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Weak type inequalities for maximal operators associated to double ergodic sums

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Abstract. Given an approach region $\Gamma \in \mathbb{Z}_+^2$ and a pair U, V of commuting nonperiodic measure preserving transformations on a probability space (Ω, Σ, μ), it is shown that either the associated multiparameter ergodic averages of any function in $L^1(\Omega)$ converge a.e. or that, given a positive increasing function ϕ on $[0, \infty)$ that is $o(\log x)$ as $x \to \infty$, there exists a function $g \in L\phi(L)\,(\Omega)$ whose associated multiparameter ergodic averages fail to converge a.e.

Contents

1. Introduction

Let U and V be two commuting measure preserving transformations on a probability space (Ω, Σ, μ). The general behavior of the multiparameter ergodic averages associated to U and V is becoming well understood. As was proven by N. Dunford in [2] and A. Zygmund in [13], if $f \in L \log L(\Omega)$ then

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} f(U^j V^k \omega)$$

converges for a.e. ω. If the pair U, V is *nonperiodic* in the sense that, for any $(m,n) \neq (0,0)$, $(m,n) \in \mathbb{Z}^2$ we have $\mu\{\omega \in \Omega : U^m V^n \omega = \omega\} = 0$, then the $L \log L$ condition is sharp: as was shown in [6], if ϕ is a positive increasing

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function on $[0,\infty)$ that is $o(\log x)$ as $x \to \infty$, then there exists $g \in L\phi(L)$ such that

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} g(U^j V^k \omega)$$

fails to converge a.e. As expected, these convergence and divergence results are reflected in the behavior of the associated ergodic strong maximal operator M_S, defined by

$$M_S f(\omega) = \sup_{m,n\geq 1} \frac{1}{mn} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \left| f\left(U^j V^k \omega\right)\right| .$$

In [3], Fava showed that M_S satisfies the weak type $(L \log L, L^1)$ inequality

$$\mu \left\{\omega \in \Omega : M_S f(\omega) > \alpha\right\} \leq \int_\Omega \frac{|f|}{\alpha}\left(1 + \log^+ \frac{|f|}{\alpha}\right).$$

The sharpness of this result was proved in [6], where it was shown that, given a pair of commuting nonperiodic measure preserving transformations U and V on Ω and an $o(\log x)$ function ϕ as above, there exists a function $g \in L\phi(L)$ such that the associated ergodic maximal operator $M_S g$ is infinite a.e.

This paper is concerned with somewhat better behaved multiparameter ergodic maximal operators, corresponding to improved a.e. convergence results. The maximal operators and corresponding ergodic averages we will be considering are associated to *rare bases*, ergodic theory analogues of bases associated to geometric rare maximal operators previously studied by Hagelstein, Hare, and Stokolos (see, e.g, [5], [7], and [11]). Being more specific, let $\Gamma \subset \mathbb{Z}_+^2$ be an unbounded region. (Such a set Γ is sometimes referred to as an *approach region* as it has a close connection to approach regions associated to boundary value problems arising in harmonic analysis, complex variables, and partial differential equations.) The corresponding ergodic maximal operator M_Γ is given by

$$M_\Gamma f(\omega) = \sup_{(m,n)\in\Gamma} \frac{1}{mn} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \left| f(U^j V^k \omega)\right| .$$

(Note if $\Gamma = \mathbb{Z}_+^2$ itself, then M_Γ is the usual strong ergodic maximal operator M_S.)

In this paper we will show that, given Γ, if U, V is a commuting pair of nonperiodic measure preserving transformations one of two possibilities must occur:

(i) M_Γ is of weak type $(1,1)$ and accordingly the associated rare ergodic averages

$$\lim_{\substack{m,n\to\infty\\(m,n)\in\Gamma}} \frac{1}{mn} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} f(U^j V^k \omega)$$

converge a.e. for every $f \in L^1(\Omega)$; or

(ii) M_Γ is of weak type $(L \log L, L^1)$ but such that, given a positive increasing function ϕ on $[0, \infty)$ that is $o(\log x)$ for $x \to \infty$, there exists $g \in L\phi(L)$ satisfying $M_\Gamma g = \infty$ a.e. and such that

$$\lim_{\substack{m,n\to\infty \\ (m,n)\in\Gamma}} \frac{1}{mn} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} g\left(U^j V^k \omega\right)$$

fails to converge a.e.

We shall see that a *monotonicity* condition on Γ determines whether case (i) or (ii) holds. The notion of monotonicity is defined as follows. For any positive integer j, let j^* be the integer satisfying $2^{j^*-1} < j \leq 2^{j^*}$. Given a set $\Gamma \in \mathbb{Z}^2_+$, we define the *dyadic skeleton* Γ^* of Γ by

$$\Gamma^* = \left\{ (2^{m^*}, 2^{n^*}) : (m,n) \in \Gamma \right\}.$$

We say that Γ is *monotonic* if, for any (m_1, n_1), (m_2, n_2) in Γ^*, $m_1 < m_2$ implies $n_1 \leq n_2$. We will prove that if Γ is contained in a finite union of monotonic sets then case (i) holds, and otherwise case (ii) will hold.

2. Weak type (1,1) bounds associated to monotonic approach regions

We now show that the ergodic maximal operator M_Γ associated to a monotonic region $\Gamma \subset \mathbb{Z}^2_+$ is of weak type $(1,1)$. To prove this theorem, we will "transfer" the known weak type $(1,1)$ bound of a geometric maximal operator associated to a monotonic basis of rectangles to a weak type $(1,1)$ bound of M_Γ. The transference mechanism will be constructed explicitly, taking advantage of a lemma of Katznelson and Weiss involving commuting nonperiodic pairs of measure preserving transformations. We hope to yield a general transference principle relating weak type bounds of "rare" multi-parameter ergodic maximal operators associated to commuting nonperiodic pairs of measure preserving transformations to weak type bounds of rare geometric maximal operators on a future occasion.

Lemma 1. *Let $\Gamma \subset \mathbb{Z}^2_+$ be a monotonic region and let U, V be a pair of commuting nonperiodic measure preserving transformations on a probability space (Ω, Σ, μ). Then the associated maximal operator M_Γ satisfies the weak type $(1,1)$ inequality*

$$\mu \left\{ \omega \in \Omega : M_\Gamma f(\omega) > \alpha \right\} \leq \frac{C}{\alpha} \int_\Omega |f| \,.$$

Proof. Let Γ^* denote the dyadic skeleton of Γ. One may readily check that $M_\Gamma f \leq 4 M_{\Gamma^*} f$, hence it suffices to show that M_{Γ^*} is of weak type (1,1).

