connect up the endpoints of the adjusted arcs in the parallel disks M with disjoint arcs

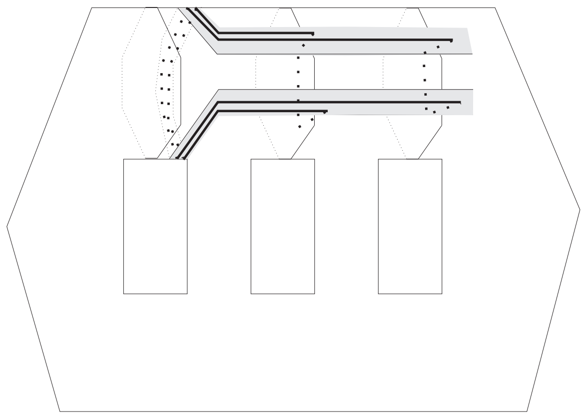

FIGURE 2. Making segments disjoint

(lips). This gives us, with the natural framing, a *framed obstruction link* of Type 3.

The above yields the obstruction links for 3-manifold complexes and handle presentations associated with products of 3-manifolds $M_*(k) \times [-1,1]$. By passing to normal handle presentations and using obstruction links in J_+ and J_- we get corresponding obstruction links and pairs in $J \times [-1,1]$. When we pass to the normal handle presentation case, we have two possible equivalences to use in the surgery calculus: equivalence either in Σ or equivalence in $J_+ \cup J_-$. We refer to *equivalence* $\approx$ in the first case and *strong equivalence* $\overset{s}{\approx}$ in the second.

The reduced surgery link, $\mathcal{T}$. For each component $R_i \subset R(0)$ of R, slide R_i over the corresponding components of L and follow this by an isotopy in J, as indicated in Figure 3, to convert R_i to T_i so that the attaching word for T_i reads the free reduction of the attaching word read for R_i. For the remaining components R_i of R set $T_i = R_i$ and denote by T the union $\bigcup T_i$. Let $\mathcal{T}$ denote the corresponding framed link in Σ where the framing on T is defined by the framing on R together with the slidings and isotopy. We call this the *reduced surgery link*. It is important to remember here that in the reduced surgery, the only curves that have been modified are the ones that contribute to algebraic handle cancellation.

An alternate way to form $\mathcal{T}$ is the following: For each cancellation segment involving the lip construction, apply the construction twice on pairs of disks parallel and very close to each other. There will be two small arcs running between the parallel disks that were contained in extended cancellation segments. Discard these. What remains are a new component of the obstruction link together with a work in progress forming the reduced surgery link. When this construction terminates, the second piece mentioned above becomes the reduced surgery link.

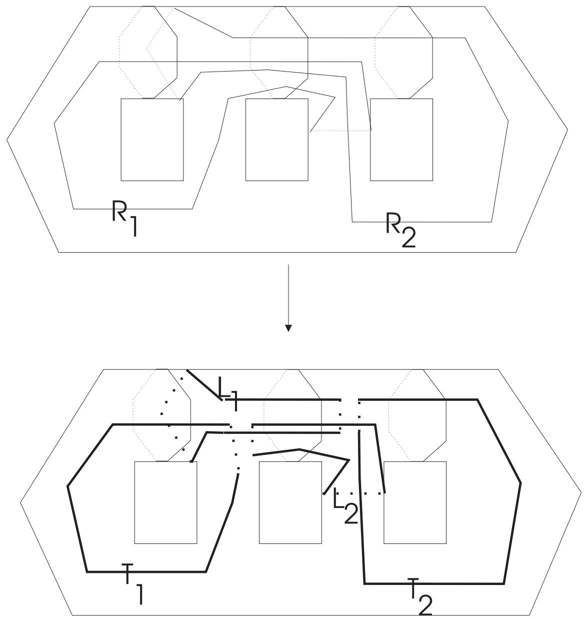

FIGURE 3. Converting R to T

Status of the framed obstruction links. The few examples we know of have the framed obstruction link of Type 2 or 3 strongly trivial. In [PK11a] and [PK11b] Kapitza attempts to come up with a minimal handle presentation for a 4-manifold bounded by the double of one of the Boileau–Zieschang manifolds. Kapitza constructs an algebraically minimal normal handle presentation for $M_*(1) \times [-1, 1]$. This presentation has 95 1-handles and 95 2-handles. Kapitza gets a framed obstruction link $\mathcal{L}$ of Type 1 and a corresponding reduced surgery link $\mathcal{T}$. Before extension, the obstruction link has 56 components involving cancellation with an average word cancellation length of 4. He shows that his $\mathcal{L}$ is strongly trivial thus reducing the existence of a minimal handle presentation for some suitable 4-manifold bounded by the double of the Boileau–Zieschang manifold to showing an equivalence $\mathcal{T} \approx \mathcal{T} \cup \mathcal{L}$.

