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Coactions and Fell bundles

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Abstract. We show that for any Fell bundle $\mathscr{A}$ over a locally compact group G, there is a natural coaction δ of G on the Fell-bundle C^*-algebra $C^*(G,\mathscr{A})$ such that the full crossed product $(C^*(G,\mathscr{A})\rtimes_\delta G)\rtimes_{\hat{\delta}} G$ by the dual action $\hat{\delta}$ of G is canonically isomorphic to $C^*(G,\mathscr{A})\otimes\mathcal{K}(L^2(G))$. Hence the coaction δ is maximal.

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Introduction

The theorem announced in the abstract, which we prove as Theorem 8.1, is part of a larger program that is inspired by the realization, which only recently has come into focus, that Fell bundles provide a natural setting for a broad range of imprimitivity theorems and equivalence theorems for C^*-dynamical systems, especially theorems involving nonabelian duality. Even in the context of group actions or coactions, the Fell bundles which arise are over groupoids, and so our approach in this program will involve the recent equivalence theorem of [16] for Fell bundles over groupoids. The present paper is a first step in this larger program: in the course of reformulating the standard C^*-dynamical system constructions in the context

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of Fell bundles, it will be crucial to know that the coactions which arise will always be maximal ones.

Very roughly, a Fell bundle $\mathscr{A}$ over a locally compact group G is a bundle over G such that the fibre A_e over the identity e of G is a C^*-algebra and such that the fibre A_s over each $s \in G$ is an $A_e - A_e$-imprimitivity bimodule with the property that $A_s \otimes_{A_e} A_t$ is isomorphic to A_{st} in such a way that tensoring gives an associative multiplication on $\mathscr{A}$.[1] The space of continuous, compactly supported cross sections of $\mathscr{A}$, denoted $\Gamma_c(G;\mathscr{A})$, carries a natural convolution-like product under which it forms a $*$-algebra. A certain completion of this algebra is a C^*-algebra, denoted $C^*(G,\mathscr{A})$. One can profitably think of $C^*(G,\mathscr{A})$ as a generalized crossed product of A_e by G. Indeed, if G acts on a C^*-algebra B via a continuous homomorphism $\alpha : G \to \mathrm{Aut}(B)$, and if $\mathscr{A}$ is defined to be $B \times G$, with product defined by the equation $(a,s)(b,t) = (a\alpha_s(b), st)$, then $\mathscr{A}$ is a Fell bundle over G, called the *semidirect-product bundle* determined by the action, and the C^*-crossed product $B \rtimes_\alpha G$ is isomorphic to the bundle C^*-algebra $C^*(G,\mathscr{A})$.[2] This point was made by Fell in his first works on the subject [9, 10] and was one of the reasons he began the theory of these bundles. Importantly, not every Fell bundle over a group G is isomorphic to such a bundle [12, §§VIII.3.16, VIII.4.7]; thus the results in the present paper properly generalize those (in [3], for example) which directly address group actions.

Coactions were introduced to give a generalization, for nonabelian groups, of the Takai–Takesaki duality for crossed products by actions of abelian groups on C^*-algebras. Subsequently Katayama proved a crossed-product duality theorem for coactions, specifically, if δ is a coaction of a group G on a C^*-algebra A, then there is a dual action $\hat{\delta}$ of G on the crossed product $A \rtimes_\delta G$ such that the *reduced* crossed product $(A \rtimes_\delta G) \rtimes_{\hat{\delta},r} G$ is isomorphic to $A \otimes \mathcal{K}(L^2(G))$. Katayama used what are now known as *reduced* coactions, which involve the reduced group C^*-algebra $C^*_r(G)$. For more information on crossed-product duality, see [4, Appendix A].

The use of the term "crossed product" both in the context of group actions and in the context of coactions may seem confusing, initially. However, in practice, it is easy to distinguish between the two.

Raeburn introduced *full* coactions, which involve the full group C^*-algebra $C^*(G)$, to take advantage of universal properties. For such coactions, there is always a canonical surjection

$$\Phi : A \rtimes_\delta G \rtimes_{\hat{\delta}} G \to A \otimes \mathcal{K}(L^2(G)),$$

and the question naturally arose, when is Φ in fact an isomorphism? When this is the case, *full crossed-product duality* is said to hold, and the coaction

[1] We follow the convention that the total space of a Banach bundle is represented in a script font, while the fibres are written in Roman font. Thus if $p : \mathscr{A} \to X$ is a bundle over a space X, then we'll write A_x for the fibre $p^{-1}(x)$ viewed as a Banach space.

[2] We shall have more to say about semidirect-product bundles in Sections 6 and 7.

δ is said to be *maximal*. For example, the dual coaction on a full crossed product by an action is always maximal [3, Proposition 3.4].

Since Fell bundle C^*-algebras are generalizations of crossed products by actions, it is natural to ask whether there exists a coaction δ of G on $C^*(G, \mathscr{A})$, and if so, whether δ is maximal. In the present paper, we settle these questions affirmatively; thus our Theorem 8.1 can be regarded as a generalization of [3, Proposition 3.4] to arbitrary Fell bundles over G.

The existence of a coaction on $C^*(G, \mathscr{A})$ was briefly presented in [13] for the case of reduced coactions. In [19] (see also [18]), the third author showed that when the group G is discrete there is in fact a bijective correspondence between Fell bundles over G and coactions of G on C^*-algebras. Further, in [5] the third author and Echterhoff observed that given a Fell bundle $\mathscr{A}$ over a discrete group G, there is a natural coaction δ of G on $C^*(G, \mathscr{A})$ and the crossed product $C^*(G, \mathscr{A}) \rtimes_\delta G$ is naturally isomorphic to the C^*-algebra of a Fell bundle $\mathscr{A} \times_{\mathrm{lt}} G$ over the discrete groupoid $G \times_{\mathrm{lt}} G$ obtained by letting G act on itself by left translation. This observation, coupled with the work of the second and fourth authors on the theory of Fell bundles over groupoids [16] (which, in turn, was inspired, in part, by [19]), was the point of departure for the current project.

Indeed, although groupoids do not appear explicitly in the statement of our main theorem, Fell bundles over groupoids are crucial in the techniques we develop for the proof: roughly speaking, crossed products by actions of groups are modeled by Fell bundles over semidirect product groupoids, and crossed products by coactions of groups are modeled by Fell bundles over transformation groupoids. We rely heavily on [16] for the theory and basic results concerning Fell bundles over groupoids. In particular, we make free use of the Disintegration Theorem for Fell bundles [16, Theorem 4.13] which is a generalization of Renault's Disintegration Theorem for groupoids [23, Proposition 4.2]. (See [17, §7] for more discussion and references on Renault's Theorem.)