Since Γ is monotonic, we may write $\Gamma^* = \{(m_1, n_1), (m_2, n_2), \ldots\}$ where $(m_j, n_j) = \left(2^{m_j^*}, 2^{n_j^*}\right)$ and where $m_j \leq m_{j+1}$, $n_j \leq n_{j+1}$ for each j. Also let

$\Gamma_j^* = \{(m_1, n_1), \ldots, (m_j, n_j)\}$. As

$$\lim_{j\to\infty} \mu\left\{\omega \in \Omega : M_{\Gamma_j^*} f(\omega) > \alpha\right\} = \mu\left\{\omega \in \Omega : M_{\Gamma^*} f(\omega) > \alpha\right\}$$

there exists N such that

$$\mu\left\{\omega \in \Omega : M_{\Gamma_N^*} f(\omega) > \alpha\right\} \geq \frac{1}{2}\mu\left\{\omega \in \Omega : M_{\Gamma^*} f(\omega) > \alpha\right\}.$$

For notational simplicity we shall denote $M_{\Gamma_N^*}$ by M^*. It suffices to show

$$\mu\left\{\omega \in \Omega : M^* f(\omega) > \alpha\right\} \leq \frac{C}{\alpha}\int_\Omega |f|\,, \tag{1}$$

where C is independent of N.

It is useful at this point to recall the following result of Katznelson and Weiss:

Lemma 2 ([9]). *Let U and V be two commuting nonperiodic measure preserving transformations on a measure space Ω of finite measure. Then for any $\epsilon > 0$ and positive integer γ there exist sets B and E in Ω such that $\mu(E) < \epsilon$ and*

$$\Omega = \left(\bigcup_{j,k=0}^{\gamma-1} B^{j,k}\right) \cup E\,,$$

where the $B^{j,k} = U^j V^k B$ are pairwise disjoint.

Let $\epsilon = \frac{1}{4}\mu\left\{\omega : M^* f(\omega) > \alpha\right\}$. We assume without loss of generality that $\epsilon > 0$. Set $R_N = \max(m_N, n_N)$. Let $\gamma \in \mathbb{Z}_+$ be such that $\frac{2R_N}{\epsilon} < \gamma$. By Lemma 2, there exists a set A such that $\left\{U^j V^k A\right\}_{j,k=0}^{\gamma-1}$ is a disjoint sequence of sets in Ω such that $\mu\left(\cup_{j,k=0}^{\gamma-1} U^j V^k A\right) > 1 - \epsilon$. Observe that $1 - \epsilon < \gamma^2 \mu(A) \leq 1$ and hence

$$\begin{aligned}\mu\left(\bigcup_{j,k=0}^{\gamma-1-R_N} U^j V^k A\right) &= (\gamma - R_N)^2\,\mu(A) \\ &\geq \gamma^2 \mu(A) - 2R_N \gamma \mu(A) \\ &> (1-\epsilon) - (\epsilon\gamma)\,\gamma\mu(A) \\ &\geq 1 - 2\epsilon\,.\end{aligned}$$

Accordingly,

$$\mu\left(\left\{\omega : M^* f(\omega) > \alpha\right\} \cap \bigcup_{j,k=0}^{\gamma-1-R_N} U^j V^k A\right) \geq \frac{1}{2}\mu\left\{\omega : M^* f(\omega) > \alpha\right\}.$$

For $s = 1, 2, \ldots, N$ let

$$E_s = \left\{ \omega \in \Omega : \frac{1}{m_s n_s} \sum_{j=0}^{m_s-1} \sum_{k=0}^{n_s-1} \left| f\left(U^j V^k \omega\right) \right| > \alpha \right\}.$$

and let $A_{s,j,k} = A \cap U^{-j} V^{-k} E_s$.

We now let $\{B_r\}_{r=1}^{\tilde{N}}$ be a disjoint collection of sets of positive measure such that:

(i) $\bigcup_{r=1}^{\tilde{N}} B_r = \bigcup_{s=1}^{N} \bigcup_{j,k=0}^{\gamma-1-R_N} A_{s,j,k}$, and

(ii) given any B_r and $A_{s,j,k}$ for $1 \le r \le \tilde{N}$; $1 \le s \le N$; and $1 \le j, k \le \gamma - 1 - R_N$, either $B_r \subset A_{s,j,k}$ or $\mu(B_r \cap A_{s,j,k}) = 0$.

In order to circumvent certain technical complications later on involving sets of measure zero, we assume without loss of generality that a slightly stronger version of (ii) holds, namely, given any B_r and $A_{s,j,k}$ for $1 \le r \le \tilde{N}$; $1 \le s \le N$; and $1 \le j, k \le \gamma - 1 - R_N$, either $B_r \subset A_{s,j,k}$ or $B_r \cap A_{s,j,k} = \emptyset$. This may be justified from removing from the space Ω the set of zero measure

$$\bigcup_{r=1}^{\tilde{N}} \bigcup_{m,n=-\infty}^{\infty} U^m V^n \left\{\omega \in B_r \cap A_{s,j,k} : \mu(B_r \cap A_{s,j,k}) = 0\right\}.$$

Note that if $M^* f(\omega) > \alpha$ and $\omega \in \cup_{j,k=0}^{\gamma-1-R_N} U^j V^k A$, then $\omega \in E_s$ for some s, and hence for some $0 \le j, k \le \gamma - 1 - R_N$ we have $U^{-j} V^{-k} \omega \in A_{s,j,k}$ Hence $U^{-j} V^{-k} \omega \in B_r$ for some r, and thus $\omega \in U^j V^k B_r$. We will frequently denote $U^j V^k B_r$ by $B_{r,j,k}$. So

$$\mu\left(\{\omega : M^* f(\omega) > \alpha\} \cap \bigcup_{j,k=0}^{\gamma-1-R_N} U^j V^k A \right)$$

$$= \mu\left(\{\omega : M^* f(\omega) > \alpha\} \cap \bigcup_{r=1}^{\tilde{N}} \bigcup_{j,k=0}^{\gamma-1-R_N} B_{r,j,k} \right)$$
$$= \sum_{r=1}^{\tilde{N}} \mu\left(\{\omega : M^* f(\omega) > \alpha\} \cap \bigcup_{j,k=0}^{\gamma-1-R_N} B_{r,j,k} \right),$$

the latter equality following from the fact that

$$\mu\left(\left(\bigcup_{j,k=0}^{\gamma-1-R_N} U^j V^k B_r \right) \bigcap \left(\bigcup_{j,k=0}^{\gamma-1-R_N} U^j V^k B_s \right) \right) = 0$$

when $r \neq s$.

