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# A generalization of Jørgensen's inequality to infinite dimension

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Abstract. In this paper, we give a generalization of Jørgensen's inequality to hyperbolic Möbius transformations in infinite dimension by using Clifford algebras. We also give an application.

Contents



## 1. Introduction

In the theory of discrete groups, the following important and useful inequality is well known as Jørgensen's inequality, see [5].

**Theorem J.** *Suppose that $f, g \in M(\overline{\mathbb{R}}^2)$ generate a discrete and nonelementary group $\langle f, g\rangle$. Then*

$$|\mathrm{tr}^2(f) - 4| + |\mathrm{tr}([f, g]) - 2| \geq 1.$$

In [4], Hersonsky gave a partial generalization of Theorem J to Möbius transformations in $\overline{\mathbb{R}}^n$ by using Clifford algebra, which is stated in the following form.

**Theorem H.** *Let $f, g \in M(\overline{\mathbb{R}}^n)$ such that $f$ and $[f, g]$ are hyperbolic, and suppose that $\langle f, g\rangle$ is a discrete and nonelementary group. Then*

$$|\mathrm{tr}^2(f) - 4| + |\mathrm{tr}([f, g]) - 2| \geq 1.$$

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In [12], Waterman generalized Jørgensen's inequality to high dimensional groups and obtained

**Theorem WA.** *Let $f$, $g \in M(\overline{\mathbb{R}}^n)$. If $\langle f, g\rangle$ is discrete and nonelementary, then*

$$\|f - I\| \cdot \|g - I\| \geq \frac{1}{32}.$$

In [11], Wang also studied the generalization of Jørgensen's inequality to hyperbolic Möbius transformations in high dimension, giving the following generalization of Theorem H.

**Theorem W.** *Let $f, g \in M(\overline{\mathbb{R}}^n)$ such that $f$ is hyperbolic and $[f, g]$ is vectorial, and suppose that $\langle f, g\rangle$ is a discrete and nonelementary group. Then*

$$|\mathrm{tr}^2(f) - 4| + |\mathrm{tr}([f, g]) - 2| \geq 1.$$

We refer to [6, 9, 10, 11, 12, 13] for related investigations in this direction.

The main aim of this paper is to establish Jørgensen's inequality in the infinite dimensional case. Our main result is Theorem 3.1, which is a generalization of Theorems H and W and a partial generalization of Theorem J to infinite dimension. We will state and prove it in Section 3. In Section 4 we will give an application of Theorem 3.1.

## 2. Preliminaries

The Clifford algebra $\ell$ is the associative algebra over the real field $\mathbb{R}$, generated by a countable family $\{i_k\}_{k=1}^{\infty}$ subject to the following relations:

$$i_h i_k = -i_k i_h \ (h \neq k), \quad i_k^2 = -1, \quad \forall h, k \geq 1$$

and no others. Every element of $\ell$ can be expressed of the following type

$$a = \sum a_I I,$$

where $I = i_{v_1} i_{v_2} \ldots i_{v_p}, 1 \leq v_1 < v_2 < \cdots < v_p$, $p \leq n$, $n$ is a fixed natural number depending on $a$, $a_I \in \mathbb{R}$ are the coefficients and $\sum_I a_I^2 < \infty$. If $I = \emptyset$, then $a_I$ is called the real part of $a$ and denoted by $\mathrm{Re}(a)$; the remaining part is called the imaginary part of $a$ and denoted by $\mathrm{Im}(a)$.

In $\ell$, the Euclidean norm is expressed by

$$|a| = \sqrt{\sum_I a_I^2} = \sqrt{|\mathrm{Re}(a)|^2 + |\mathrm{Im}(a)|^2}.$$

The algebra $\ell$ has three important involutions:

(1) "′": replacing each $i_k$ ($k \geq 1$) of $a$ by $-i_k$, we get a new number $a'$. $a \mapsto a'$ is an isomorphism of $\ell$:

$$(ab)' = a'b', \quad (a + b)' = a' + b',$$

for $a, b \in \ell$.