11. Main results

Proof of Theorem 2. The strong triviality of $\mathcal{L}$ implies that $\mathcal{R} \cup \mathcal{L} \approx \mathcal{R}$. Just push $\mathcal{L}$ into $\mathrm{Int}(J^+ \cup J^-)$ and do the trivialization of $\mathcal{L}$ there holding $\mathcal{R}$ fixed. By construction, $\mathcal{R} \cup \mathcal{L} \approx \mathcal{T} \cup \mathcal{L}$. The triviality of the obstruction pair implies thus that $\mathcal{R} \approx \mathcal{T}$, and so by Proposition 2, $\mathcal{R}$ and $\mathcal{T}$ result in the same 3-manifold up to connected sum with copies of $S^2 \times S^1$.

The framed link surgery $\mathcal{R}$ converts Σ to the double $2M_*(1)$. This is a rational homology sphere, and the product

$$N = J \times [-1,1] + \text{ 2-handles}$$

associated with the attaching spheres in R, the manifold N is a rational homology ball. It follows now that the 4-manifold N' associated with the 2-handle attachment along T is also a rational homology ball, and so again by duality, the result of the framed surgery $\mathcal{T}$ is a rational homology sphere M'. Thus neither $2M_*(1)$ nor M' has a connected summand that is $S^2 \times S^1$. By Proposition 2 we have $M' = 2M_*(1)$. Let $\mathcal{H}$ denote the algebraically minimal handle presentation for N. The associated handle presentation $\mathcal{H}'$ for N' has 1 0-handle and n 1-handles and n 2-handles where n is the genus of the cube with handles associated with $\mathcal{T}$. All but $\mathrm{en}(M_*(1))$ of the 2-handles cancel 1-handles. When that cancelling is done, we have a 2-handle presentation $\mathcal{H}''$ for N' with 1 0-handle, and $\mathrm{en}(M_*(1))$ 1-handles and the same number of 2-handles. By construction, the attaching curves T_i are homotopic to the attaching curves R_i; so the 2-complexes naturally associated with $\mathcal{H}_R$ and $\mathcal{H}'$ 3-deform to each other. It now follows that the handle presentation $\mathcal{H}''$ satisfies the conditions for a minimal handle presentation for N'. We have already observed that $\mathrm{Bd}(N') = 2M_*(1)$. □

Proof of Theorem 3. Given the hypotheses for this theorem, we know from the discussion in §2 that there exist minimal handle presentations if and only if M is the 3–sphere. Suppose that M is the 3-sphere and that $k = 1$. We know that $\mathcal{R}$ surgery produces the 3-sphere. An application of Proposition 3 shows that $\mathcal{T}$ is equivalent to a framed link $\mathcal{T} \cup \mathcal{R}'$ where $\mathcal{R} \subset \mathcal{R}'$, and the components of $\mathcal{R}'\backslash\mathcal{R}$ are homotopically trivial in Σ. Do the surgery corresponding to $\mathcal{R}$. This converts Σ to the 3-sphere, and it converts the relative surgery $\mathcal{T} \cup (\mathcal{R}'\backslash\mathcal{R})$ to a surgery turning the 3-sphere back into itself. It is easy to check that this relative framed link in the 3-sphere has as associated symmetric bilinear linking form, a unimodular form of even type and signature 0. This follows from the fact that the linking form restricted to half a basis, $(\mathcal{R}'\backslash\mathcal{R})$, is trivial. By Remark 3 on page 52 of [RK78], or after drawing the diagrams in [TW64a]and [TW64b], it follows that the relative framed link is equivalent to the empty link. That implies, back in Σ, that the two framed links $\mathcal{R}$ and $\mathcal{T}$ are equivalent. But then we have the following chain of equivalences relating $\mathcal{R}$ and $\mathcal{T}$:

$$\mathcal{T} \approx \mathcal{R} \approx \mathcal{R} \cup \mathcal{L} \approx \mathcal{T} \cup \mathcal{L}$$

establishing the equivalence $\mathcal{T} \cup \mathcal{L} \approx \mathcal{T}$.