Fix an $r \in \left\{1, \ldots, \tilde{N}\right\}$. It suffices to show

$$\mu\left(\{\omega : M^* f(\omega) > \alpha\} \cap \bigcup_{j,k=0}^{\gamma-1-R_N} B_{r,j,k}\right) \leq \frac{C}{\alpha} \int_{\cup_{j,k=0}^{\gamma-1} B_{r,j,k}} |f| \, d\mu \, .$$

For our convenience, we set $\rho_r = \sqrt{\mu(B_r)}$. Define g_r on

$$Q_r := [0, \gamma\rho_r] \times [0, \gamma\rho_r]$$

by

$$g_r(\xi, \eta) = \frac{1}{\mu(B_r)} \sum_{j,k=0}^{\gamma-1} \left(\int_{B_{r,j,k}} |f| \, d\mu\right) \chi_{[j\rho_r, (j+1)\rho_r) \times [k\rho_r, (k+1)\rho_r)}(\xi, \eta) \, .$$

Note that

$$\int_{Q_r} g_r(\xi, \eta) d\xi d\eta = \int_{\cup_{j,k=0}^{\gamma-1} B_{r,j,k}} |f| \, d\mu \, .$$

Let now the collection of rectangles $\beta_{\Gamma^*_{N,r}}$ be defined by

$$\beta_{\Gamma^*_N, r} = \left\{[j\rho_r, (j + m_\ell)\rho_r] \times [k\rho_r, (k + n_\ell)\rho_r] : j, k \in \mathbb{Z}, 1 \leq \ell \leq N\right\} .$$

We define the geometric maximal operator $\mathcal{M}_r$ associated to $\beta_{\Gamma_N, r}$ by

$$\mathcal{M}_r f(\xi, \eta) = \sup\left\{\frac{1}{|R|} \int_R |f(u, v)| \, du dv : (\xi, \eta) \in R, R \in \beta_{\Gamma^*_N, r}\right\} .$$

Suppose $M^* f(\omega) > \alpha$ and $\omega \in B_{r,j,k}$ for some $0 \leq j, k \leq \gamma - 1 - R_N$. Then $\omega \in E_s$ and $U^{-j} V^{-k} \omega \in A_{s,j,k}$ for some s, and hence $B_r \subset A_{s,j,k}$, implying $U^j V^k B_r \subset E_s$, i.e. $B_{r,j,k} \subset E_s$. So

$$\frac{1}{\mu(B_r)} \frac{1}{m_s n_s} \int_{B_{r,j,k}} \sum_{a=0}^{m_s-1} \sum_{b=0}^{n_s-1} \left|f\left(U^a V^b w\right)\right| d\mu(w) > \alpha \, .$$

Hence if $(\xi, \eta) \in [j\rho_r, (j+1)\rho_r) \times [k\rho_r, (k+1)\rho_r)$ for $1 \leq j, k \leq \gamma - 1 - R_N$ we have

$$\begin{aligned}
\mathcal{M}_r g_r(\xi, \eta) &\geq \frac{1}{m_s n_s \mu(B_r)} \int_{u=j\rho_r}^{(j+m_s)\rho_r} \int_{v=k\rho_r}^{(k+n_s)\rho_r} g_r(u, v) du dv \\
&= \frac{1}{m_s n_s \mu(B_r)} \sum_{a=j}^{j+m_s-1} \sum_{b=k}^{k+n_s-1} \rho_r^2 \frac{1}{|B_r|} \int_{B_{r,a,b}} |f| \, d\mu \\
&= \frac{1}{m_s n_s \mu(B_r)} \int_{B_{r,j,k}} \sum_{a=0}^{m_s-1} \sum_{b=0}^{n_s-1} \left|f\left(U^a V^b w\right)\right| d\mu(w) \; > \alpha \, .
\end{aligned}$$

So $\{\omega : M^* f(\omega) > \alpha\} \cap \left(\cup_{j,k=0}^{\gamma-1-R_N} B_{r,j,k}\right)$ is a disjoint union of a subcollection of the $B_{r,j,k}$'s, and if $B_{r,j,k} \subset \{\omega : M^* f(\omega) > \alpha\} \cap \left(\cup_{j,k=0}^{\gamma-1-R_N} B_{r,j,k}\right)$

then

$$[j\rho_r, (j+1)\rho_r) \times [k\rho_r, (k+1)\rho_r) \subset \{(x,y) : \mathcal{M}_r g_r(x,y) > \alpha\} .$$

As the sets $B_{r,j,k}$ are of the same measure $\mu(B_r)$ and disjoint, as well as the sets of the form $[j\rho_r, (j+1)\rho_r) \times [k\rho_r, (k+1)\rho_r)$, we realize

$$\mu\left(\{\omega : M^* f(\omega) > \alpha\} \cap \bigcup_{j,k=0}^{\gamma-1-R_N} B_{r,j,k}\right) \le |\{(\xi,\eta) : \mathcal{M}_r g_r(\xi,\eta) > \alpha\}| .$$

Hence it suffices to show

$$|\{(\xi,\eta) : \mathcal{M}_r g_r(\xi,\eta) > \alpha\}| \le \frac{C}{\alpha} \int_{\cup_{j,k=0}^{\gamma-1} B_{r,j,k}} |f| \, d\mu \, .$$

The rectangles in $\beta_{\Gamma_N^*,r}$ satisfy the following monotonicity property: if $R_1, R_2 \in \beta_{\Gamma_N^*,r}$, then there exists a translate τR_1 of R_1 such that either $\tau R_1 \subset 2 \cdot R_2$ or $R_2 \subset 2 \cdot \tau R_1$ where multiplication by 2 means the doubling of the dimensions of the rectangle. This follows from the monotonicity property of Γ_N.

Any geometric maximal operator associated to a basis of such rectangles in $\mathbb{R}^2$ with sides parallel to the axes is automatically of weak type (1,1), as may be readily seen by the proof of the Vitali covering theorem. (See [12] for more details.) Hence

$$\begin{aligned} |\{(\xi,\eta) : \mathcal{M}_r g_r(\xi,\eta) > \alpha\}| &\le \frac{C}{\alpha} \int_{\mathbb{R}^2} g_r(\xi,\eta) d\xi d\eta \\ &\le \frac{C}{\alpha} \int_{\cup_{j,k=0}^{\gamma-1} B_{r,j,k}} |f| \, d\mu \, , \end{aligned}$$

as desired. □

Theorem 1. *Let U and V be a pair of commuting nonperiodic measure preserving transformations on a probability space (Ω, Σ, μ), and let $\Gamma \subset \mathbb{Z}_+^2$ be contained in a finite number of monotonic sets. Then the associated maximal operator M_Γ satisfies the weak type $(1,1)$ inequality*

$$\mu\{\omega \in \Omega : M_\Gamma f(\omega) > \alpha\} \le \frac{C}{\alpha} \int_\Omega |f| \, ,$$

and the associated rare ergodic averages

$$\lim_{\substack{m,n\to\infty \\ (m,n)\in\Gamma}} \frac{1}{mn} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} f(U^j V^k \omega)$$

converge a.e. for every $f \in L^1(\Omega)$.

Proof. Since $\Gamma \subset \mathbb{Z}_+^2$ is contained in a finite number of monotonic sets, there exists subsets $\Gamma^1, \ldots, \Gamma^N$ of $\mathbb{Z}_+^2$ that are monotonic such that $\Gamma \subset \cup_{j=1}^N \Gamma^j$. As each M_{Γ^j} is of weak type (1,1) by Lemma 1 and as by sublinearity we have $M_\Gamma f \le M_{\Gamma^1} f + \cdots + M_{\Gamma^N} f$, the weak type (1,1) estimate follows.

