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A new record for the canonical height on an elliptic curve over $\mathbb{C}(t)$

Sonal Jain

Abstract. We exhibit an elliptic curve $E/\mathbb{C}(t)$ of discriminant degree 84 with a nontorsion point P of canonical height 2987/120120 (a new record). We also prove that if (E, P) has Szpiro ratio $\sigma \leq 4$, then $\hat{h}(P)$ must exceed this value, providing some evidence that our example may yield the smallest height possible over $\mathbb{C}(t)$. Using the same strategy, we find other $E/\mathbb{C}(t)$ with nontorsion points of small canonical height, including Elkies' previous record.

Contents

1. Introduction

Let E/K be an elliptic curve over a number field or complex function field K. A conjecture by Lang postulates a uniform lower bound for the canonical height of nontorsion points $P \in E(K)$:

Conjecture (Lang)**.** *There exists a constant $C = C(K) > 0$ such that for all pairs (E, P) one has*

$$\hat{h}(P) \geq C \log |N_{K/\mathbb{Q}} \Delta_{E/K}|.$$

Over $\mathbb{C}(t)$ or $\mathbb{C}(\mathcal{C})$ for $\mathcal{C}$ a curve, the same bound holds with $\log |N_{K/\mathbb{Q}} \Delta_{E/K}|$ replaced by the discriminant degree $d = 12n$.

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In [5], Hindry and Silverman prove Lang's conjecture under the hypothesis of a conjecture of Lucien Szpiro [12]. Szpiro's conjecture is equivalent to the ABC conjecture of Masser–Oersterlé, showing that Lang's conjecture is true over function fields. In the case that $K = \mathbb{C}(t)$, Hindry and Silverman determine an explicit value for $C \approx 7 \times 10^{-10}$. In [4] Elkies improves the value of C to $\approx 10^{-7}$, and conjectures that the correct value of C should be $3071/10810800 \approx 2.84 \times 10^{-4}$. It is natural to ask: What is the smallest possible canonical height $\hat{h}(P)$ of a rational nontorsion point P on a curve $E/\mathbb{C}(t)$?

In this paper we exhibit explicit equations for an elliptic curve $E/\mathbb{C}(t)$ of discriminant degree $d = 84$ with a rational point P of canonical height $2987/120120$, which breaks the previous record of $261/10010$ held by Elkies. Our example comes very close to Elkies' conjectural lower bound for $12nC$. In fact, $2987/120120$ is only 4.2% larger than $(12 \cdot 7)\, 3071/10810800$. This leads us to ask: Could this value be a global minimum for the canonical height of a nontorsion point on an elliptic curve over $\mathbb{C}(t)$?

The conjectural value of the constant C relies on a heuristic improvement of the *Szpiro ratio* σ, which is defined for $E/\mathbb{C}(t)$ as the ratio of the degrees of the discriminant of E and the conductor of E [12]. One always has $\sigma \leq 6$ (see Hindry–Silverman [5, Thm. 5.1]). In general, most curves over $\mathbb{C}(t)$ will not have a section of small canonical height. A parameter count suggests that an $E/\mathbb{C}(t)$ with Szpiro ratio $\sigma > 4$ and a section of height less than n should be rare. (see Section 4.3). We prove the following:

Theorem 1.1. *Suppose that E is a nonconstant elliptic curve over $\mathbb{C}(t)$ with Szpiro ratio $\sigma \leq 4$, and P is a nontorsion point in $E(\mathbb{C}(t))$. Then the canonical height $\hat{h}(P) > 2987/120120$.*

In fact our record example is the first and only known example with a nontorsion integral point P that has Szpiro ratio $\sigma > 4$. Our attempts at constructing such elliptic curves as well as the heuristic discussed in Section 4.3 make us believe it is highly unlikely that another such curve, if it exists, will also have a point of very small canonical height. Also, as d grows, the only way $\hat{h}(P)$ could be smaller than our record is for σ to be significantly larger than 4. We conjecture:

Conjecture 1.2. *The minimum canonical height of a nontorsion point on an elliptic curve over $\mathbb{C}(t)$ is $2987/120120$.*

Although we provide some evidence for the conjecture, the evidence is not conclusive. A proof of this conjecture is not within reach using the techniques of this paper, as the heuristic improvement of the Szpiro ratio on which the conjecture depends puts a strong combinatorial constraint on the fibration (see Section 4.3).