The plan for our proof of Theorem 8.1 is as follows: The initial two sections are preparatory. Section 1 establishes notation and collects some results that will be used in the sequel. Section 2 addresses some fine points regarding the problem of "promoting" a Fell bundle over a group to a Fell bundle over the product of the group with itself. The first real step in our analysis is taken in Section 3. There we prove in Proposition 3.1 that if $\mathscr{A}$ is a Fell bundle over a locally compact group G, then there is a natural coaction δ of G on $C^*(G, \mathscr{A})$ analogous to the dual coaction on a crossed product.

We note in passing that in [7], Exel and Ng prove a result that is similar to our Proposition 3.1. However, their setting is somewhat different from ours in that it uses an older and no-longer-used definition of "full coaction" that was advanced by Raeburn in [20]. Also, their proof is different in certain important respects. So, to keep this note self-contained we present full details.

The second substantial step taken in our analysis is Theorem 5.1, which asserts that there is a natural isomorphism θ from the crossed product $C^*(G,\mathscr{A}) \rtimes_\delta G$ to the C^*-algebra $C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G)$ of the Fell bundle $\mathscr{A} \times_{\mathrm{lt}} G$ over the transformation groupoid $G \times_{\mathrm{lt}} G$. As we mentioned above, this isomorphism theorem was inspired by [5]. Section 4 provides the necessary prerequisites for the formulation and proof of Theorem 5.1.

The third major step is Theorem 7.1, which establishes, in the general context of a Fell bundle $\mathscr{B}$ over a groupoid $\mathcal{G}$, an isomorphism between the C^*-algebra of a semidirect-product bundle $\mathscr{B} \times_\alpha G$ (the theory of which is developed in Section 6) and the crossed product of $C^*(\mathcal{G},\mathscr{B})$ by a corresponding action of G.

The remainder of the argument occupies Section 8. There, we show that the isomorphism θ established in Theorem 5.1 is equivariant for the dual action $\hat{\delta}$ of G on $C^*(G,\mathscr{A}) \times_\delta G$ and a natural action of G on

$$C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G).$$

Using this, θ is promoted to an isomorphism between the two crossed products. We then apply the result of Section 7 to this natural action to see that the crossed product can be realized as the C^*-algebra of a certain semidirect-product bundle; this bundle turns out to be isomorphic to one whose C^*-algebra is easily recognized as $C^*(G,\mathscr{A}) \otimes \mathcal{K}(L^2(G))$. Finally, we show that these isomorphisms combine to give the canonical surjection Φ, and this completes our proof of Theorem 8.1.

1. Preliminaries

If A is a C^*-algebra, then its maximal unitization $M(A)$ ([21, Definition 2.46]) is called the *multiplier algebra* of A. Traditionally, $M(A)$ is realized as the collection of double centralizers. Here we adopt the approach taken in [21], regarding $M(A)$ as the algebra $\mathcal{L}(A)$ of bounded adjointable operators on A viewed as a right-Hilbert module over itself. (That any two maximal unitizations are naturally isomorphic is guaranteed by [21, Theorem 2.47].) As usual, we let $\tilde{A}$ be the C^*-subalgebra of $M(A)$ generated by A and $1_{M(A)}$. (Thus $\tilde{A} = A$ if A is unital, and $\tilde{A}$ is A with an identity adjoined otherwise.) We use minimal tensor products of C^*-algebras throughout.

Let G be a locally compact group. We use $u : G \to M(C^*(G))$ to denote the canonical embedding, although sometimes we will simply identify $s \in G$ with its image $u(s) \in M(C^*(G))$. Similarly, we will usually not distinguish between a strictly continuous unitary homomorphism of G and its unique nondegenerate extension to $C^*(G)$. As a general reference for group actions we use [24], and for coactions we refer to [4, Appendix A].

1.1. Group Actions. An *action* of G on a C^*-algebra A is a homomorphism $\alpha : G \to \mathrm{Aut}\, A$ such that the map $s \mapsto \alpha_s(a)$ is norm continuous from

G to A for each $a \in A$. A *covariant representation* of (A, G, α) on a Hilbert space $\mathcal{H}$ is a pair (π, U), where $\pi : A \to B(\mathcal{H})$ is a nondegenerate representation and $U : G \to B(\mathcal{H})$ is a strongly continuous unitary representation, which satisfies the *covariance condition*

$$\pi(\alpha_s(a)) = U_s \pi(a) U_s^* \quad \text{for } a \in A \text{ and } s \in G. \tag{1.1}$$

More generally, for a C^*-algebra B, a *covariant homomorphism* of (A, G, α) into $M(B)$ is a pair (π, U), where $\pi : A \to M(B)$ is a nondegenerate homomorphism and $U : G \to M(B)$ is a strictly continuous unitary homomorphism, which satisfies (1.1).

A *crossed product* for (A, G, α) is a C^*-algebra $A \rtimes_\alpha G$, together with a covariant homomorphism (i_A, i_G) of (A, G, α) into $M(A \rtimes_\alpha G)$ which is universal in the sense that for any covariant homomorphism (π, U) of (A, G, α) into $M(B)$ there is a unique nondegenerate homomorphism

$$\pi \rtimes U : A \rtimes_\alpha G \to M(B),$$

called the *integrated form* of (π, U), such that

$$\pi = (\pi \rtimes U) \circ i_A \quad \text{and} \quad U = (\pi \rtimes U) \circ i_G.$$

The crossed product is generated by the universal covariant homomorphism in the sense that

$$A \rtimes_\alpha G = \overline{\operatorname{span}}\{\, i_A(a) i_G(f) : a \in A \text{ and } f \in C_c(G) \,\}.$$