Let $f \in L^1(\Omega)$ and $\epsilon > 0$. To prove the convergence result, it suffices to show

$$\mu\left\{\omega \in \Omega : \left(\limsup_{\substack{m,n\to\infty \\ (m,n)\in\Gamma}} - \liminf_{\substack{m,n\to\infty \\ (m,n)\in\Gamma}}\right) \frac{1}{mn}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1} f(U^j V^k \omega) > \epsilon\right\} < \epsilon\ .$$

Let $\epsilon_1 > 0$, where ϵ_1 is to be determined later. Since $L\log L(\Omega)$ is dense in $L^1(\Omega)$, there exists $g \in L\log L(\Omega)$ such that $\|f-g\|_{L^1(\Omega)} < \epsilon_1$. As

$$\lim_{m,n\to\infty} \frac{1}{mn}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1} g(U^j V^k \omega)$$

converges a.e. as was shown by Dunford and Zygmund, we necessarily have

$$\lim_{\substack{m,n\to\infty \\ (m,n)\in\Gamma}} \frac{1}{mn}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1} g(U^j V^k \omega)$$

converges a.e. Hence

$$\begin{aligned}
&\mu\left\{\omega \in \Omega : \left(\limsup_{\substack{m,n\to\infty \\ (m,n)\in\Gamma}} - \liminf_{\substack{m,n\to\infty \\ (m,n)\in\Gamma}}\right) \frac{1}{mn}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1} f(U^j V^k \omega) > \epsilon\right\} \\
&= \mu\left\{\omega \in \Omega : \left(\limsup_{\substack{m,n\to\infty \\ (m,n)\in\Gamma}} - \liminf_{\substack{m,n\to\infty \\ (m,n)\in\Gamma}}\right) \frac{1}{mn}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1} (f-g)(U^j V^k \omega) > \epsilon\right\} \\
&< \frac{C}{\epsilon}\|f-g\|_{L_1(\Omega)} < \frac{C\epsilon_1}{\epsilon}\ .
\end{aligned}$$

As ϵ_1 is arbitrarily small, the desired result holds. □

We remark that an alternative proof of this result may be obtained using techniques of A. Zygmund in [13]. In this paper Zygmund states, without providing details, a result that encompasses the above theorem even in the case of noncommuting measure preserving transformations. However, the transference methods we have constructed in our proof are effectively "reversible" and enable us in the next section to show that certain weak-type bounds on multiparameter ergodic maximal operators are indeed sharp.

3. Nonmonotonic approach regions

In this section we shall show that if the approach region Γ is not monotonic, then the weak type $(L\log L, L^1)$ estimate on M_Γ is sharp and moreover that the rare ergodic averages associated to Γ will converge a.e. for all functions in $L\log L(\Omega)$ but not for all functions in any larger Orlicz class. Observe that the weak type $(L\log L, L^1)$ estimate on M_Γ follows from bounding M_Γ by the strong ergodic maximal operator M_S and applying De Guzmán's $(L\log L, L^1)$ estimate for M_S. That the rare ergodic averages associated to Γ converge for all functions in $L\log L(\Omega)$ follows immediately from Dunford and Zygmund's result that the strong ergodic averages of any function in $L\log L(\Omega)$ converge a.e.

Analogous sharpness results for $(L\log L, L^1)$ bounds have been found previously for geometric maximal operators by the second author (see in particular [12].) The strategy here will be to "transfer" the associated techniques of proof used by Stokolos to the ergodic setting, and the means of transference will be the Katznelson–Weiss lemma.

Let I and I' be two rectangles in the plane whose sides are parallel to the coordinate axes. If there exists a translation placing one of them inside the other, we say I and I' are *comparable*. If such a translation does not exist we say I and I' are *incomparable*.

Lemma 3. *Let $I_1, \dots, I_k$ be pairwise incomparable rectangles in the plane whose sides are parallel to the axes and whose sidelengths are dyadic. Then there are two sets Θ and Y in the plane such that*

$$|Y| \geq k2^{k-3}|\Theta|$$

and such that for every $(x,y) \in Y$ there is a shift τ such that for some j,

$$(x,y) \in \tau(I_j) \quad \text{and} \quad |\tau(I_j) \cap \Theta| \geq 2^{1-k}|\tau(I_j)|.$$

Moreover, each $\tau(I_j)$ is a dyadic rectangle, $\Theta \subset Y$, and Y is contained in a dyadic rectangle $H_{\Theta,Y}$ such that

$$\frac{|Y|}{|H_{\Theta,Y}|} \geq k2^{-k-1}.$$

Proof. Without loss of generality we assume that $I_1, \dots, I_k$ have a common lower left vertex. Let $I_j = I_j^1 \times I_j^2$, with $\left|I_j^1\right| = 2^{-m_j}$ and $\left|I_j^2\right| = 2^{-n_j}$. We also assume without loss of generality that $I_1^1 \subset I_2^1 \subset \cdots \subset I_k^1$ while $I_1^2 \supset I_2^2 \cdots \supset I_k^2$, corresponding to $m_1 > m_2 > \cdots > m_k$ and $n_1 < n_2 < \cdots < n_k$.

We define Θ^1 and Θ^2 by

$$\Theta^1 = \left\{ x_1 \in I_k^1 : \prod_{j=1}^{k-1} \sum_{s=0}^{2^{m_j - m_{k-1}}-1} \chi_{I_j^1}\left(x_1 - 2s\left|I_j^1\right|\right) = 1 \right\},$$

$$\Theta^2 = \left\{ x_2 \in I_1^2 : \prod_{j=2}^{k} \sum_{s=0}^{2^{n_j - n_{1}-1}-1} \chi_{I_j^2}\left(x_2 - 2s\left|I_j^2\right|\right) = 1 \right\}.$$

Observe that $\left|\Theta^1\right| = 2^{1-k}\left|I_k^1\right|$ and $\left|\Theta^2\right| = 2^{1-k}\left|I_1^2\right|$. Set $\Theta = \Theta^1 \times \Theta^2$. Then $|\Theta| = 2^{2-2k}\left|I_k^1\right| \cdot \left|I_1^2\right|$.