1.3. Integral multiples. There is an established connection (see Elkies [3]) between points of low canonical height and points with *integral* multiples. For $E/\mathbb{C}(t)$, P is integral if it does not meet the zero section of E, i.e., if the coordinates of P with respect to a minimal Weierstrass equation have no poles.

Our (E, P) has mP integral for *every* $m \in [1, 10] \cup \{12, 14, 15\}$. Previously, the maximal known m for which mP is integral was 12, for a curve of conductor 60. The maximum possible such m is 42, and in the case that $\sigma \leq 4$ the maximum possible m is 15 (see [4] p.21). As an (E, P) with P integral and $\sigma > 4$ is rare, it is likely that 15 is maximum possible m. We also find another example (E, P) with mP integral for $m \in [1, 9] \cup \{12, 13, 15\}$.

2. Strategy

2.1. A special family of K3 elliptic surfaces. In our paper [7], we found the unique $K3$ elliptic surface that attains the smallest possible regulator $R(P, Q) = 1/100$ for a rank 2 sublattice $\mathbb{Z}\,P \oplus \mathbb{Z}\,Q$ of its Mordell–Weil lattice. We located this surface (as $q = 3$) in the following family $K3$ ellipic surfaces of Picard number 19 with two independent sections, parametrized by $\mathbb{P}^1$:

$$
\begin{aligned}
(1) \qquad E_q(t): \quad & Y^2 + q(-t^2 + (q+1)t - 1)XY \\
& + (qt(t-q)(t-q+1)(qt-1)(qt-t-1))Y \\
& \quad = X^3 - qt(t-q+1)(qt-t-1)X^2 \\
Q_q(t): \quad & \big(qt(t-q)(qt-t-1), -q^2t^2(t-q)(qt-t-1)\big) \\
P_q(t): \quad & (0, 0)\,.
\end{aligned}
$$

2.2. Small heights over $\mathbb{Q}$. In [7], where we found this family, we used to following strategy to produce the elliptic curves $E/\mathbb{Q}$ with smallest known nonzero canonical height. There are 28 pairs of points on $E_3(t)$ of the form $mP + m'Q$, with m and m' both nonzero, that have naive height 6. This means that these points meet the zero section for exactly one value of t. For each such (m, m') we specialized $E_3(t)$ to this value t_0, forcing $mP_3(t_0) + m'Q_3(t_0) = 0$ on the curve $E_3(t_0)/\mathbb{Q}$. This gave us a point P_g generating $\mathbb{Z}P_3(t_0) + \mathbb{Z}Q_3(t_0)$ of potentially small canonical height on this curve:

$$\hat{h}(P_g) = \frac{\gcd(m', m)^2}{m'^2}\hat{h}(P_3(t_0)).$$

We expected this height to be small, both because several multiples of the point P_g would be integral on $E_3(t_0)$, and also because of specialization theorems of Silverman and Tate (cf. chapter III section 11 of [11]). Our prediction was correct, and we recovered the five smallest known nonzero canonical heights over $\mathbb{Q}$ in this way.

For example The point $7P + Q$ has x-coordinate

$$x(7P + Q) = \frac{72t^6 + 426t^5 - 501t^4 - 1233t^3 - 198t^2 + 216t}{(7t + 6)^2}.$$

Specializing to $t = -6/7$ yields the elliptic curve over $\mathbb{Q}$ of conductor 3990 with the point of smallest known canonical height, $\hat{h}(P_g) \approx 0.00445716$ (see Section 8, [7]).

2.3. Specializing to a curve. We apply a similar strategy to the family in (1), in an attempt to produce elliptic surfaces over $\mathbb{P}^1$ with sections of very small canonical height. Considering both t and q as parameters, we view this family as an elliptic three-fold fibred over $\mathbb{P}^1 \times \mathbb{P}^1$. If we specialize $t = f(q)$ to some rational function of q, we obtain an elliptic surface (in general not $K3$) over the q-line. The generic member of the family (1) has 22 points $R = mP + m'Q$, with both m and m' nonzero, with naive height 6. For each such pair (m, m'), the section $mP_q(t) + m'Q_q(t)$ meets the zero section at one value $t = f(q)$, where f is some rational function. Specializing to $t = f(q)$ forces $mP_q + m'Q_q = 0$ on the elliptic curve over $\mathbb{C}(q)$. This will yield a point P_g generating $\mathbb{Z}P + \mathbb{Z}Q$ of potentially small canonical height on E_q:

$$\hat{h}(P_g) = \frac{\gcd(m', m)^2}{m'^2}\hat{h}(P).$$

If $a, b \in \mathbb{Z}$ are such that $am + bm' = \gcd(m, m')$, then

$$P_g = bP - aQ = \frac{\gcd(m, m')}{m'}P = -\frac{\gcd(m, m')}{m}Q.$$

Our prediction that $\hat{h}(P_g)$ should be small is correct, and using this strategy we obtain a new record for the canonical height on an elliptic surface over $\mathbb{P}^1$. In addition, this strategy recovers the elliptic surface attaining the previous record of Elkies.