(2) "$*$": replacing each $i_{v_1}i_{v_2}\dots i_{v_p}$ of $a$ by $i_{v_p}i_{v_{p-1}}\dots i_{v_1}$. We know that $a \mapsto a^*$ is an anti-isomorphism of $\ell$:

$$(ab)^* = b^*a^*, \quad (a+b)^* = b^* + a^*.$$

(3) "$-$": $\bar{a} = (a^*)' = (a')^*$. It is obvious that $a \mapsto \bar{a}$ is also an anti-isomorphism of $\ell$.

We refer to elements of the following type as vectors:

$$x = x_0 + x_1 i_1 + \cdots + x_n i_n + \cdots \in \ell.$$

The set of all such vectors is denoted by $\ell_2$ and we let $\overline{\ell_2} = \ell_2 \bigcup \{\infty\}$. For any $x \in \ell_2$, we have $x^* = x$ and $\bar{x} = x'$. For $x, y \in \ell_2$, the inner product $(x \cdot y)$ of $x$ and $y$ is given by

$$(x \cdot y) = x_0 y_0 + x_1 y_1 + \cdots + x_n y_n + \dots,$$

where $x = x_0 + x_1 i_1 + \cdots + x_n i_n + \dots,\ y = y_0 + y_1 i_1 + \cdots + y_n i_n + \dots.$

Obviously, any nonzero vector $x$ is invertible in $\ell$ with $x^{-1} = \frac{\bar{x}}{|x|^2}$. The inverse of a vector is invertible too. Since any product of nonzero vectors is invertible, we conclude that any product of nonzero vectors is invertible in $\ell$. The set of products of finitely many nonzero vectors is a multiplicative group, called Clifford group and denoted by $\Gamma$.

**Definition 2.1.** If a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies:

(1) $a, b, c, d \in \Gamma \bigcup \{0\}$,
(2) $\triangle(g) = ad^* - bc^* = 1$,
(3) $ab^*,\ d^*b,\ cd^*,\ c^*a \in \ell_2$,

then we call $g$ a Clifford matrix in infinite dimension; the set of all such matrices is denoted by $\mathrm{SL}(\Gamma)$.

Let

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ g^{-1} = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}.$$

Obviously, $gg^{-1} = g^{-1}g = I$, that is, $g^{-1}$ is the inverse of $g$. By a simple computation, we know that $\mathrm{SL}(\Gamma)$ is a multiplicative group of matrices.

For any $g = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(\Gamma)$, the corresponding mapping

$$g(x) = (ax + b)(cx + d)^{-1}$$

is a bijection of $\overline{\ell_2}$ onto itself, which we call a Möbius transformation in infinite dimension. Correspondingly, the set of all such mappings is also a group, which is still denoted by $\mathrm{SL}(\Gamma)$.

Now, we give a classification of the nontrivial elements of $\mathrm{SL}(\Gamma)$ as follows:

- $f$ is *loxodromic* if it is conjugate in $\mathrm{SL}(\Gamma)$ to $\begin{pmatrix} r\lambda & 0 \\ 0 & r^{-1}\lambda' \end{pmatrix}$, where $r \in \mathbb{R}\backslash\{\pm 1,\ 0\}$, $\lambda \in \Gamma$ and $|\lambda| = 1$; if $\lambda = \pm 1$, then $f$ is called *hyperbolic*.
- $f$ is *parabolic* if it is conjugate in $\mathrm{SL}(\Gamma)$ to $\begin{pmatrix} a & b \\ 0 & a' \end{pmatrix}$, where $a,\ b \in \Gamma$, $|a| = 1$, $b \neq 0$ and $ab = ba'$; if $a = \pm 1$, then $f$ is called *strictly parabolic*.
- Otherwise we say $f$ is *elliptic*.

**Definition 2.2.** For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(\Gamma)$, we define the trace of $g$ as

$$\mathrm{tr}(g) = a + d^*.$$

For a nontrivial element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(\Gamma)$, if $b^* = b$, $c^* = c$ and $\mathrm{tr}(g) \in \mathbb{R}$, then we call $g$ vectorial.

For the trace, we have the following result (see [8]).

**Lemma 2.3.** *Let* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(\Gamma)$. *Then* $\mathrm{Re}(\mathrm{tr}(g))$ *is invariant under conjugation.*

The following two lemmas come from [8].