Set now $Y_k^1 = I_k^1$, $Y_1^2 = I_1^2$, and

$$Y_i^1 = \left\{ x_1 \in I_k^1 : \prod_{j=i}^{k-1} \sum_{s=0}^{2^{m_j - m_{k-1}}-1} \chi_{I_j^1}\left(x_1 - 2s\left|I_j^1\right|\right) \right\},$$

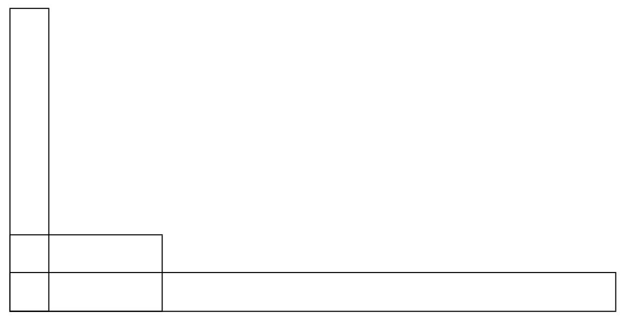

FIGURE 1.

$$Y_i^2 = \left\{ x_2 \in I_1^2 : \prod_{j=2}^{i} \sum_{s=0}^{2^{n_j - n_1 - 1} - 1} \chi_{I_j^2}\left(x_2 - 2s\left|I_j^2\right|\right) = 1 \right\}$$

for $i = 1, \dots, k-1$ and $i = 2, \dots, k$ respectively. We let $Y_i = Y_i^1 \times Y_i^2$. Note that $\left|Y_i^1\right| = 2^{-(k-i)}\left|I_k^1\right|$ and $\left|Y_i^2\right| = 2^{1-i}\left|I_1^2\right|$. So $|Y_i| = 2^{1-k}\left|I_k^1\right| \cdot \left|I_1^2\right|$.

Let now $Y = Y_1 \cup \cdots \cup Y_k$. For $j = 1, \dots, k$, Y_j is a disjoint union of translates of I_j, with at least one-quarter of each translate not intersecting any of the other Y_i's. So

$$|Y| \geq \frac{1}{4}\sum_{i=1}^{k} |Y_i| = k2^{-1-k}\left|I_k^1\right| \cdot \left|I_1^2\right| = k2^{k-3}\left|\Theta\right| .$$

Moreover, if $(x, y) \in Y$, then $(x, y) \in \tau(I_j)$ for some $1 \leq j \leq k$ and shift τ, where

$$\frac{|\tau(I_j) \cap \Theta|}{|\tau(I_j)|} = \frac{|I_j \cap \Theta|}{|I_j|} = \frac{|Y_j \cap \Theta|}{|Y_j|} = \frac{|\Theta|}{|Y_j|} = \frac{2^{2-2k}\left|I_k^1\right| \cdot \left|I_1^2\right|}{2^{1-k}\left|I_k^1\right| \cdot \left|I_1^2\right|} = 2^{1-k} .$$

Let now $H_{\Theta,Y} = I_k^1 \times I_1^2$. By construction $\Theta \subset Y \subset H_{\Theta,Y}$. Moreover,

$$\frac{|Y|}{|H_{\Theta,Y}|} \geq \frac{k2^{k-3}\left|\Theta\right|}{\left|I_k^1 \times I_1^2\right|} = \frac{k2^{k-3}2^{2-2k}\left|I_k^1\right| \cdot \left|I_1^2\right|}{\left|I_k^1\right| \cdot \left|I_1^2\right|} = k2^{-k-1} ,$$

completing the proof of the lemma. □

Figures 1 and 2 should aid the understanding of the proof of the above lemma. Figure 1 illustrates three incomparable rectangles I_1, I_2, and I_3. Figure 2 features the set Θ (what is shaded in black) as well as the corresponding Y (the union of the rectangles in the figure).

We now introduce some new notation that will be helpful to us. Given an approach region $\Gamma \subset \mathbb{Z}_+^2$, associate to the dyadic skeleton Γ^* of Γ the collection of dyadic rectangles $\mathcal{R}_{\Gamma^*}$, where

$$\mathcal{R}_{\Gamma^*} = \left\{ \left[0, 2^{m^*}\right] \times \left[0, 2^{n^*}\right] : \left(2^{m^*}, 2^{n^*}\right) \in \Gamma^* \right\} .$$

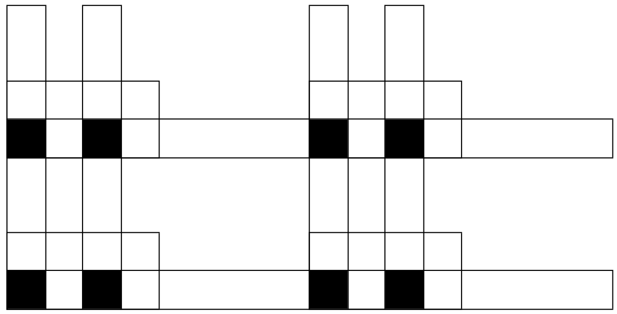

FIGURE 2.

A crucial observation at this point is that, if Γ is is not contained in a finite number of monotonic sets, given any positive integer k and positive number α there exists a collection of k pairwise incomparable rectangles in $\mathcal{R}_{\Gamma^*}$ all of whose sidelengths exceed α.

Given $\Gamma \subset \mathbb{Z}_+^2$ and the associated collection of rectangles $\mathcal{R}_{\Gamma^*}$, we now let $\tilde{\mathcal{R}}_{\Gamma^*}$ be the collection of dyadic rectangles in the plane consisting of all the shifts of members of $\mathcal{R}_{\Gamma^*}$. We define the associated maximal operator $\tilde{M}_{\Gamma^*}$ by

$$\tilde{M}_{\Gamma^*} f(x,y) = \sup_{(x,y)\in R \in \tilde{R}_{\Gamma^*}} \frac{1}{|R|}\int_R |f| \; .$$

Lemma 4. *Suppose $\Gamma \subset \mathbb{Z}_+^2$ is not contained in a finite number of monotonic sets. Let $\epsilon > 0$. For $0 < \lambda < \frac{1}{100}$, let $k \in \mathbb{Z}$ be such that $2^{-k} \leq \lambda < 2^{1-k}$. Then there exist sets $\Theta_{\lambda,\epsilon} \subset Y_{\lambda,\epsilon} \subset H_{\lambda,\epsilon}$ in the plane, all being unions of dyadic squares of sidelength 1 and such that $H_{\lambda,\epsilon}$ is a dyadic square itself, such that*

$$\tilde{M}_{\Gamma^*} \chi_{\Theta_{\lambda,\epsilon}} > \lambda \text{ on } Y_{\lambda,\epsilon} \; ,$$

$$|Y_{\lambda,\epsilon}| \geq k 2^{k-3} |\Theta_{\lambda,\epsilon}| \; ,$$

and

$$\frac{|H_{\lambda,\epsilon} - Y_{\lambda,\epsilon}|}{|H_{\lambda,\epsilon}|} < \epsilon \; .$$

Proof. Since Γ is not contained in a finite number of monotonic sets, there exist a collection $I_{1,1}, \ldots, I_{1,k}$ of pairwise incomparable rectangles in $\mathcal{R}_{\Gamma^*}$. By the previous lemma, there are two sets $\tilde{\Theta}_1$ and $\tilde{Y}_1$ in the plane such that

$$\left|\tilde{Y}_1\right| \geq k 2^{k-3} \left|\tilde{\Theta}_1\right|$$

and such that for every $(x,y) \in \tilde{Y}_1$ there is a shift τ such that for some j,

$$(x,y) \in \tau\left(I_{1,j}\right) \text{ and } \left|\tau\left(I_{1,j}\right) \cap \tilde{\Theta}_1\right| \geq 2^{1-k} \left|\tau\left(I_{1,j}\right)\right| .$$

Moreover, each $\tau(I_{1,j})$ is a dyadic rectangle and $\tilde{\Theta}_1$ and $\tilde{Y}_1$ lie in a dyadic rectangle H_1 such that

$$\frac{\left|\tilde{Y}_1\right|}{|H_1|} \geq k2^{-1-k} .$$

Observe that $\tilde{M}_{\Gamma^*}\chi_{\tilde{\Theta}_1} > 2^{1-k} > \lambda$ on $\tilde{Y}_1$.