2.4. Computing canonical heights. Let P be a point on an elliptic curve E over $\mathbb{C}(C)$, where C is a complex algebraic curve of genus g. The canonical height $\hat{h}(P)$ can be written as a sum of the naive height $h(P)$ and some local correction terms:

$$\hat{h}(P) = h(P) + \sum_{\upsilon} \lambda_{\upsilon}(P),$$

where the sum is taken over singular fibers υ. The local correction term $\lambda_\upsilon(P)$ depends only on the type of the singular fiber E_υ at υ, and the component c_υ of E_υ that meets the section s_P corresponding to P. The naive height $h(P)$ is equal to $2n + 2s_P \cdot s_0$, where $s_P \cdot s_0$ is equal to the intersection number of s_P with the zero section. In the case that $C = \mathbb{P}^1$, $2s_P \cdot s_0$ is the number of poles of $x(P)$. We list explicit formulas for the local correction terms for each possible singular fiber. These formulas have been worked out by Cox and Zucker in [1]. One can use Tate's algorithm [13] to compute the type of each singular fiber E_υ, and thus computing exact canonical heights in this setting is a straightforward calculation.

- If the section s_P intersects the identity component of E_υ, then

$$\lambda_\upsilon(P) = 0.$$

- If E_v is an additive fiber of type III, IV, I_0^*, III^* or IV^*, and s_P intersects a nonidentity component of E_v, then $\lambda_v(P) = -d_v/6$.
- Suppose E_v is an additive fiber of type $I_\nu^*(\nu > 0)$ and s_P passes through a nonidentity component. If ν is odd and s_P meets the distinguished 2-torsion component, then $\lambda_v(P) = -1$. Otherwise we have $\lambda_v(P) = -\nu/4 - 1$.
- Finally, if E_v is a multiplicative fiber of type I_ν and s_P passes through component a, then

$$\lambda_v(P) = \frac{(a-\nu)a}{\nu}.$$

3. Examples

In [7], we parametrized the set of triples (E, P, Q) of an elliptic curve E/Q with rational points P, Q such that $P, 2P, Q, P \pm Q$ and $2P + Q$ are all integral by an open subset of $\mathbb{P}^3$. We located the one parameter family of $K3$ elliptic surfaces (1) as a one parameter family of conics in this $\mathbb{P}^3$. By its definition, the moduli space has a symmetry interchanging Q and $-P - Q$. Thus, although the generic member of the family in (1) has 22 points $R = mP + m'Q$ (both m and m' nonzero) of naive height 6, by this symmetry we need only consider 11 pairs (m, m') The naive height of $mP + m'Q$ equals 6 for the following pairs (m, m'):

$$(1,2), (1,3), (2,3), (3,3), (5,1), (5,2), (6,1), (6,2), (6,3), (7,1), (8,2),$$

as well as their images under the symmetry $(m, m') \leftrightarrow (m - m', -m')$. We specialize the family at these eleven pairs. We find very small values for the canonical height at $(5, 2)$, $(6, 1)$, and $(7, 1)$.

3.1. Example 1. Specializing $t = -q^2 + 2q$ forces $5P + 2Q = 0$. The resulting model is not minimal at $q = 0$, and we change coordinates to a obtain a global minimal model $y^2 + A_1(q)xy + A_3(q)y = x^3 + A_2(q)x^2$, where

$$\begin{aligned}
A_1 &= -(q-1)^3(q^3 - 2q^2 + q - 1),\\
A_2 &= (q-2)(q^2 - q - 1)(q^3 - 3q^2 + 2q + 1),\\
A_3 &= -(q-2)(q-1)^2(q^2 - q - 1)^2(q^3 - 3q^2 + 2q + 1).
\end{aligned}$$

We then use Tate's Algorithm [13] to compute the Kodaira fiber types, and compute the component of each fiber meeting the section $P_t = (0, 0)$. We sum the local contributions to the canonical height $\hat{h}(P_t)$. We list the places at which we there a nonzero contribution to the height in Table 1. We obtain $h(P_t) = 205/308$, and then divide this by 25 to get the height of the generator for $\mathbb{Z}\,P \oplus \mathbb{Z}\,Q$ to be $41/1540$. This curve was previously known by Elkies, and is believed to attain the minimum canonical height for $d = 48$ [3].