Finally, if the homotopy 3-sphere M is not a 3-sphere, then, by Proposition 2, $\mathcal{R}$ and $\mathcal{T}$ cannot be equivalent, and neither can $\mathcal{T} \cup \mathcal{L}$ and $\mathcal{T}$. □

Here is a good place to discuss the effect of the Poincaré Conjecture on the main results, Theorems 2 and 3. First, the homotopy 3-sphere M is always a 3-sphere. This implies that $\mathcal{T} \cup \mathcal{L} \approx \mathcal{T}$ always holds. Thus by

Theorem 3, there are always minimal handle presentations and $\mathcal{R}$ and $\mathcal{T}$ are equivalent. Given our goal of reducing the Poincaré Conjecture to the Andrews–Curtis Conjecture, we are trying to show that when $\mathrm{en}(M) = 0$, we have $\mathcal{T} \cup \mathcal{L} \approx \mathcal{T}$. We are trying to prove something that at least we know is true. Turning that knowledge into a proof of the equivalence leaves us with the task of establishing $\mathcal{L} \overset{s}{\approx} \emptyset$ for some suitable algebraicially minimal handle presentation.

12. Other schemes for turning algebraic into geometric handle cancellation

The form of linking used in [RC89] can sometimes be used to turn algebraic into geometric handle cancellation via a change of basis in circumstances where free reduction obstructions are present. Consider an ordinary handle presentation with a complete system of meridian disks $\{E_\ell\}$ for J. Let there be given an irreducible factoring of the cancellation segments $\{A_i\}$ in R, a disjoint union of simple closed curves in $\mathrm{Bd}(J)$. The factoring yields a pairing of the points of $A_i \cap E_\ell$. In fact, the pairs of points are the boundaries of the irreducible cancellation segments in the factoring. List the pairs $P_1, \ldots, P_q$. Form a symmetric $q \times q$–matrix C over $\mathbb{Z}/2\mathbb{Z}$ where c_{ij} is given by 0 if any of the following conditions is met and 1 otherwise:

(1) $i = j$.
(2) P_i and P_j belong to different meridian disks E_ℓ.
(3) P_i and P_j belong to the same meridian disk E_ℓ, but P_i and P_j do not link in $\mathrm{Bd}(E_\ell)$.

A result of Zieschang's on geometrically realizing Whitehead length minimizing automorphisms of free groups [HZ65] and [TK82] says that a change of geometric basis will allow the chosen attaching curves to read free generators. The result below shows that in some cases, one can make this geometric change of basis algebraically the identity.

Theorem 5. *Suppose that an ordinary handle presentation $\mathcal{H}$ for $N = M_* \times [-1, 1]$ is given along with an irreducible factoring of the cancellation segments A_{ij} contained in $R(0) \subset R$, and suppose that the associated bilinear form above, C, has rank q.*

Then there is a change of geometric basis corresponding to new meridian disks $\{E'_\ell\}$ with respect to which each of the curves $R_i \subset R(0)$ reads the reduced version of what it read before.

Proof. This is essentially the Kaneto version [TK82] of Zieschang's theorem [HZ65] mentioned above. It is convenient to switch from meridian disks E_ℓ to meridians e_ℓ, that is, from disks E_ℓ to boundaries e_ℓ. It is also convenient to regard each e_i as a disjoint union of curves in $\mathrm{Bd}(J)$ where the number of curves in each e_i starts out at 1 but grows or shrinks according to the construction below.