The space $C_c(G, A)$ of compactly supported continuous functions from G into A is a $*$-algebra with (convolution) multiplication and involution given by

$$(f * g)(s) = \int_G f(t) \alpha_t(g(t^{-1}s))\, dt \quad \text{and} \quad f^*(s) = f(s^{-1})^* \Delta(s)^{-1},$$

where Δ denotes the modular function of G. The algebra $C_c(G, A)$ embeds as a dense $*$-subalgebra of $A \rtimes_\alpha G$ via the map

$$f \mapsto \int_G i_A(f(s)) i_G(s)\, ds,$$

so that if (π, U) is a covariant homomorphism of (A, G, α), then

$$\pi \rtimes U(f) = \int_G \pi(f(s)) U(s)\, ds.$$

1.2. Coactions. A *coaction* of G on a C^*-algebra A is a nondegenerate injective homomorphism $\delta : A \to M(A \otimes C^*(G))$ which satisfies the *coaction identity*

$$(\delta \otimes \mathrm{id}_G) \circ \delta = (\mathrm{id} \otimes \delta_G) \circ \delta, \tag{1.2}$$

and which is *nondegenerate as a coaction* in the sense that

$$\overline{\operatorname{span}}\{\delta(A)(1 \otimes C^*(G))\} = A \otimes C^*(G). \tag{1.3}$$

Here $\delta_G : C^*(G) \to M(C^*(G) \otimes C^*(G))$ is the homomorphism determined by the unitary homomorphism of G given by $s \mapsto u(s) \otimes u(s)$. Note that condition (1.3) implies nondegeneracy of δ as a map into $M(A \otimes C^*(G))$.

A *covariant representation* of (A, G, δ) on a Hilbert space $\mathcal{H}$ is a pair (π, μ), where $\pi : A \to B(\mathcal{H})$ and $\mu : C_0(G) \to B(\mathcal{H})$ are nondegenerate representations which satisfy the *covariance condition*

$$\operatorname{Ad}(\mu \otimes \mathrm{id})(w_G)(\pi(a) \otimes 1) = (\pi \otimes \mathrm{id})(\delta(a)) \quad \text{for } a \in A. \tag{1.4}$$

Here w_G is the element of $M(C_0(G) \otimes C^*(G))$ which corresponds to the canonical embedding $u : G \to M(C^*(G))$ under the natural isomorphism of $M(C_0(G) \otimes C^*(G))$ with the strictly continuous bounded maps from G to $M(C^*(G))$. More generally, for any C^*-algebra B, a *covariant homomorphism* of (A, G, δ) into $M(B)$ is a pair (π, μ), where $\pi : A \to M(B)$ and $\mu : C_0(G) \to M(B)$ are nondegenerate homomorphisms satisfying (1.4).

A *crossed product* for (A, G, δ) is a C^*-algebra $A \rtimes_\delta G$, together with a covariant homomorphism (j_A, j_G) of (A, G, δ) into $M(A \rtimes_\delta G)$ which is universal in the sense that for any covariant homomorphism (π, μ) of (A, G, δ) into $M(B)$ there is a unique nondegenerate homomorphism

$$\pi \rtimes \mu : A \rtimes_\delta G \to M(B),$$

called the *integrated form* of (π, μ), such that

$$\pi = (\pi \rtimes \mu) \circ j_A \quad \text{and} \quad \mu = (\pi \rtimes \mu) \circ j_G.$$

The crossed product is generated by the universal covariant homomorphism in the sense that

$$A \rtimes_\delta G = \overline{\operatorname{span}}\{\, j_A(a) j_G(f) : a \in A \text{ and } f \in C_0(G)\,\}.$$

The *dual action* of G on $A \rtimes_\delta G$ is the homomorphism $\hat{\delta} : G \to \operatorname{Aut}(A \rtimes_\delta G)$ given on generators by

$$\hat{\delta}_s(j_A(a) j_G(f)) = j_A(a) j_G(\mathrm{rt}_s(f)),$$

where rt denotes the action of G on $C_0(G)$ by right translation: $\mathrm{rt}_s(f)(t) = f(ts)$.

Given a representation π of A on a Hilbert space $\mathcal{H}$, the associated *regular representation* Λ of $A \rtimes_\delta G$ on $\mathcal{H} \otimes L^2(G)$ is the integrated form

$$\Lambda = \big((\pi \otimes \lambda) \circ \delta\big) \rtimes (1 \otimes M),$$

where λ is the left regular representation of G on $L^2(G)$ and M is the representation of $C_0(G)$ on $L^2(G)$ by multiplication: $(M_f \xi)(s) = f(s)\xi(s)$. When π is faithful, the associated regular representation is always faithful [4, Remark A.43(3)], and thus gives an isomorphism between $A \rtimes_\delta G$ and the concrete C^*-algebra

$$\Lambda(A \rtimes_\delta G) = \overline{\operatorname{span}}\{\, (\pi \otimes \lambda) \circ \delta(a)(1 \otimes M_f) : a \in A \text{ and } f \in C_0(G)\,\}.$$

The *canonical surjection* associated to δ is the map

$$\Phi = \big((\mathrm{id} \otimes \lambda) \circ \delta \rtimes (1 \otimes M)\big) \rtimes (1 \otimes \rho) : A \rtimes_\delta G \rtimes_{\hat{\delta}} G \to A \otimes \mathcal{K}(L^2(G)),$$

where ρ is the right regular representation of G on $L^2(G)$. (It almost goes without saying that, by convention, $(\lambda_s\xi)(t) = \xi(s^{-1}t)$ and $(\rho_s\xi)(t) = \xi(ts)\Delta(s)^{1/2}$.) On the generators, Φ is given by

$$\Phi\big(i_{A\rtimes_\delta G}(j_A(a)j_G(f))i_G(g)\big) = (\mathrm{id}\otimes\lambda)\circ\delta(a)\big(1\otimes M_f\rho(g)\big)$$

for $a \in A$, $f \in C_0(G)$, and $g \in C^*(G)$. The coaction δ is *maximal* if the canonical surjection Φ associated to δ is injective; thus the maximal coactions are precisely those coactions for which full crossed-product duality holds in the sense that Φ is an isomorphism of $A\rtimes_\delta G\rtimes_{\hat\delta} G$ onto $A\otimes\mathcal{K}(L^2(G))$.