TABLE 1.

Fiber	Type	$c_q(P)$	$\lambda_q(P)$
$q=1$	I_4	2	-1
$q=2$	I_7	1	$-6/7$
$q=(-1\pm\sqrt{5})/2$	$2I_5$	2	$2\cdot-6/5$
$q=1\pm\sqrt{2}$	$2I_2$	1	$2\cdot-1/2$
$q=\infty$	I_{11}	2	$-11/18$
$q^3-3q^2+2q+1=0$	$3I_3$	0	0
$q^2-q^2-9q+13=0$	$3I_1$	0	0

TABLE 2.

Fiber	Type	$c_q(P)$	$\lambda_q(P)$
$q=1$	I_{13}	3	$-30/13$
$q=\infty$	I_{11}	1	$-10/11$
$q=2$	I_7	1	$-6/7$
$q=(1\pm\sqrt{5})/2$	$2I_5$	2	$2\cdot-6/5$
$q^3-q^2-2q^2+1=0$	$3I_2$	1	$3\cdot-1/2$
$q^3-2q^2+q-1=0$	$3I_3$	1	$3\cdot-2/3$
$q^4-9q^3+28q^2-34q+13=0$	$4I_1$	0	0

3.2. Example 2. Specializing to $t=(-q^2+2q)/(q^2-2q+1)$ forces $6P+Q=0$. The equations we obtain are not minimal at $q=0$ or $q=\infty$, and we change coordinates to obtain the minimal model

$$y^2+A_1(q)xy+A_3(q)y=x^3+A_2(q)x^2,$$

with

$$A_1(q)=q^5-q^4-7q^3+13q^2-6q+1,$$
$$A_2(q)=(q-2)(q-1)^3(q^2-q-1)(q^3-2q^2+q-1),$$
$$A_3(q)=(q-2)(q-1)^3(q^2-q-1)^2(q^3-2q^2+q-1)(q^3-q^2-2q+1).$$

The curve has $d=60$. Table 2 lists all of the local data for the elliptic curve and its section P. We calculate $\hat{h}(P)=261/10010$, which was previously the smallest known nonzero canonical height on an elliptic curve over $\mathbb{C}(t)$. This curve was found using different methods by Elkies, and is believed to attain the minimum canonical height for $d=60$ [3].

3.3. Example 3 (Record height). Finally, specializing to

$$t=(-q^3+3q^2-2q)/(q^3-3q^2+2q+1)$$

forces $7P+Q=0$ and yields our new record. The model we obtain is not minimal at $q=0$ and the roots of $q^3-3q^2+2q+1=0$. We change

TABLE 3.

Fiber	Type	$c_q(P)$	$\lambda_q(P)$
$q=1$	I_{11}	3	$-24/11$
$q=\infty$	I_{13}	1	$-12/13$
$q=2$	I_8	1	$-7/8$
$q=(1\pm\sqrt{5})/2$	$2I_7$	3	$2\cdot -12/7$
$q^3-3q^2+2q+1=0$	$3I_5$	1	$3\cdot -4/5$
$q^3-2q^2-q+3=0$	$3I_2$	1	$3\cdot -1/2$
$q^4-3q^3+2q^2+1=0$	$4I_3$	1	$4\cdot -2/3$
$q^5-9q^4+28q^3-33q^2+7q+7=0$	$5I_1$	$5I_1$	0

coordinates to attain a minimal model

$$y^2+A_1(q)xy+A_3(q)y=x^3+A_2(q)x^2,$$

with

$$\begin{aligned}A_1(q)&=q^7-3q^6-5q^5+28q^4-32q^3+5q^2+6q+1,\\ A_2(q)&=(q-2)(q-1)^3(q^2-q-1)(q^3-3q^2+2q+1)(q^4-3q^3+2q^2+1),\\ A_3(q)&=(q-2)(q-1)^3(q^2-q-1)^3(q^3-3q^2+2q+1)\\ &\quad\cdot(q^3-2q^2-q+3)(q^4-3q^3+2q^2+1).\end{aligned}$$

All the local information is compiled in Table 3. Computing the canonical height of P_t, we find that $\hat{h}(P_t)=2987/120120$. This is a new record for the canonical height over $\mathbb{C}(t)$.