**Lemma 2.4.** $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(\Gamma)$ $(c \neq 0)$ *is hyperbolic if and only if* $\mathrm{tr}(g) \in \mathbb{R}$, $\mathrm{tr}^2(g) > 4$ *and* $c \in \ell_2$. *If $g$ is hyperbolic, then the two fixed points of $g$ are*

$$u, v = -\frac{1}{2}(c^{-1}d - ac^{-1}) \pm \frac{1}{2}c^{-1}((a + d^*)^2 - 4)^{\frac{1}{2}}.$$

**Lemma 2.5.** $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{SL}(\Gamma)$ $(b \neq 0)$ *is hyperbolic if and only if* $\mathrm{tr}(g) \in \mathbb{R}$, $\mathrm{tr}^2(g) > 4$ *and* $b \in \ell_2$. *If $g$ is hyperbolic, then the two fixed points of $g$ are $\infty$ and $-b(a - d)^{-1}$.*

**Definition 2.6.** For a subgroup $G \subset \mathrm{SL}(\Gamma)$, we call $G$ elementary if $G$ has a finite $G$-orbit, that is, there exists a point $x \in \overline{\ell_2}$ such that

$$G(x) = \{g(x) | g \in G\}$$

is finite; otherwise, we call $G$ nonelementary.

We say that $G$ is discrete if $g,\ f_1, f_2, \cdots \in G$ and $f_i \to g$ imply $f_i = g$ for all sufficiently large $i$. Otherwise, $G$ is not discrete.

**Lemma 2.7.** *Let $f \in \mathrm{SL}(\Gamma)$ be not elliptic, and let $\theta : \mathrm{SL}(\Gamma) \to \mathrm{SL}(\Gamma)$ be defined by*

$$\theta(g) = gfg^{-1}.$$

*Suppose that there exists $n$ such that $\theta^n(g) = f$, then the group $\langle f, g\rangle$ generated by $f$ and $g$ is elementary.*

**Proof.** Define $g_0 = g$ and $g_n = \theta^n(g)$. So for some $m \geq 0$,

$$g_{m+1} = g_m f g_m^{-1}.$$

Suppose first that $f$ is parabolic. Since $f$ has exactly one fixed point, we may assume that $f(\infty) = \infty$. As $g_1, \ldots, g_n$ are conjugate to $f$, they are each parabolic and so have a unique fixed point. Thus if $g_{r+1}$ fixes $\infty$, then so does $g_r$, where $r \geq 0$. As $g_n(= f)$ fixes $\infty$, we deduce that each $g_j$ $(j = 0, 1, \ldots, n)$ fixes $\infty$. This shows that $\langle f, g\rangle$ is elementary.

Suppose now that $f$ is loxodromic and the two fixed points of $f$ are $x$ and $y$. Clearly, $g_1, \ldots, g_n$ each have exactly two fixed points. Now suppose that $g_{r+1}$ fixes $x$ and $y$ (as does $g_n$): then

$$\{x, y\} = \{g_r(x), g_r(y)\}.$$

Since $g_r$ cannot interchange $x$ and $y$ for $r \geq 1$, we know that if $g_{r+1}$ fixes $x$ and $y$, then so does $g_r$ for $r \geq 1$. It follows that $g_1, \ldots, g_n$ each fix $x$ and $y$. This shows that $f$ and $g$ leave the set $\{x, y\}$ invariant and so $\langle f, g\rangle$ is elementary. □

## 3. The main result and its proof

Now we come to state and prove our main result.

**Theorem 3.1.** *Let $f, g \in \mathrm{SL}(\Gamma)$ such that $f$ is hyperbolic and $[f, g]$ is vectorial, and suppose that $\langle f, g\rangle$ is discrete and nonelementary, then*

$$|\mathrm{tr}^2(f) - 4| + |\mathrm{tr}([f, g]) - 2| \geq 1. \tag{3.1}$$

**Proof.** By Lemmas 2.4, 2.5 and 2.3, we know that $\mathrm{tr}(f) \in \mathbb{R}$, and $\mathrm{tr}(f)$ and $\mathrm{tr}([f, g])$ are invariant under conjugation. Without loss of generality, we may assume that

$$f = \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $\tau > 0$ and $\tau \neq 1$. Let $\kappa$ denote the left side of relation (3.1) and suppose that (3.1) fails. Then

$$\kappa = (\tau - \tau^{-1})^2(1 + |bc|) < 1. \tag{3.2}$$

We let

$$g_0 = g, \quad g_{m+1} = g_m f {g_m}^{-1}, \quad g_m = \begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix}, \quad m = 0, 1, \ldots.$$