Some of our coaction calculations will involve the *Fourier–Stieltjes* algebra $B(G)$. (see [4, §§A.4–A.5] for brief survey or [8] for a more detailed treatment). In simple terms the Fourier–Stieltjes algebra $B(G)$ is a space of bounded continuous functions on G which can be identified with the dual space $C^*(G)^*$ via the formula

$$f(g) = \int_G f(s)g(s)\,ds \quad \text{for } f \in B(G) \text{ and } g \in C_c(G) \subseteq C^*(G).$$

By [8, Propositions 3.4 and 3.7], the intersection $B(G)\cap C_0(G)$ is norm dense in $C_0(G)$. For $f \in B(G)$, the *slice map* $\mathrm{id}_A\otimes f : A\otimes C^*(G) \to A$ determined by

$$(\mathrm{id}_A \otimes f)(a\otimes b) = af(b) \quad \text{for } a \in A \text{ and } b \in C^*(G)$$

extends uniquely to a strictly continuous linear map

$$\mathrm{id}_A\otimes f : M(A\otimes C^*(G)) \to M(A),$$

and moreover such slice maps separate the points of $M(A\otimes C^*(G))$ ([4, Lemma A.30]).

1.3. Fell Bundles. A Fell bundle over a groupoid is a natural generalization of Fell's C^*-algebraic bundles over groups treated in detail in [12, Chap. VIII] and discussed briefly in the introduction. We will refer to [16] for the particulars of Fell bundles over groupoids. Generally speaking, a *Fell bundle* $p : \mathscr{B} \to \mathcal{G}$ is a upper semicontinuous Banach bundle over a locally compact Hausdorff groupoid $\mathcal{G}$ satisfying the axioms laid out in [16, Definition 1.1].[3] It was observed in [1, Lemma 3.30] that the underlying Banach bundle of an upper semicontinuous Fell bundle over a *group* is necessarily continuous. (The authors of [1] attribute this observation to Exel.) Since all the Fell bundles in this work originate from Fell bundles over groups, they will necessarily be built on continuous Banach bundles.[4] We will assume all the Fell bundles here are *separable* in that $\mathcal{G}$ is second countable and the

[3] There are a number of equivalent definitions of Fell bundles over groupoids in the literature starting with Yamagami's original in [25, Definition 1.1], as well as [14, Definition 6] and [2, Definition 2.1].

[4] An exception is that in sections 6 and 7 we work with general Fell bundles over groupoids, and there it is not necessary to assume that the underlying Banach bundles are continuous.

Banach space $\Gamma_0(\mathcal{G};\mathscr{B})$ of sections is separable. (This hypothesis is not only a sign of good taste, but it will also ensure that the results of [16] apply.)

We are only interested in groupoids $\mathcal{G}$ with a continuous Haar system $\{\lambda^u\}_{u\in\mathcal{G}^{(0)}}$. Then the set $\Gamma_c(\mathcal{G};\mathscr{B})$ of continuous compactly supported sections of $\mathscr{B}$ has the structure of a $*$-algebra:

$$f*g(x):=\int_{\mathcal{G}} f(y)g(y^{-1}x)\,d\lambda^{r(x)}(y) \quad \text{and} \quad f^*(x):=f(x^{-1})^*.$$

Then we can define a norm, $\|\cdot\|_I$, on $\Gamma_c(\mathcal{G};\mathscr{B})$ via

$$\|f\|_I=\max\Big\{\sup_{u\in\mathcal{G}^{(0)}}\int_{\mathcal{G}}\|f(x)\|\,d\lambda^u(x),\ \sup_{u\in\mathcal{G}^{(0)}}\int_{G}\|f(x)\|\,\lambda_u(x)\Big\}.$$

If $\mathcal{H}$ is a Hilbert space, then a $*$-homomorphism $L:\Gamma_c(\mathcal{G};\mathscr{B})\to B(\mathcal{H})$ is called $\|\cdot\|_I$-*decreasing* if $\|L(f)\|\le\|f\|_I$ for all f. We say that L is a $\|\cdot\|_I$-decreasing *representation* if it is also *nondegenerate* in the sense that

$$\overline{\operatorname{span}}\{\,L(f)\xi : f\in\Gamma_c(\mathcal{G};\mathscr{B}) \text{ and } \xi\in\mathcal{H}\,\}=\mathcal{H}.$$

Then, by definition, the *universal norm* on $\Gamma_c(\mathcal{G};\mathscr{B})$ is

$$\|f\|:=\sup\{\,\|L(f)\| : L \text{ is a } \|\cdot\|_I\text{-decreasing representation of } \Gamma_c(\mathcal{G};\mathscr{B})\,\}.$$

The completion $\overline{\big(\Gamma_c(\mathcal{G};\mathscr{B}),\|\cdot\|\big)}$ is the C^*-algebra $C^*(\mathcal{G},\mathscr{B})$ of the Fell bundle $p:\mathscr{B}\to\mathcal{G}$.[5]

More generally, a nondegenerate $*$-homomorphism $L:\Gamma_c(\mathcal{G},\mathscr{B})\to B(\mathcal{H})$ is called simply a *representation* if L is continuous when $\Gamma_c(\mathcal{G};\mathscr{B})$ is equipped with the inductive limit topology and $B(\mathcal{H})$ is given the weak operator topology. It is a nontrivial result — a consequence of the Disintegration Theorem ([16, Theorem 4.13]) — that every representation of $\Gamma_c(\mathcal{G};\mathscr{B})$ is $\|\cdot\|_I$-decreasing. Since $\|\cdot\|_I$-decreasing representations are clearly representations, we see that

$$\|f\|=\sup\{\,\|L(f)\| : L \text{ is a representation of } \Gamma_c(\mathcal{G};\mathscr{B})\,\}$$

(see [16, Remark 4.14]).

Lemma 1.1. *Suppose that $p:\mathscr{B}\to\mathcal{G}$ is a Fell bundle over a locally compact groupoid $\mathcal{G}$. If $\mathcal{H}$ is a locally compact groupoid and $\varphi:\mathcal{H}\to\mathcal{G}$ is a continuous groupoid homomorphism, then the pull-back $q:\varphi^*\mathscr{B}\to\mathcal{H}$ is a Fell bundle over $\mathcal{H}$ with multiplication and involution given by*

$$(a,h)(b,t)=(ab,ht) \quad \textit{and} \quad (a,h)^*=(a^*,h^{-1}).$$

Proof. The proof is routine. For example, $q:\varphi^*\mathscr{B}\to\mathcal{H}$ is clearly a Banach bundle (see [11, §II.13.7] where pull-backs are called retractions). The fibre over h is isomorphic to $B_{\varphi(h)}$. The Fell bundle structure from $\mathscr{B}$ makes the latter into a $B_{r(\varphi(h))}-B_{s(\varphi(h))}$-imprimitivity bimodule. Since the fibre over

[5]It might be helpful to look over the examples in [16, §2] at this point.