Let now $I_{2,1}, \ldots, I_{2,k}$ be a collection of pairwise incomparable rectangles in $\mathcal{R}_\Gamma$ all of whose sidelengths exceed the longest sidelength of H_1. Applying the previous lemma again we obtain two sets Θ_2 and Y_2 in the plane such that

$$|Y_2| \geq k2^{k-3} |\Theta_2|$$

and such that for every $(x,y) \in Y_2$ there is a shift τ such that for some j,

$$(x,y) \in \tau(I_{2,j}) \text{ and } |\tau(I_{2,j}) \cap \Theta_2| \geq 2^{1-k} |\tau(I_{2,j})| .$$

Moreover, each $\tau(I_{2,j})$ is a dyadic rectangle, $\Theta_2 \subset Y_2$, and Y_2 lies in a dyadic rectangle H_2 such that

$$\frac{|Y_2|}{|H_2|} \geq k2^{-1-k} .$$

Assuming without loss of generality that the construction of Θ_2 and Y_2 from the $I_{2,j}$ was like the one described in the proof of the previous lemma, $H_2 - Y_2$ consists of an a.e. disjoint union of dyadic rectangles, each being a translate of H_1. (This follows from the method of construction and the fact that each $I_{2,j}$ has sidelengths exceeding the largest sidelength of H_1.) Defining the shift operators $\tau_{2,1}, \ldots, \tau_{2,\ell_2}$ such that $H_2 - Y_2$ is the a.e. disjoint union of the $\tau_{2,j}H_1$, we set

$$\tilde{Y}_2 = Y_2 \cup \left(\bigcup_{j=1}^{\ell_2} \tau_{2,j}\tilde{Y}_1\right),$$

$$\tilde{\Theta}_2 = \Theta_2 \cup \left(\bigcup_{j=1}^{\ell_2} \tau_{2,j}\tilde{\Theta}_1\right).$$

An important observation here is that

$$\frac{\left|H_2 - \tilde{Y}_2\right|}{|H_2|} \leq \left(1 - k^{-1-k}\right)^2$$

and

$$\left|\tilde{Y}_2\right| \geq k2^{k-3} \left|\tilde{\Theta}_2\right| .$$

Also note that $\tilde{M}_{\Gamma^*}\chi_{\tilde{\Theta}_2} > \lambda$ on $\tilde{Y}_2$.

We proceed by induction. Suppose $\tilde{Y}_n$, $\tilde{\Theta}_n$, and H_n have been constructed, all being unions of rectangles in $\tilde{R}_\Gamma$. Moreover, suppose $M_{\Gamma^*}\chi_{\tilde{\Theta}_n} > \lambda$ on $\tilde{Y}_n$, and

$$\frac{\left|H_n - \tilde{Y}_n\right|}{|H_n|} \leq \left(1 - k^{-1-k}\right)^n .$$

Let $I_{n+1,1}, \ldots, I_{n+1,k}$ be a collection of incomparable rectangles in $\mathcal{R}_\Gamma$ all of whose sidelengths exceed the longest sidelength of H_n. Applying the techniques of the previous lemma we obtain two sets Θ_{n+1}, Y_{n+1} in the plane such that

$$|Y_{n+1}| \geq k2^{k-3} |\Theta_{n+1}|$$

and such that for every $(x, y) \in Y_{n+1}$ there is a shift τ such that for some j, $(x, y) \in \tau(I_{n+1,j})$ and $|\tau(I_{n+1,j}) \cap \Theta_{n+1}| \geq 2^{1-k} |\tau(I_{n+1,j})|$. Moreover, each $\tau(I_{n+1,j})$ is a dyadic rectangle, $\Theta_{n+1} \subseteq Y_{n+1}$, and Θ_{n+1} and Y_{n+1} lie in a dyadic rectangle H_{n+1} such that $\frac{|Y_{n+1}|}{|H_{n+1}|} \geq k2^{-1-k}$. Now, $H_{n+1} - Y_{n+1}$ is an a.e. disjoint union of dyadic rectangles each being a translate of H_n, due to the nature of construction of Θ_{n+1} and Y_{n+1} and the fact that each $I_{n+1,j}$ has sidelengths exceeding the largest sidelength of H_n. Defining $\tau_{n+1,1}, \ldots, \tau_{n+1,\ell_{n+1}}$ such that $H_{n+1} - Y_{n+1}$ is an a.e. disjoint union of the $\tau_{n+1,j}H_n$, we set

$$\tilde{Y}_{n+1} = Y_{n+1} \cup \left(\bigcup_{j=1}^{\ell_{n+1}} \tau_{n+1,j} \tilde{Y}_n \right)$$

and

$$\tilde{\Theta}_{n+1} = \Theta_{n+1} \cup \left(\bigcup_{j=1}^{\ell_{n+1}} \tau_{n+1,j} \tilde{\Theta}_n \right) .$$

Note that

$$\frac{\left|H_{n+1} - \tilde{Y}_{n+1}\right|}{|H_{n+1}|} \leq \left(1 - k^{-1-k}\right)^{n+1} ,$$

$$\tilde{M}_{\Gamma^*} \chi_{\tilde{\Theta}_{n+1}} > \lambda \text{ on } \tilde{Y}_{n+1} ,$$

and

$$\left|\tilde{Y}_{n+1}\right| \geq k2^{k-3} \left|\tilde{\Theta}_{n+1}\right| .$$