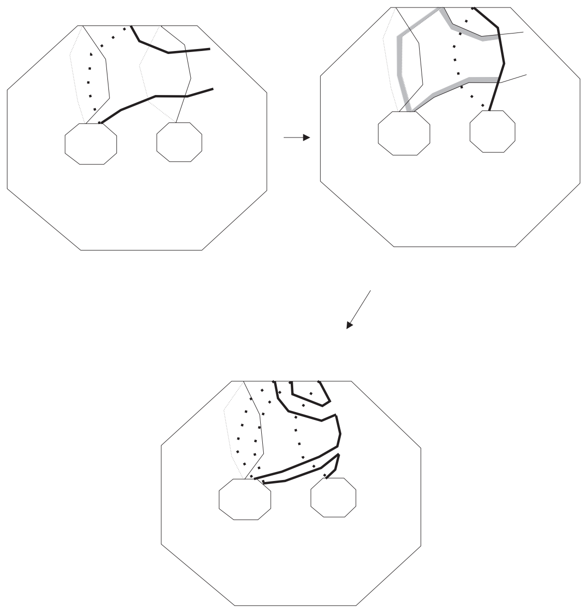

FIGURE 4. Breaking a meridian in two

Let A be an innermost segment in the factoring. This means that no interior points of the segment lie on any meridian e_ℓ. Let the segment begin and end on e_ℓ. Thicken the segment slightly in $\mathrm{Bd}(J)$ on one side of e_ℓ to get a disk E intersecting e_ℓ in two arcs on $\mathrm{Bd}(E)$. See Figure 4. Replace the arc intersection of e_ℓ with E by the remaining part of the boundary of E. This gives a pair of new meridians. Follow Kaneto in continuing to call the union of both pieces e_ℓ even though e_ℓ is disconnected by this action. Repeat this step now for a new shortest segment. The replacement operation may disconnect a component of e_ℓ or it may connect two components. Eventually we arrive at a final revision of the curves e'_ℓ so that all the pairs of points of intersection P_i have been removed. Each curve labeled e'_ℓ is a union of components that are contractible in J.

Kaneto observes that it is possible to choose components of the curves e'_ℓ so that a complete meridian system is obtained. In the circumstances here, it is not necessary to pick components. We claim that for each e_ℓ, the union of 1-spheres e'_ℓ has in fact a single component. To prove this, it is most convenient for us to abstract away the 3-manifold information.

We begin with a disjoint collection of 1-spheres e_ℓ and a collection of 0-spheres $\{P_i\}$ on the 1-spheres. For each 0-sphere P_i we have a very narrow band joining small neighborhoods of the two points so that band intersects the 1-spheres in the union of two small arcs in its boundary. In the order described above, the interiors of the bands never intersect the current stage of the meridians; however we are going to reorder the band moves. Reordering will introduce temporary intersections that we would prefer to ignore. For this reason we regard the bands as abstract. Each band is attached so that the union of the band and a disk bounded by the meridian is a planar surface. For each band, the two arcs on the meridian are traded for the two remaining arc pieces of the boundary of the band. The claim is that after all the band operations are done, each simple closed curve e_ℓ is transformed into a simple closed curve. To show this, we will show that the changes in the meridians $\{e_\ell\}$ can be rearranged in steps $\{e_\ell(k)\} \to \{e_\ell(k+1)\}$ with $\{e_\ell(0)\} = \{e_\ell\}$ with corresponding steps for the matrix $A(k) \to A(k+1)$ so that each step involves a pair of bands attached to the same simple closed curve $e_\ell(k)$ and the bands are attached to thickening of linked 0-spheres $P_i(k)$ and $P_j(k)$ in $e_\ell(k)$. Observe that this does not disconnect a meridian. Since each step does not disconnect the meridian being revised, we see that the end result will indeed be a complete collection of meridians $e_\ell(q)$ as desired.

The matrix A representing the linking form is congruent to a direct sum of matrices of the forms $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and (1). But A represents a bilinear form of even type; so there can be no matrices of the second form in the direct sum. Thus the dimension of A is even.

By reindexing the 0-spheres P_i, arrange things so that the upper left hand corner of the matrix A is $\begin{smallmatrix}0&1\\1&0\end{smallmatrix}$. By subtracting the first two rows and first two columns from suitable rows and columns, we arrive at a congruent matrix with no additional 1's in the first two rows or columns other than the entries in the upper left hand block. The resulting matrix is the direct sum of $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and a matrix $A(1)$ of dimension two smaller than before. The corresponding pair of band operations does not disconnect the meridian e_ℓ that was modified. But the matrix $A(1)$ is the linking matrix for the 0-spheres $P_3, P_4, \ldots P_q$ in the curves $e_\ell(1)$ that result from the first paired band operation. The new matrix has rank and dimension $q-2$. After $q/2$ iterations of this process we end up with the desired connected curves $e_\ell(q/2) = e'_\ell(q)$. □

13. Questions and conjectures

The main results of this paper say that provided that certain surgery conditions are met, algebraic handle cancellation associated with certain 4-manifolds can be turned into geometric handle cancellation for handle presentations of different 4-manifolds that have the same boundary. We examine now Theorem 4 to see if that result can be explained by the framed link obstruction approach here.