$s(h)$ is isomorphic to $B_{\varphi(s(h))}$ and $\varphi(s(h)) = s(\varphi(h))$, the rest is easy. (Note that when $\mathcal{G}$ and $\mathcal{H}$ are groups, this result is [12, §VIII.3.17].) □

1.4. Fell Bundles over Groups. However, to begin with, we are interested in a (separable, of course) Fell bundle $p : \mathscr{A} \to G$ where G is a locally compact *group*. This case affords a number of simplifications, and also allows us to avoid some of the overhead coming from [16]. Note that a Fell bundle $p : \mathscr{A} \to G$ over a group is what Fell and Doran call a C^*-algebraic bundle over G (see [12, Definitions VIII.16.2 and VIII.3.1]). Since we ultimately treat Fell bundles over groups as a special case of a Fell bundle over a groupoid, our axioms require that $p : \mathscr{A} \to G$ is *saturated* in the sense that $\overline{\operatorname{span}}\{A_s A_t\} = A_{st}$ for all $s, t \in G$ (see [12, §VIII.2.8]). We will often write a_s for an element of A_s; that is, $a_s \in \mathscr{A}$ and $p(a_s) = s$.

We do make one deviation from the groupoid treatment when building the associated C^*-algebra, $C^*(G, \mathscr{A})$. In order that we can easily obtain the usual group C^*-algebra construction as well as the usual crossed-product construction as special cases, it is convenient to add the modular function, Δ, on G to the definition of the involution on $\Gamma_c(G; \mathscr{A})$:

$$f^*(s) = \Delta(s)^{-1} f(s^{-1})^*$$

(see [12, §VIII.5.6]). Then the somewhat unsatisfactory $\|\cdot\|_I$ reduces to the normal analog of the L^1-norm:

$$\|f\|_1 := \int_G \|f(s)\|\, ds,$$

and the universal norm on $\Gamma_c(G; \mathscr{A})$ is given as the supremum over $\|\cdot\|_1$-decreasing representations. As we shall see shortly (see Remark 1.5), the isomorphism class of $C^*(G, \mathscr{A})$ is the same as that obtained using the definition of the involution given for groupoids where no modular function is available.

Assuming $p : \mathscr{A} \to G$ is a Fell bundle over a group, a $*$-homomorphism $\pi : \mathscr{A} \to M(B)$ is just a map with the obvious algebraic properties. We call π *nondegenerate* if

$$\overline{\operatorname{span}}\{\pi(A_e)B\} = B.$$

The next lemma shows that $\mathscr{A}$ comes with a canonical nondegenerate strictly continuous embedding $\iota : \mathscr{A} \to M(C^*(G, \mathscr{A}))$. Then Lemma 1.3 shows that the pair $(C^*(G, \mathscr{A}), \iota)$ is in fact universal for strictly continuous nondegenerate $*$-homomorphisms of $\mathscr{A}$ into multiplier algebras.

Lemma 1.2. *Let $p : \mathscr{A} \to G$ be a separable Fell bundle over a locally compact group G. There exists a strictly continuous nondegenerate $*$-homomorphism $\iota : \mathscr{A} \to M(C^*(G, \mathscr{A}))$ such that for $a_s \in A_s$ and $f \in \Gamma_c(G; \mathscr{A})$, we have $\iota(a_s)f \in \Gamma_c(G; \mathscr{A})$, with*

$$(\iota(a_s)f)(t) = a_s f(s^{-1}t). \tag{1.5}$$

Proof. For each $a_s \in A_s$, (1.5) clearly defines a linear map $\iota(a_s)$ of $\Gamma_c(G;\mathscr{A})$ into itself. Here we will view $\Gamma_c(G;\mathscr{A})$ as a dense subspace of $C^*(G,\mathscr{A})$ viewed as a Hilbert module over itself. Then the inner product $\langle f\,,\,g\rangle = f^* {*} g$ is $\Gamma_c(G;\mathscr{A})$-valued on $\Gamma_c(G;\mathscr{A})$. It is easy to check that $\iota(a_s)\iota(a_t) = \iota(a_s a_t)$, and a straightforward computation shows that

$$\big\langle \iota(a_s)f\,,\,g\big\rangle = \big\langle f\,,\,\iota(a_s^*)g\big\rangle \tag{1.6}$$

(a similar, but more involved computation is given in detail in the proof of Theorem 5.1). Since $\|a_s\|^2 1_{A_e} - a_s^* a_s \geq 0$ in $\tilde{A}_e$, there is a $b_e \in \tilde{A}_e$ such that $\|a_s\|^2 1_{A_e} - a_s^* a_s = b_e^* b_e$. Then, since (1.5) makes sense and ι is multiplicative for elements of $\tilde{A}_e$, and since (1.6) also holds for $b_e \in \tilde{A}_e$, we see that

$$\begin{aligned}\|a_s\|^2\big\langle f\,,\,f\big\rangle - \big\langle \iota(a_s)f\,,\,\iota(a_s)f\big\rangle &= \big\langle \iota(\|a_s\|^2 1_{A_e} - a_s^* a_s)f\,,\,f\big\rangle \\ &= \big\langle \iota(b_e)f\,,\,\iota(b_e)f\big\rangle \geq 0\end{aligned}$$

for all $f \in \Gamma_c(G;\mathscr{A})$. It follows that $\iota(a_s)$ is bounded and extends to a bounded operator on $C^*(G,\mathscr{A})$ with adjoint $\iota(a_s^*)$. It is routine to verify that the resulting map $\iota : \mathscr{A} \to M(C^*(G,\mathscr{A}))$ is a $*$-homomorphism.

To see that ι is nondegenerate, first note that A_s is an A_e – A_e-imprimitivity bimodule. Thus if $\{\, a_i \,\}_{i\in I}$ is an approximate identity in A_e, then $a_i a_s \to a_s$ for any $a_s \in A_s$. Then a messy compactness argument similar to that given in the proof of Theorem 5.1 shows that $\iota(a_i)f \to f$ in the inductive limit topology on $\Gamma_c(G;\mathscr{A})$ for any $f \in \Gamma_c(G;\mathscr{A})$. Since convergence in the inductive limit topology implies convergence in the C^*-norm, this establishes nondegeneracy.

It only remains to prove strict continuity. Our separability assumptions on $p : \mathscr{A} \to G$ allow us to invoke [11, Proposition II.13.21] to see that $\mathscr{A}$ is second countable. Thus, it suffices to show that if $\{\, a_{s_n} \,\}$ is a sequence in $\mathscr{A}$ converging to a_s, then $\iota(a_{s_n}) \to \iota(a_s)$ strictly.