Let now $N = N(\lambda, \epsilon) \in \mathbb{Z}_+$ be such that $\left(1 - k^{-1-k}\right)^N < \epsilon$. H_N is not necessarily a dyadic square. However, there exist a collection of shift operators $\tau_{H_N,j}$ for $1 \leq j \leq r_{H_N}$ such that the a.e. disjoint union of the $\tau_{H_{N,j}}$ forms a dyadic square. Defining $\Theta_{\lambda,\epsilon}$, $Y_{\lambda,\epsilon}$, and $H_{\lambda,\epsilon}$ by

$$\Theta_{\lambda,\epsilon} = \bigcup_{j=1}^{r_{H_N}} \tau_{H_N,j}(\Theta_N) ,$$

$$Y_{\lambda,\epsilon} = \bigcup_{j=1}^{r_{H_N}} \tau_{H_N,j}(Y_N) ,$$

and

$$H_{\lambda,\epsilon} = \bigcup_{j=1}^{r_{H_N}} \tau_{H_N,j}(H_N) \ ,$$

we obtain the lemma. □

We now consider some pleasantries associated to the fact that, although M_Γ is a "centered" maximal operator, $\tilde{M}_{\Gamma^*}$ is not. We define the four "quasi-centered" maximal operators $\tilde{M}_{\Gamma^*,I}$, $\tilde{M}_{\Gamma^*,II}$, $\tilde{M}_{\Gamma^*,III}$, and $\tilde{M}_{\Gamma^*,IV}$ by

$$\tilde{M}_{\Gamma^*,I} f(x,y) = \sup_{R\in\mathcal{R}_{\Gamma^*}} \frac{1}{|R|}\int_R f((\lfloor x\rfloor, \lfloor y\rfloor) + (u,v))\, dudv \ ,$$

$$\tilde{M}_{\Gamma^*,II} f(x,y) = \sup_{R\in\mathcal{R}_{\Gamma^*}} \frac{1}{|R|}\int_R f((\lceil x\rceil, \lfloor y\rfloor) + (-u,v))\, dudv \ ,$$

$$\tilde{M}_{\Gamma^*,III} f(x,y) = \sup_{R\in\mathcal{R}_{\Gamma^*}} \frac{1}{|R|}\int_R f((\lceil x\rceil, \lceil y\rceil) + (-u,-v))\, dudv \ ,$$

and

$$\tilde{M}_{\Gamma^*,IV} f(x,y) = \sup_{R\in\mathcal{R}_{\Gamma^*}} \frac{1}{|R|}\int_R f((\lfloor x\rfloor, \lceil y\rceil) + (u,-v))\, dudv \ .$$

Note that $\tilde{M}_{\Gamma^*} f \leq \tilde{M}_{\Gamma^*,I} f + \tilde{M}_{\Gamma^*,II} f + \tilde{M}_{\Gamma^*,III} f + \tilde{M}_{\Gamma^*,IV} f$. We may assume without loss of generality that on a set within $Y_{\lambda,\epsilon}$ of measure at least $\frac{1}{4}|Y_{\lambda,\epsilon}|$ that $\tilde{M}_{\Gamma^*,I}\chi_{\Theta_{\lambda,\epsilon}} \geq \frac{1}{4}\tilde{M}_{\Gamma^*}\chi_{\Theta_{\lambda,\epsilon}}$. To see this, suppose it had been that, say, $\tilde{M}_{\Gamma^*,II}\chi_{\Theta_{\lambda,\epsilon}} \geq \frac{1}{4}\tilde{M}_{\Gamma^*}\chi_{\Theta_{\lambda,\epsilon}}$ on a set within $Y_{\lambda,\epsilon}$ of measure at least $\frac{1}{4}|Y_{\lambda,\epsilon}|$. Assuming without loss of generality that $H_{\lambda,\epsilon}$ were situated such that its lower left hand corner were at the origin, we could replace $\Theta_{\lambda,\epsilon}$, $Y_{\lambda,\epsilon}$ by sets $\Theta'_{\lambda,\epsilon}$ and $Y'_{\lambda,\epsilon}$, where

$$\chi_{\Theta'_{\lambda,\epsilon}}(x,y) = \chi_{\Theta_{\lambda,\epsilon}}(|H_{\lambda,\epsilon}|^{1/2} - x, y) \ ,$$

$$\chi_{Y'_{\lambda,\epsilon}}(x,y) = \chi_{Y_{\lambda,\epsilon}}(|H_{\lambda,\epsilon}|^{1/2} - x, y) \ .$$

Observe that $\tilde{M}_{\Gamma^*,II}\chi_{\Theta_{\lambda,\epsilon}} \geq \frac{1}{4}\tilde{M}_{\Gamma^*}\chi_{\Theta_{\lambda,\epsilon}}$ on a set of measure at least $\frac{1}{4}|Y_{\lambda,\epsilon}|$ implies that $\tilde{M}_{\Gamma^*,I}\chi_{\Theta'_{\lambda,\epsilon}} \geq \frac{1}{4}\tilde{M}_{\Gamma^*}\chi_{\Theta'_{\lambda,\epsilon}}$ on a set of measure at least $\frac{1}{4}|Y_{\lambda,\epsilon}|$. Relabeling $\Theta'_{\lambda,\epsilon}$ and $Y'_{\lambda,\epsilon}$ by $\Theta_{\lambda,\epsilon}$ and $Y_{\lambda,\epsilon}$ we would obtain the desired result. Similar symmetries apply if we replace $\tilde{M}_{\Gamma^*,II}$ by $\tilde{M}_{\Gamma^*,III}$ or $\tilde{M}_{\Gamma^*,IV}$.

We summarize these considerations with the following.

Lemma 5. *Suppose $\Gamma \subset \mathbb{Z}^2_+$ is not contained in a finite number of monotonic sets. Let $\epsilon > 0$. For $0 < \lambda < \frac{1}{100}$, let $k \in \mathbb{Z}$ be such that $2^{-k} \leq \lambda < 2^{1-k}$. Then there exist sets $\Theta_{\lambda,\epsilon} \subset Y_{\lambda,\epsilon} \subset H_{\lambda,\epsilon}$ in the plane, all being unions of dyadic squares of sidelength 1 and such that $H_{\lambda,\epsilon}$ is a dyadic square itself, such that*

$$\tilde{M}_{\Gamma^*,I}\chi_{\Theta_{\lambda,\epsilon}}(x,y) > \frac{1}{4}\lambda \text{ for any } (x,y) \in Y_{\lambda,\epsilon} \ ,$$

$$|Y_{\lambda,\epsilon}| \geq \frac{1}{4}k2^{k-3}\,|\Theta_{\lambda,\epsilon}| \ ,$$

and

$$\frac{|H_{\lambda,\epsilon} - Y_{\lambda,\epsilon}|}{|H_{\lambda,\epsilon}|} < 3/4 + \epsilon \ .$$

By means of transference we now obtain an ergodic analogue of Lemma 5.

Lemma 6. *Let U and V be two commuting nonperiodic measure preserving transformations on a probability space (Ω, Σ, μ), and suppose $\Gamma \subset \mathbb{Z}^2_+$ is not contained in a finite union of monotonic sets. Let $0 < \lambda < \frac{1}{100}$, $0 < \epsilon < 1$. Then there exists a set $A_{\lambda,\epsilon} \subset \Omega$ such that:*

(i) *$M_{\Gamma^*}\chi_{A_{\lambda,\epsilon}} > \frac{1}{4}\lambda$ on Ω on a set of measure greater than $1/4 - 2\epsilon$, and*
(ii) *$|A_{\lambda,\epsilon}| \leq \frac{100\lambda}{\log(\frac{1}{\lambda})}$.*

Proof. Let $k \in \mathbb{Z}$ be such that $2^{-k} \leq \lambda < 2^{1-k}$ and let $\Theta_{\lambda,\epsilon}$, $Y_{\lambda,\epsilon}$, and $H_{\lambda,\epsilon}$ be as is provided by Lemma 5. For notational convenience let $\rho_{\lambda,\epsilon} = |H_{\lambda,\epsilon}|^{1/2}$. Applying the Katznelson–Weiss lemma (Lemma 2) we obtain sets $B_{\lambda,\epsilon}$ and $E_{\lambda,\epsilon}$ in Ω such that $|E_{\lambda,\epsilon}| < \epsilon$ and

$$\Omega = \left(\bigcup_{j,k=0}^{\rho_{\lambda,\epsilon}-1} U^j V^k B_{\lambda,\epsilon} \right) \cup E_{\lambda,\epsilon} \ ,$$

where the $U^j V^k B_{\lambda,\epsilon}$ are pairwise a.e. disjoint.