Conjecture 1. *Let $\mathcal{H}$ be a normal handle presentation for a 4-manifold $N = M_* \times [-1,1]$ and suppose that for some basis for $\pi_1(N(1))$, the presentation $\mathcal{H}$ allows the algebraic cancelling of k 1-handles. Here $N(1) = J \times [-1,1]$. Let the basis for $\pi_1(N(1))$ correspond geometrically to a complete set of* *meridian disks in J, and suppose that L in the obstruction link $\mathcal{L}$ is trivial in the sense of* §5. *Then both obstructions, $\mathcal{L}$ and $(\mathcal{T} \cup \mathcal{L}, \mathcal{T})$ vanish.*

Partial proof. To show that $\mathcal{L}$ is trivial, it is sufficient to show that the natural framing on the components of L is the 0-framing. This is Proposition 6. To show that $(\mathcal{T} \cup \mathcal{L}, \mathcal{T})$ is trivial, given the disjoint 2-cells property of the curves in the obstruction link, it would be sufficient to show that the disks bounded by the components of L can be arranged so that they do not intersect T.

Conjecture 2. *If $\mathcal{L}$ is a framed obstruction link of Type* 2 *or Type* 3 *then $\mathcal{L}$ is strongly trivial.*

Conjecture 3. *For any normal handle presentation for $M_* \times [-1, 1]$, we have the equivalence $\mathcal{T} \cup \mathcal{L} \approx \mathcal{T}$ provided that $\mathcal{L}$ is a framed obstruction link of Type* 1*, or Type* 2*, or Type* 3 *and is strongly trivial.*

What evidence we have points to the truth of the first conjecture. Although the second conjecture seems forbidding, we feel that somehow the two conjectures are related. A relation is suggested by the fact that Σ can be thought of in two ways:

(1) the boundary of the 4-manifold that results from adding 1-handles to the 4-ball;
(2) the result of doing 0-framed surgery on an unlink in the 3-sphere.

The second way of looking at things also corresponds to one of the stabilization moves in the surgery calculus we use here.

The correct definition of $\mathcal{L}$. The choice of the framed obstruction link $\mathcal{L}$ involves a compromise. The link needs to be simple enough that it is trivial and yet complex enough that it affords free reduction of attaching words by sliding. The link has to link the attaching spheres in a simple enough way that the equivalence $\mathcal{T} \cup \mathcal{L} \approx \mathcal{T}$ can be shown. We remark that it is easy to use Proposition 3 to define a framed link $\mathcal{L} \approx \emptyset$ that permits the desired free reduction. But then showing $\mathcal{T} \cup \mathcal{L} \approx \mathcal{T}$ becomes the problem. In the end, there is only one real test for the correctness of the definition of the framed obstruction link: Can Conjecture 2 be proved, and, when minimal handle presentations exist, can their existence be established by the main construction here?

Other surgery theories. To what extent is our choice of surgery calculus critical to the results here? Is it possible to get some of these results using a theory with the Kirby move, say with the zero framed unknot operation discarded? Some caution seems in order here. It seems probable that when manifolds are produced using even surgeries, one copy, at least, of the odd framed surgery on the unknot must remain behind in the sense that stably, that surgery can be recovered from the presentation. Our choice of moves is made in part because we have not seen any examples to indicate that any more moves are needed. Also it seems a shame to give up the even framing on $\mathcal{L}$ by exposure to Kirby moves since the evenness is free.

Graph manifolds. In the Introduction it was pointed out that the source of the significant examples was Seifert fibered manifolds or more generally graph manifolds. It would seem to be worthwhile to try to produce the surgeries $\mathcal{R}$ and $\mathcal{L}$ using the graph manifold structure.

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