The convergent sequence $\{\, a_{s_n} \,\}$ must lie in a norm-bounded subset of $\mathscr{A}$, so the image $(\iota(a_{s_n}))$ is a bounded sequence in $M(C^*(G,\mathscr{A}))$ (because $\|\iota(a_s)\| \leq \|a_s\|$). Thus, it suffices to show that $\iota(a_{s_n}) \to \iota(a_s)$ $*$-strongly; and since $a_{s_n}^* \to a_s^*$ and ι is $*$-preserving, it suffices to show strong convergence. Finally, since $\{\, \iota(a_{s_n}) \,\}$ is bounded, it suffices to show that $\iota(a_{s_n})f \to \iota(a_s)f$ in the inductive limit topology, for each $f \in \Gamma_c(G;\mathscr{A})$.

Suppose not; so there is $f \in \Gamma_c(G;\mathscr{A})$ such that $\iota(a_{s_n})f$ does not converge to $\iota(a_s)f$ in the inductive limit topology. Note that since $s_n \to s$ in G, we can find a compact set $K \subseteq G$ such that the supports of $\iota(a_s)f$ and all the $\iota(a_{s_n})f$ are contained in K, so it must be that the convergence is not uniform on K. So, passing to a subsequence and relabeling, we can find $\varepsilon > 0$ and $t_n \to t$ in K such that for all n,

$$\|\iota(a_{s_n})f(t_n) - \iota(a_s)f(t_n)\| \geq \varepsilon.$$

But by joint continuity of multiplication in $\mathscr{A}$, we have

$$\iota(a_{s_n})f(t) = a_{s_n} f(s_n^{-1}t) \to a_s f(s^{-1}t) = \iota(a_s)f(t)$$

in $\mathscr{A}$. Since this implies that the norm of the difference goes to zero, we have a contradiction. $\square$

Lemma 1.3. *Let $p : \mathscr{A} \to G$ be as in Lemma 1.2. If B is a C^*-algebra and $\pi_0 : \mathscr{A} \to M(B)$ is a strictly continuous nondegenerate $*$-homomorphism, then there is a unique nondegenerate homomorphism $\pi : C^*(G, \mathscr{A}) \to M(B)$, called the* integrated form *of π_0, such that $\pi \circ \iota = \pi_0$. Moreover,*

$$\pi(f) = \int_G \pi_0(f(s))\, ds \quad \text{for } f \in \Gamma_c(G; \mathscr{A}). \tag{1.7}$$

Conversely, every nondegenerate $*$-homomorphism of $C^*(G, \mathscr{A})$ is the integrated form of some such π_0.

Remark 1.4. Note that the integral in (1.7) makes sense since $\pi_0 \circ f$ is strictly continuous so that we can apply, for example, [21, Lemma C.11].

Proof. It is straightforward to check that (1.7) defines a $*$-homomorphism $\pi : \Gamma_c(G; \mathscr{A}) \to M(B)$.

To see that π is nondegenerate, we need to see that

$$\operatorname{span}\{\, \pi(f)b : f \in \Gamma_c(G; \mathscr{A}) \text{ and } b \in B \,\}$$

is dense in B. To this end, fix $a \in A_e$ and choose $f \in \Gamma_c(G; \mathscr{A})$ such that $f(e) = a$. Let $\{\, \varphi_k \,\}$ be a sequence in $C_c^+(G)$ with integral one whose supports shrink to the identity. Let $f_k(s) = \varphi_k(s)f(s)$. Then it is not hard to see that $\pi(f_k)b \to \pi_0(a)b$. Therefore, the nondegeneracy of π follows from that of π_0.

If $L : B \to B(\mathcal{H})$ is a faithful representation, then $L\circ\pi$ is a $\|\cdot\|_1$-decreasing representation of $\Gamma_c(G; \mathscr{A})$. By the definition of the universal norm,

$$\|L \circ \pi(f)\| \le \|f\|.$$

Since the extension of L to $M(B)$ is isometric, $\|\pi(f)\| \le \|f\|$. Therefore, π extends to $C^*(G, \mathscr{A})$.

To prove uniqueness, we need to establish that

$$\int_G \iota\big(f(s)\big)\, ds = f, \tag{1.8}$$

where the equality in (1.8) is meant in $M\big(C^*(G, \mathscr{A})\big)$. Therefore, it suffices to see that

$$\Big(\int \iota\big(f(s)\big)\, ds\Big) g = \int_G \iota\big(f(s)\big) g\, ds = f * g \quad \text{for all } g \in \Gamma_c(G; \mathscr{A}). \tag{1.9}$$

Thus we need to establish that the $C^*(G, \mathscr{A})$-valued integral in the middle of (1.9) takes values in (the image of) $\Gamma_c(G; \mathscr{A})$ in $C^*(G, \mathscr{A})$ and coincides with $f*g$. This can be verified almost exactly as in the proof of [24, Lemma 1.108].

Now, if $\rho : C^*(G, \mathscr{A}) \to M(B)$ is a homomorphism such that $\rho \circ \iota = \pi_0$, then by (1.8), for each $f \in \Gamma_c(G; \mathscr{A})$ we must have

$$\rho(f) = \rho\Big(\int_G \iota(f(s))\, ds\Big) = \int_G \rho(\iota(f(s)))\, ds = \int_G \pi_0(f(s))\, ds = \pi(f).$$

For the converse, let $\pi : C^*(G, \mathscr{A}) \to M(B)$ be a nondegenerate $*$-homomorphism. By nondegeneracy, π extends to a strictly continuous homomorphism of $M(C^*(G, \mathscr{A}))$, so that $\pi \circ \iota$ is a strictly continous nondegenerate $*$-homomorphism of $\mathscr{A}$ whose integrated form, by uniqueness, is π. $\square$