3.4. Another small family. We apply the same strategy to another one parameter family of $K3$ surfaces of Picard number 19. The generic member of the family below is a $K3$ elliptic surface of conductor degree 9 with a rank 2 subgroup $\mathbb{Z}\,P\oplus\mathbb{Z}\,Q$, such that the volume of the sublattice generated by P and Q is $1/48$. This is the smallest possible regulator for a rank 2 subgroup of an elliptic $K3$ that is attained by a one parameter family.

$$\begin{aligned}&E_q(t):\\ &Y^2-q\left(q^4t^2-q^3t^2+qt^2-t^2-2q^3t+3q^2t-2qt-t+q^2-2q+1\right)XY\\ &-(q-1)^2\left(q^2-q+1\right)(t-1)^2\,t\,(qt-t-q)\left(q^2t-qt+t-q^2+q\right)Y\\ &\qquad=X^3+q^2\left(q^2-q+1\right)t\,(qt-t-1)\left(q^2t-qt+t-q+1\right)X^2,\end{aligned}$$

$$\begin{aligned}&Q_q(t):\\ &\Big(-(q-1)\,q^2\left(q^2-q+1\right)t\,(qt-1)\,(qt-t-1)\left(q^2t-qt+t-q+1\right),\\ &-(q-1)\,q^3\left(q^2-q+1\right)^2t^2\,(qt-1)\,(qt-t-1)^2\left(q^2t-qt+t-q+1\right)\Big)\end{aligned}$$

$$P_q(t):\quad (0,0)\,.$$

TABLE 4.

Fiber	Type	$c_q(R)$	$\lambda_q(R)$
$q=1$	I_5	1	$-4/5$
$q=\infty$	I_{13}	6	$-42/13$
$q=0$	I_{11}	1	$-10/11$
$q^2-q+1=0$	$2I_7$	1	$2\cdot -6/7$
$q^2+1=0$	$2I_5$	2	$2\cdot -6/5$
$q^3-2q^2+q-1=0$	$3I_4$	1	$3\cdot -3/4$
$q^4-2q^3+2q^2-q+1=0$	$4I_3$	1	$4\cdot -2/3$
$q^7-4q^5+5q^4-8q^3+6q^2-5q+1=0$	$7I_1$	0	0

This family also has several points of naive height 6, including $2P+5Q$.

3.5. Example 4 (Several integral multiples). We specialize

$$t=(q^2-q+1)/q,$$

which forces $2P+5Q=0$. Again the model is not minimal at $q=1$ and we change coordinates to obtain the minimal model

$$Y^2+A_1(q)XY+A_3(q)Y=X^3+A_2(q)X^2:$$

(2)

$$A_1(q)=-\frac{q^8-3\,q^7+3\,q^6+q^5-6\,q^4+6\,q^3-5\,q^2+2\,q-1}{q-1}$$

$$A_2(q)=\frac{q\,\left(q^2-q+1\right)^2\,\left(q^3-2\,q^2+q-1\right)\,\left(q^4-2\,q^3+2\,q^2-q+1\right)}{\left(q-1\right)^2}$$

$$A_3(q)=-\left(q-1\right)\,q^5\,\left(q^2-q+1\right)^2\,\left(q^3-2\,q^2+q-1\right)\left(q^4-2\,q^3+2\,q^2-q+1\right)$$

$$Q(q)=\Bigg(\left(-q^2\,\left(q^2-q+1\right)^2\,\left(q^3-2\,q^2+q-1\right)\,\left(q^4-2\,q^3+2\,q^2-q+1\right),\right.\\ \left.-\frac{q^2\,\left(q^2-q+1\right)^4\,\left(q^3-2\,q^2+q-1\right)^2\,\left(q^4-2\,q^3+2\,q^2-q+1\right)}{q-1}\right)\Bigg).$$

The curve has discriminant degree 84 and conductor $N=3\cdot 7$. All of the correction terms for the generator $R=P+2Q$ of $\mathbb{Z}P+\mathbb{Z}Q$ are compiled in Table 4. Adding up the correction terms gives us a height $\hat{h}(R)=1753/60600$. This curve also has a very large number of integral multiples. The point mR is integral for $m\in[1,9]\cup\{12,13,15\}$.

4. Proof of Theorem

Our proof of Theorem 1.1 uses two main ingredients. The first is applying linear programming to find asymptotic lower bounds for the canonical height. The second is searching through combinatorial configurations of fibers and sections that could correspond to an elliptic surface.