Then, we have

$$a_{m+1} = \tau a_m {d_m}^* - \tau^{-1} b_m {c_m}^*,$$
$$b_{m+1} = (\tau^{-1} - \tau) a_m {b_m}^*,$$
$$c_{m+1} = -(\tau^{-1} - \tau) c_m {d_m}^*,$$
$$d_{m+1} = \tau^{-1} d_m {a_m}^* - \tau c_m {b_m}^*,$$
$$b_{m+1} {c_{m+1}}^* = -(\tau^{-1} - \tau)^2 (1 + b_m {c_m}^*) b_m {c_m}^*. \tag{3.3}$$

Let $f : [0, +\infty) \longrightarrow [0, +\infty)$ be defined by

$$f(x) = x(1 + x)(\tau^{-1} - \tau)^2.$$

Let $r = (\tau^{-1} - \tau)^{-2} - 1$. It is obvious that $f(x)$ is an increasing function on $[0, +\infty)$ such that $f(x) \leq x$ on $[0, r]$. It follows from (3.2) that $|bc| < r$. The above facts and relations (3.3) show that

$$|b_{m+1} {c_{m+1}}^*| \leq f(|b_m {c_m}^*|) \leq \cdots \leq f^{m+1}(|bc^*|) \leq |bc^*|,$$
$$|b_{m+1} {c_{m+1}}^*| \leq (\tau^{-1} - \tau)^2 (1 + |b_m {c_m}^*|) |b_m {c_m}^*|$$
$$\leq (\tau^{-1} - \tau)^2 (1 + |bc^*|) |b_m {c_m}^*| = \kappa |b_m {c_m}^*|,$$
$$|b_{m+1} {c_{m+1}}^*| \leq \kappa^{m+1} |bc|.$$

So

$$\lim_{m\to\infty} b_m {c_m}^* = 0, \quad \lim_{m\to\infty} a_m {d_m}^* = 1.$$

The above relation and (3.3) imply that

$$\lim_{m\to\infty} a_m = \tau, \quad \lim_{m\to\infty} d_m = \tau^{-1}.$$

Now

$$|b_m^{-1} b_{m+1}| = |(\tau^{-1} - \tau) {a_m}^*| \to |\tau(\tau^{-1} - \tau)| < \sqrt{\kappa}\tau.$$

So for sufficiently large $m$, we have

$$\left| \frac{b_{m+1}}{\tau^{m+1}} \right| \leq \sqrt{\kappa} \left| \frac{b_m}{\tau^m} \right|.$$

It follows that

$$\left| \frac{b_m}{\tau^m} \right| \to 0.$$

In a very similar way, we get that

$$\lim_{m\to\infty} c_m \tau^m = 0.$$

It follows that

$$\lim_{m\to\infty} f^{-m} g_{2m} f^m = f.$$

Since $\langle f, g \rangle$ is discrete, we must have $g_{2m} = f$ for some $m$. By Lemma 2.7, $\langle f, g \rangle$ must be elementary, which violates the assumption. The contradiction shows that $\kappa$ cannot be less than 1. □

**Remark 3.2.** Theorem 3.1 is a generalization of Theorem $B$ in [4] and the corresponding result in [11].

## 4. An application

For $f_r = \begin{pmatrix} a_r & b_r \\ c_r & d_r \end{pmatrix}$, where $a_r, b_r, c_r, d_r \in \Gamma \cup \{0\}$ and $r = 1,\ 2$, define

$$\|f_r\| = \sqrt{|a_r|^2 + |b_r|^2 + |c_r|^2 + |d_r|^2},$$

$$\|f_1 - f_2\| = \sqrt{|a_1 - a_2|^2 + |b_1 - b_2|^2 + |c_1 - c_2|^2 + |d_1 - d_2|^2}.$$

Then

**Lemma 4.1** ([7])**.** *For any* $U = \begin{pmatrix} a & b \\ -b' & a' \end{pmatrix} \in \mathrm{SL}(\Gamma)$ *($U$ is called unitary),* $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$*, we have* $\|g\| = \|gU\| = \|Ug\|$*, where* $\alpha,\ \beta,\ \gamma,\ \delta \in \Gamma \cup \{0\}$.