Remark 1.5 (Modular Differences)**.** If $p : \mathscr{A} \to G$ is a Fell bundle over a locally compact *group*, then we could just as well have formed the C^*-algebra $C^*_{\mathrm{Gr}}(G, \mathscr{A})$ by treating G as a groupoid. (That is, by leaving the modular function off the involution.) To see that $C^*_{\mathrm{Gr}}(G, \mathscr{A})$ and $C^*(G, \mathscr{A})$ are naturally isomorphic, we first observe that Lemma 1.2 and Lemma 1.3 remain valid for $C^*_{\mathrm{Gr}}(G, \mathscr{A})$ using virtually the same proofs; the only difference is that Equations (1.5) and (1.7) must be modified to deal with the lack of modular function in the involution:

$$\big(\iota'(a_s)f\big)(t) = \Delta(s)^{\frac{1}{2}} a_s f(s^{-1}t) \quad \text{and} \tag{1.5$'$}$$

$$\pi(f) = \int_G \pi_0'\big(f(s)\big)\Delta(s)^{-\frac{1}{2}}\, ds. \tag{1.7$'$}$$

Then notice that there is a $*$-isomorphism $\varphi : \Gamma_c^{\mathrm{Gr}}(G, \mathscr{A}) \to \Gamma_c(G; \mathscr{A})$ given by $\varphi(f)(s) = \Delta(s)^{-\frac{1}{2}} f(s)$. We just need to see that φ is isometric with respect to the universal norm $\|\cdot\|_{\mathrm{Gr}}$ on $C^*_{\mathrm{Gr}}(G, \mathscr{A})$ and $\|\cdot\|$ on $C^*(G, \mathscr{A})$. To verify this, let M be a faithful representation of $C^*(G, \mathscr{A})$. Then M is the integrated form of $M_0 : \mathscr{A} \to B(\mathcal{H})$. But if L is the representation of $C^*_{\mathrm{Gr}}(G, \mathscr{A})$ which is the integrated form of M_0, then

$$\|\varphi(f)\| = \|M(\varphi(f))\| = \|L(f)\| \le \|f\|_{\mathrm{Gr}}.$$

On the other hand, if L is a faithful representation of $C^*_{\mathrm{Gr}}(G, \mathscr{A})$ which is the integrated form of L_0, then we can let M be the representation of $C^*(G, \mathscr{A})$ that is integrated up from L_0. Then

$$\|\varphi(f)\| \ge \|M(\varphi(f))\| = \|L(f)\| = \|f\|_{\mathrm{Gr}}.$$

Thus φ is isometric.

Remark 1.6. The same comments about modular functions apply to the standard group C^*-algebra and crossed product constructions; that is, one can omit the modular function in the definition of the involution and arrive at isomorphic algebras. However, you have pay for the luxury of modular-free involutions by adding the modular function to the integrated form of any representation as in (1.7′).

Proposition 1.7. *Let $p : \mathscr{B} \to \mathcal{G}$ be a separable Fell bundle over a locally compact groupoid $\mathcal{G}$, and let X_0 be a dense subspace of a right Hilbert A-module X. Suppose that L is a algebra homomorphism of $\Gamma_c(\mathcal{G}; \mathscr{B})$ into the linear operators,* $\mathrm{Lin}(\mathsf{X}_0)$*, on X_0 such that for all $x, y \in \mathsf{X}_0$:*

(i) $\big\langle L(f)x\, ,\, y\big\rangle_A = \big\langle x\, ,\, L(f^*)y\big\rangle_A$,
(ii) $f \mapsto \big\langle L(f)x\, ,\, y\big\rangle_A$ *is continuous in the inductive limit topology, and*
(iii) $\mathrm{span}\{\, L(f)x : f \in \Gamma_c(\mathcal{G}; \mathscr{B})$ *and* $x \in \mathsf{X}_0 \,\}$ *is dense in* X.

Then L is bounded with respect to the universal C^*-norm on $\Gamma_c(\mathcal{G};\mathscr{B})$ and extends to a nondegenerate homomorphism $L : C^*(\mathcal{G},\mathscr{B}) \to \mathcal{L}(\mathsf{X})$.

Proof. This proposition is a consequence of the disintegration result [16, Theorem 4.13] for Fell bundles. To see this, let ρ be a state on A. Then

$$(x \mid y)_\rho := \rho\bigl(\langle y\,,\,x\rangle_A\bigr)$$

is a pre-inner product on X_0. After modding out by the subspace $\mathcal{N}$ of vectors of length zero, we get a pre-Hilbert space $\mathcal{H}_0 := \mathsf{X}_0/\mathcal{N}$ which we view as a subspace of its completion $\mathcal{H}$. Since

$$(L(f)x \mid L(f)x)_\rho = (x \mid L(f^* * f)x)_\rho,$$

it follows from the Cauchy Schwartz inequality that $L(f)$ maps $\mathcal{N}$ to itself. Therefore $L(f)$ defines a linear operator $L^\rho(f)$ on $\mathcal{H}_0$ via $L^\rho(f)(x+\mathcal{N}) = L(f)x + \mathcal{N}$. It is clear that L^ρ defines a pre-representation of $\mathscr{B}$ on $\mathcal{H}_0$ as in [16, Definition 4.1]. Then [16, Theorem 4.13] implies that

$$(L(f)x \mid L(f)x)_\rho \le \|f\|^2 (x \mid x)_\rho.$$

Since this holds for all states ρ, we have $\|L(f)\| \le \|f\|$. The rest is straightforward. □

Proposition 1.8. *Let $\mathscr{A}$ be a separable Fell bundle over a groupoid $\mathcal{G}$. Every $*$-homomorphism from $\Gamma_c(\mathcal{G};\mathscr{A})$ into a C^*-algebra which is continuous from the inductive limit topology into the norm topology is bounded for the universal norm, and hence has a unique extension to $C^*(\mathcal{G},\mathscr{A})$.*

Proof. Suppose that $\pi : \Gamma_c(\mathcal{G};\mathscr{A}) \to B$ is such a homomorphism, and that $\rho : B \to B(\mathcal{H})$ is a faithful representation of B on a Hilbert space $\mathcal{H}$. Let

$$\mathcal{H}_1 = \overline{\operatorname{span}}\{\, \rho\circ\pi(f)\xi : f \in \Gamma_c(\mathcal{G};\mathscr{A}), \xi \in \mathcal{H}\,\}.$$

Then $f \mapsto \rho\circ\pi(f)|_{\mathcal{H}_1}$ is a representation of $\mathscr{A}$ on $\mathcal{H}_1$ in the sense of [16, Definition 4.7], since the operator norm topology is stronger than the weak operator topology. By [16, Remark 4.14],

$$\|\pi(f)\| = \|\rho\circ\pi(f)\| = \|\rho\circ\pi(f)|_{\mathcal{H}_1}\| \le \|f\| \quad \text{for all } f \in \Gamma_c(\mathcal{G};\mathscr{A}). \qquad \square$$

2. Product bundles

If $p : \mathscr{A} \to G$ is a Fell bundle over a locally compact group G, then the Cartesian product, $\mathscr{A}\times G$, carries a natural Fell bundle structure over $G\times G$. The bundle projection $q : \mathscr{A}\times G \to G\times G$ is given by $q(a,t) = (p(a),t)$ and the multiplication and involution are given by

$$(a_s,t)(b_r,u) = (a_s b_r, tu) \quad \text{and} \quad (a_s,t)^* = (a_s^*, t^{-1}).$$

(Indeed, the map $(a,t) \mapsto (a,(p(a),t))$ is a bijection of $\mathscr{A}\times G$ onto the pull-back Fell bundle $\varphi^*\mathscr{A}$ — see Lemma 1.1 — where $\varphi : G\times G \to G$ is the projection onto the first factor.)