4.1. Hindry–Silverman. In [5], Hindry–Silverman prove Lang's conjecture for elliptic curves over function fields. Their basic approach in approximating $\hat{h}(P)$ is:

1. Replace P by $12P$ so that P meets the identity component of any additive fiber.
2. Carefully choose coefficients c_m so that

$$\sum c_m \hat{h}(mP) \gg (12n). \tag{3}$$

Since $\sum c_m \hat{h}(mP) = (\sum m^2 c_m)\hat{h}(P)$, they are able to obtain an explicit constant $C \approx 7 \times 10^{-10}$.

4.2. Elkies' approach. One may use the following to greatly improve the constant obtained by Hindry–Silverman [4]:

1. In order to find lower bounds for the canonical height, it is only necessary to search through configurations consisting entirely of fibers of type I_ν. This eliminates a factor of $12^2 = 144$ in the denominator of the lower bound.
2. Any choice of c_m's inside a particular polytope ensures that (3) holds. Thus one may minimize the linear form $\sum m^2 c_m$ on this polytope to obtain a better bound.

The feasible region for Elkies' linear program depends on the Szpiro ratio σ. One always has $\sigma \leq 6$, and setting $\sigma = 6$ yields

$$C = 39086299807/99005116318560 \approx 1/25330.$$

If E has a small section, however, one expects (heuristically) that $\sigma \leq 4$:

4.3. Heuristic improvement of $\boldsymbol{\sigma}$. One can improve the upper bound on σ via the following parameter counting argument. Let E be an elliptic curve over $\mathbb{C}(t)$ of discriminant degree $12n$, with minimal Weierstrass equation:

$$y^2 = x^3 + a_4(t)x + a_6(t).$$

Semistability is equivalent to $a_4(t)$ and $a_6(t)$ being coprime polynomials of degrees $\leq 4n$ and $\leq 6n$, respectively. The discriminant $\Delta(t)$ is given by

$$\Delta(t) = a_4(t)^3 - 27a_6(t)^2.$$

By changing coordinates on the base $\mathbb{P}^1$, we may assume that E has good reduction at ∞, and hence $\Delta(t)$ is polynomial of degree $12n$. The polynomials $a_4(t)$ and $a_6(t)$ vary in an affine space $\mathbb{A}^{10n+2}$ of dimension $10n+2$. After accounting for the four dimensions of symmetry given by rescaling $(a_4, a_6) \mapsto (u^4 a_4, u^6 a_6)$ and the action of $\mathrm{PGL}(2)$ on the base $\mathbb{P}^1$, one sees that the space of E of discriminant degree $12n$ is parametrized by an open subset of $\mathbb{A}^{10n-2}$.

Consider the space of (E, P), where E has $d = 12n$ and $P = (X(t), Y(t))$ where $X(t)$ and $Y(t)$ are coprime polynomials of degree $2n$ and $3n$. This is an affine space of dimension $10n - 2 + 5n + 2 = 15n$. The condition that P is a rational point on E amounts to the $6n + 1$ equations given by the Weierstrass equation. This determines a subvariety of dimension $9n - 1$. The polynomial $\Delta(t)$, which depends on $9n - 1$ parameters, has generically $12n$ disctinct roots and therefore $\sigma = 1$. However, imposing conditions that collapse roots, one may descend to $3n + 1 = 12n - (9n - 1)$ distinct roots. Therefore we obtain the heuristic inequaltiy $\sigma \leq 12n/(3n + 1) < 4$. Examples with $3n$ roots, which give $\sigma = 4$, are unlikely. Those with $3n - 1$ roots, which give $\sigma > 4$, are even worse.

Bounding $\sigma \leq 4$ increases the feasible region of the linear program, and yields the conjectural value of $C = 3071/10810800$.

4.4. More linear programming. We modify the linear program to compute lower bounds useful to us. The record height of $2987/120120 = .02486...$ is smaller than $(12n)3071/10810800$ for $n > 7$. Thus if (E, P) has Szpiro ratio $\sigma \leq 4$ and $n > 7$, the canonical height $\hat{h}(P)$ is larger than our new record.

For $n = 1, \ldots, 5$, one can search through possible configurations of fibers and sections corresponding to elliptic curves. Restricting to $\sigma \leq 4$ puts enough constraints on the fibration that there is no (E, P) with

$$\hat{h}(P) < 2987/120120.$$

For $n = 1, 2, 3$ the minimum heights are known (see Oguiso–Shioda [9], Shioda [10], Nishiyama [8], Elkies [3]), and for $n = 4, 5$ they are known for curves with $\sigma \leq 4$ (Elkies [3]).