Let $S_{\lambda,\epsilon} = \left\{(j,k) : (j+\frac{1}{2}, k+\frac{1}{2}) \in \Theta_{\lambda,\epsilon}\right\}$ and $A_{\lambda,\epsilon} = \cup_{(j,k)\in S_{\lambda,\epsilon}} U^j V^k B_{\lambda,\epsilon}$. Let $T_{\lambda,\epsilon} = \left\{(j,k) : (j+\frac{1}{2}, k+\frac{1}{2}) \in Y_{\lambda,\epsilon}\right\}$ and $W_{\lambda,\epsilon} = \cup_{(j,k)\in T_{\lambda,\epsilon}} U^j V^k B_{\lambda,\epsilon}$. Observe that $|A_{\lambda,\epsilon}| \leq \frac{|\Theta_{\lambda,\epsilon}|}{|H_{\lambda,\epsilon}|}$ and $|W_{\lambda,\epsilon}| > (1-\epsilon)\frac{|Y_{\lambda,\epsilon}|}{|H_{\lambda,\epsilon}|} \geq \frac{|Y_{\lambda,\epsilon}|}{|H_{\lambda,\epsilon}|} - \epsilon$. By Lemma 5 we then have

$$|A_{\lambda,\epsilon}| \leq 4k^{-1}2^{3-k} \leq \frac{100\lambda}{\log(\frac{1}{\lambda})}$$

and

$$|W_{\lambda,\epsilon}| > \frac{1}{4} - 2\epsilon \ .$$

Note also that, as $\tilde{M}_{\Gamma^*,I}\chi_{\Theta_{\lambda,\epsilon}}(x,y) > \frac{1}{4}\lambda$ for any $(x,y) \in Y_{\lambda,\epsilon}$, we must have that $M_{\Gamma^*}\chi_{A_{\lambda,\epsilon}} > \frac{1}{4}\lambda$ on $W_{\lambda,\epsilon}$, completing the proof of the lemma. □

We now are in position to show that, if the approach region $\Gamma \subset \mathbb{Z}^2_+$ is not contained in a finite union of monotonic sets, then $L \log L(\Omega)$ is the largest Orlicz class of functions for which we have a.e. convergence.

Theorem 2. *Let U and V be a commuting pair of nonperiodic measure preserving transformations on a probability space (Ω, Σ, μ), and suppose $\Gamma \subset \mathbb{Z}^2_+$ is not contained in a finite union of monotonic sets. Let ϕ be a positive* *increasing function on* $[0,\infty)$ *that is* $o(\log x)$ *as* $x \to \infty$. *Then there exists a function* $f \in L\phi(L)(\Omega)$ *such that*

$$\lim_{\substack{m,n\to\infty \\ (m,n)\in\Gamma}} \frac{1}{mn} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} f\left(U^j V^k \omega\right)$$

does not exist on a set of positive measure in Ω.

Proof. For each positive integer n, choose $0 < \lambda_n < \frac{1}{100}$ such that

$$\frac{\phi\left(\frac{n}{\lambda_n}\right)}{\log\left(\frac{1}{\lambda_n}\right)} < \frac{1}{n \cdot 2^n} \ .$$

Note that such a λ_n exists since $\phi(x) = o(\log x)$ as $x \to \infty$. By Lemma 6, there exists a set $E_n \subset \Omega$ such that $M_\Gamma \chi_{E_n} \geq \frac{1}{16}\lambda_n$ on Ω on a set of measure at least $\frac{1}{8}$, where $|E_n| \leq \frac{100\lambda_n}{\log(\frac{1}{\lambda_n})}$.

Let now $f_n = \frac{n}{\lambda_n}\chi_{E_n}$. Note that $M_\Gamma f_n > \frac{n}{16}$ on Ω on a set of measure at least $\frac{1}{8}$. Moreover,

$$\begin{aligned}
\int_\Omega f_n \phi\left(f_n\right) &= |E_n| \cdot \frac{n}{\lambda_n} \phi\left(\frac{n}{\lambda_n}\right) \\
&\leq \frac{100\lambda_n}{\log\left(\frac{1}{\lambda_n}\right)} \frac{n}{\lambda_n} \phi\left(\frac{n}{\lambda_n}\right) \\
&\leq 100 \frac{n\phi\left(\frac{n}{\lambda_n}\right)}{\log\left(\frac{1}{\lambda_n}\right)} \\
&< \frac{100}{2^n} \ .
\end{aligned}$$

Set now $f = \sup_n f_n$. Observe that $M_\Gamma f = \infty$ in Ω on a set of measure at least $\frac{1}{8}$ and hence for each ω in a set of measure $\frac{1}{8}$ in Ω there exist sequences of positive integers $j_{\omega,1}, j_{\omega,2}, j_{\omega,3}, \ldots,\ k_{\omega,1}, k_{\omega,2}, k_{\omega,3}, \ldots$ tending to infinity with each $(j_{\omega,n}, k_{\omega,n}) \in \Gamma$ such that

$$\lim_{n\to\infty} \frac{1}{j_{\omega,n} k_{\omega,n}} \sum_{j=0}^{j_{\omega,n}-1} \sum_{k=0}^{k_{\omega,n}-1} f\left(U^j V^k \omega\right) = \infty \ .$$

Moreover, $f \in L\phi(L)(\Omega)$ since

$$\sum_{n=1}^{\infty} \int_\Omega f_n \phi\left(f_n\right) < \sum_{n=1}^{\infty} \frac{100}{2^n} = 100 \ .$$

As accordingly $f \in L^1(\Omega)$ we also have

$$\int_\Omega \frac{1}{mn} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} f\left(U^j V^k\right) \leq \|f\|_{L^1(\Omega)}$$

for all positive integers m, n, and hence it is not possible for

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} f\left(U^j V^k \omega\right) = \infty$$

to hold for all ω in a set in Ω of measure $\frac{1}{8}$ (even though on such a set we may have $\limsup_{\substack{m,n\to\infty \\ (m,n)\in\Gamma}} \frac{1}{mn} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} f\left(U^j V^k \omega\right) = \infty$). The theorem follows. □

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