**Lemma 4.2.** *Let* $f \in \mathrm{SL}(\Gamma)$ *be hyperbolic. Then*

$$\|f - I\|^2 \geq \frac{1}{2}|\mathrm{tr}^2(f) - 4|.$$

**Proof.** Since $\|f - I\|$ and $\mathrm{tr}^2(f)$ are invariant under conjugation by unitary transformations by Lemmas 2.3, 2.4, 2.5 and 4.1, without loss of generality, we may assume that

$$f = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix},$$

where $u > 1$. By a simple computation, the conclusion follows. □

**Lemma 4.3.** *Let* $f, g \in \mathrm{SL}(\Gamma)$ *be hyperbolic such that* $[f, g]$ *is vectorial. Then*

$$\|f - I\|^2 \cdot \|g - I\|^2 \geq |\mathrm{tr}([f, g]) - 2|.$$

**Proof.** Since the two sides of the above inequality are invariant under conjugation by unitary transformations, we may assume that

$$f = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix},\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $u > 1$. By computation, we see that

$$[f, g] = \begin{pmatrix} ad^* - u^2 bc^* & (u^2 - 1)ab^* \\ (u^{-2} - 1)cd^* & da^* - u^{-2}cb* \end{pmatrix},\quad |\mathrm{tr}([f, g]) - 2| = (u - u^{-1})^2 |bc^*|,$$

$$\|f - I\|^2 \cdot \|g - I\|^2 = [(u - 1)^2 + (u^{-1} - 1)^2][|a - 1|^2 + |b|^2 + |c|^2 + |d - 1|^2].$$

Therefore, we have

$$\|f - I\|^2 \cdot \|g - I\|^2 \geq (u - u^{-1})^2 |bc| = |\mathrm{tr}([f, g]) - 2|.$$ □

We will use Theorem 3.1 to prove

**Theorem 4.4.** *Let $f, g \in \mathrm{SL}(\Gamma)$ be hyperbolic such that $[f, g]$ and $[g, f]$ are vectorial. If $\langle f, g\rangle$ is discrete and nonelementary, then*

$$\|f - I\| \cdot \|g - I\| \geq \sqrt{2} - 1.$$

**Proof.** Let $x = \min\{|\mathrm{tr}^2(f) - 4|,\ |\mathrm{tr}^2(g) - 4|\}$.

We first suppose that $x \leq 2\sqrt{2} - 2$. By assumptions and Theorem 3.1,

$$|\mathrm{tr}^2(f) - 4| + |\mathrm{tr}([f, g]) - 2| \geq 1, \quad |\mathrm{tr}^2(g) - 4| + |\mathrm{tr}([g, f]) - 2| \geq 1.$$

Therefore, by Lemma 4.3, we have that

$$\|f - I\|^2 \cdot \|g - I\|^2 \geq |\mathrm{tr}([f, g]) - 2| \geq 1 - |\mathrm{tr}^2(f) - 4|,$$

and

$$\|g - I\|^2 \cdot \|f - I\|^2 \geq |\mathrm{tr}([g, f]) - 2| \geq 1 - |\mathrm{tr}^2(g) - 4|.$$

Thus,

$$\|f - I\|^2 \cdot \|g - I\|^2 \geq 1 - (2\sqrt{2} - 2) = (\sqrt{2} - 1)^2.$$

Now we suppose that $x \geq 2\sqrt{2} - 2$. By Lemma 4.2, we have

$$\|f - I\|^2 \geq \frac{1}{2}|\mathrm{tr}^2(f) - 4|, \quad \|g - I\|^2 \geq \frac{1}{2}|\mathrm{tr}^2(g) - 4|.$$

We hence know that

$$\|f - I\|^2 \cdot \|g - I\|^2 \geq \frac{1}{4}|\mathrm{tr}^2(f) - 4||\mathrm{tr}^2(g) - 4| \geq (\sqrt{2} - 1)^2. \qquad \square$$

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