For $n = 7$, the lower bound for the canonical height on curves with $\sigma \leq 4$ is $(12n)3071/1081080 = .02386...$, which is slightly smaller than our record height. We consider $\sigma \leq (12 \cdot 7)/(3 \cdot 7 + 1) = 42/11$, which is the largest possible value of the Szpiro ratio that is less than 4 in the case that $n = 7$. This further shrinks the feasible of the linear program. Solving the linear program and computing a lower bound with this restriction on σ, we obtain $84 \cdot C_\sigma = 10561/360360 = .02930...$, which is larger than our record height.

Similary in the case that $n = 6$ we compute a lower bound with the restriction that $\sigma \leq (12 \cdot 6/3 \cdot 6 + 1) = 72/19$. We obtain the lower bound $72 \cdot C_\sigma = 46663/1801800 = .02589...$, which is again larger than our record height.

Thus we are left to consider the case that $\sigma = 4$, and $n = 6$ or $n = 7$.

4.5. Combinatorial search. For $n = 6$ and $n = 7$, we search through configurations of fibers and sections, restricting to $\sigma = 4$. We use the conditions explained in [3] and [7] to eliminate unattainable configurations. For $n = 7$, we find no configurations that could yield an (E, P) with $\hat{h}(P)$ smaller than our record. For $n = 6$, however, we find one configuration:

(4) $[1/11]+[4/9]+[3/8]+[1/7]+[2/7]+2[1/5]+[1/4]+2[1/3]+2[1/2]+6[0]$.

Here $[a/\nu]$ denotes a fiber of type I_ν with the section meeting the ath component. This configuration, if realized by an elliptic curve (E, P) over $\mathbb{C}(t)$, would have $\hat{h}(P) = 683/27720 = 0.02463... < 2987/120120 = .02486....$ In addition, it would have mP integral for $m \in [1, 11]$. We show that this configuration cannot be attained.

4.6. Integral points. Elkies, in [2], parametrizes the moduli space of elliptic curves (E, P) with $P, \ldots, 8P$ integral by $\mathbb{P}^1 \times \mathbb{P}^1$:

$$\begin{aligned} A_1 &= Au^5 + \left(2A^2 + 3A - 2\right) u^4 + (-4A - 8)\, u^3 \\ &\quad + \left(-A^2 - 10A - 10\right) u^2 + \left(-4A^2 - 10A - 6\right) u \\ &\quad + \left(-A^2 - 2A - 1\right) \\ A_2 &= u(u+1)(u+A+1)(Au-A-1)(Au^2+u^2+u+A+1) \\ &\quad \cdot (Au^3 - Au^2 - 2u^2 - Au - 2u - A - 1) \\ A_3 &= u(u+1)^3(u+A+1)^2(Au-A-1)(Au-A-1) \\ &\quad \cdot (Au^2+u^2+u+A+1)(Au^2-Au-2u-A-1) \\ &\quad \cdot (Au^3 - Au^2 - 2u^2 - Au - 2u - A - 1). \end{aligned}$$

Here A_1, A_2 and A_3 are the nonzero Weierstrass coefficients of $E : y^2 + A_1xy + A_3y = x^3 + A_2x^2$, and $P = (0\,,\,0)$. The coordinates A, u are the affine coordinates on the two copies of $\mathbb{P}^1$.

We consider the discriminant locus of E in $\mathbb{P}^1 \times \mathbb{P}^1$:

(5)

$$\begin{aligned} &u^{10}\, (u+1)^7\, (A+u+1)^5\, (uA-A-1)^7\, \left(u^2\, A + A + u^2 + u + 1\right)^4 \cdot \\ &\left(u^3\, A - u^2\, A - u\, A - A - 2\, u^2 - 2\, u - 1\right)^3\, \left(u^2\, A - u\, A - A - 2\, u - 1\right)^2 \cdot \\ &\left(11\, u^3\, A^3 - u^2\, A^3 - 3\, u\, A^3 + A^3 + 9\, u^4\, A^2 + 18\, u^3\, A^2 - 9\, u^2\, A^2 - 4u\, A^2 \right. \\ &\quad \left. +2\, A^2 - u^5\, A + 5\, u^4\, A + 6\, u^3\, A - 8\, u^2\, A - u\, A + A + 2\, u^4 + 2\, u^3 - u^2\right). \end{aligned}$$

If the configuration (4) were to correspond to an elliptic curve over $\mathbb{P}^1$ of discriminant degree 72, we would be able to locate this curve as a $(1, 1)$-curve l in the $\mathbb{P}^1 \times \mathbb{P}^1$ above.

We compute the slope of the line through $4P$ and $5P$, and find it is equal to $f(u) - (u^4 + u^3)/(A + 1)$ for some polynomial $f(u)$. In order for $9P$ to be integral, the slope of this line must be integral, which happens when the curve l goes through $(A, u) = (-1, -1)$ or $(0, -1)$. However l cannot go through $(0, -1)$, for at this point the I_{10} fiber along $u = 0$ merges with the I_5 fiber along $A + u + 1 = 0$ to form a I_{15} fiber. Similarly for $10P$ to be integral, l must go through $(-1/2, -1)$. This implies that l is in fact a $(1, 0)$-curve.

Finally, we see from the factorization of the discriminant in (5) that the fiber of type [1/11] in the configuration would have to occur somewhere along the line $u = 0$. The only other component of the discriminant locus that $u = 0$ meets to order 1 is the sextic factor in (5), and only at $A = 1$. This forces our $(1, 0)$-curve through $(1, 0)$, which is impossible. This completes the proof of the Theorem.

5. Further directions

5.1. Base curves of higher genus. Fixing the genus of a base curve $\mathcal{C}$, there is a minimum canonical height for elliptic curves $E/\mathbb{C}(\mathcal{C})$. One may attempt to use techniques similar to those of this paper to produce elliptic curves over higher genus curves with points of low height. For example, one could look for points of naive height 8 in a family of elliptic surfaces similar the one in Section 2.1. Applying the technique in Section 2.3 to such a family would yield an elliptic surface fibred over a hyperelliptic curve of genus 2 with several integral multiples of a nontorsion section, and potentially a point of small canonical height. It would be interesting to explore how small the canonical height could be over higher genus base curves.

5.2. A uniform bound for any genus? It is interesting to ask whether or not there is a minimum canonical height if one allows the genus of the base curve to vary. Given a curve (E, P) defined over $\mathbb{C}(t)$, a first thought might be to take the curve $(E, \frac{1}{N}P)$ defined over $\mathbb{C}(t, \frac{1}{N}P)$, which is isomorphic to $\mathbb{C}(\mathcal{C})$ for some curve $\mathcal{C}$. However it is not the case that $\hat{h}(\frac{1}{N}P)$ equals $N^{-2}\hat{h}(P)$: In general this basechange has degree N^2 which eliminates the factor of N^2 in the denominator. In the number field setting, one typically uses the absolute canonical height, so that $\hat{h}(\frac{1}{N}P)$ does equal $N^{-2}\hat{h}(P)$. In the function field setting we use the height relative to the field $K = \mathbb{C}(\mathcal{C})$.

For example, let E_1 be an elliptic curve over $\mathbb{C}$, let E be the constant curve $E_1 \times E_1$, and let P be the section coming from the identity map of E_1. Then P has height (both canonical and naive) equal 2. To define $\frac{1}{N}P$, one needs a base change to a curve E_N with $\deg(E_N/E_1) \geq N^2$. In the case of equality E_N is E_1 itself but the map $E_N \to E_1$ is multiplication by N. Then $\frac{1}{N}P$ is again the identity map, and thus again of height 2.

In fact a uniform version of Lehmer's conjecture over function fields asserts that for an elliptic curve $E/k(C)$, there is an absolute lower bound for $\hat{h}(P)$ for nontorsion points in $E(k(C'))$ for finite covers $C' \to C$. Over number fields, Lehmer's conjecture says that $[K : \mathbb{Q}]\hat{h}(P)$ is bounded below. Hence the above question, even for a single elliptic curve, is not known to be true.

5.3. Higher rank. In [7] we computed minimal discriminants for rank 2 sublattices of $E/\mathbb{C}(t)$ of discriminant degree 12 and 24 (rational or K3 elliptic surfaces). In [6], we compute nontrivial asymptotic lower bounds for rank 2 sublattices of any $E/\mathbb{C}(t)$, and conjectured the best possible bound

$12nC$. It would be interesting to find examples of elliptic curves $E/\mathbb{C}(t)$ with rank 2 sublattices whose volume is very close to the conjectural bound.

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Departent of Mathematics, New York University Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, NY 10012
jain@cims.nyu.edu

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