Every section $h \in \Gamma_c(G \times G; \mathscr{A} \times G)$ is of the form

$$h(s,t) = (h_1(s,t), t),$$

where $h_1 \in C_c(G \times G, \mathscr{A})$ satisfies $h_1(s,t) \in A_s$ for $s, t \in G$. For $f \in \Gamma_c(G; \mathscr{A})$ and $g \in C_c(G)$ we let $f \boxtimes g$ denote the element of $\Gamma_c(G \times G; \mathscr{A} \times G)$ defined by

$$(f \boxtimes g)(s,t) = (f(s)g(t), t).$$

Lemma 2.1. *With the above notation,*

$$\operatorname{span}\{\, f \boxtimes g : f \in \Gamma_c(G; \mathscr{A}) \text{ and } g \in C_c(G)\,\}$$

is inductive-limit dense in $\Gamma_c(G \times G; \mathscr{A} \times G)$.

Proof. Put $\mathcal{S} = \{\, f \boxtimes g : f \in \Gamma_c(G; \mathscr{A}), g \in C_c(G)\,\}$. Then for each $(s,t) \in G \times G$, $\{\, h(s,t) : h \in \mathcal{S}\,\}$ is easily seen to be dense in $A_s \times \{t\}$, which is the fibre of the bundle $\mathscr{A} \times G$ over (s,t). Furthermore if $u, v \in C_c(G)$ and $u \otimes v$ is the function in $C_c(G \times G)$ given by $u \otimes v(s,t) = u(s)v(t)$, then $(u \otimes v)h \in \mathcal{S}$ for all $u, v \in C_c(G)$ and $h \in \mathcal{S}$. Then, because the $u \otimes v$'s span an inductive-limit dense subspace of $C_c(G \times G)$, a straightforward partition of unity argument implies that $\operatorname{span}\mathcal{S}$ is dense as required (see [11, Proposition II.14.6 and its remark] or [24, Proposition C.24]). □

For the study of the coaction associated to a Fell bundle over a group (specifically, in Section 5) we will need the following slight variation on Lemma 2.1:

Lemma 2.2. *Let $\mathscr{A} \to G$ be a Fell bundle. For $f \in \Gamma_c(G; \mathscr{A})$ and $g \in C_c(G)$ define $f \bullet g \in C_c(G \times G, \mathscr{A})$ by*

$$f \bullet g(s,t) = f(s)g(s^{-1}t),$$

and define

$$f \star g(s,t) = (f \bullet g(s,t), t).$$

Then $f \star g \in \Gamma_c(G \times G; \mathscr{A} \times G)$, and such sections have inductive-limit-dense span.

Proof. It is obvious that $f \star g \in \Gamma_c(G \times G; \mathscr{A} \times G)$. For the second statement, let

$$\mathcal{S} = \operatorname{span}\{\, f \star g : f \in \Gamma_c(G; \mathscr{A}) \text{ and } g \in C_c(G)\,\}.$$

To show that $\mathcal{S}$ is dense, we want to invoke a partition of unity argument exactly as in Lemma 2.1; thus it suffices to establish the following two assertions:

(i) For each $(s,t) \in G \times G$, the set $\{\, h(s,t) : h \in \mathcal{S}\,\}$ is dense in $A_s \times \{t\}$.
(ii) For each $\kappa, \eta \in C_c(G)$ and $h \in \mathcal{S}$ we have $(\kappa \bullet \eta)h \in \mathcal{S}$, where similarly to the above we define $\kappa \bullet \eta(s,t) = \kappa(s)\eta(s^{-1}t)$.

(Note that (ii) suffices since the set of functions of the form $\kappa \bullet \eta$ have dense span in $C_c(G \times G)$ for the inductive limit topology, because this set is the image of the set $\{ u \otimes v : u, v \in C_c(G) \}$ under the linear homeomorphism $\Psi : C_c(G \times G) \to C_c(G \times G)$ defined by

$$\Psi(\varphi)(s,t) = \varphi(s, s^{-1}t),$$

and the functions $u \otimes v$ have dense span in the inductive limit topology.)

For (i), if $a_s \in A_s$ we can choose $f \in \Gamma_c(G; \mathscr{A})$ and $g \in C_c(G)$ such that $f(s) = a_s$ and $g(s^{-1}t) = 1$, and then

$$f \star g(s,t) = (a_s, t).$$

For (ii), just observe that

$$(\kappa \bullet \eta)(f \star g) = (\kappa f) \star (\eta g). \qquad \square$$

3. Coactions from Fell bundles

As mentioned in the introduction, if α is an action of a locally compact group G on a C^*-algebra B, then $\mathscr{A} = B \times G$ has a natural Fell-bundle structure such that $C^*(G, \mathscr{A}) \cong B \rtimes_\alpha G$. Then the dual coaction on $B \rtimes_\alpha G$ gives us a coaction on $C^*(G, \mathscr{A})$. In this section, we show that if $p : \mathscr{A} \to G$ is any Fell bundle, then $C^*(G, \mathscr{A})$ admits a natural coaction δ generalizing the dual coaction construction just described.

Proposition 3.1. *Let $\mathscr{A}$ be a separable Fell bundle over a group G. There is a unique coaction δ of G on $C^*(G, \mathscr{A})$ such that*

$$\delta(\iota(a_s)) = \iota(a_s) \otimes s \quad \text{for } a_s \in A_s \text{ and } s \in G. \tag{3.1}$$

Proof. For the proof we will make explicit the canonical map $u : G \to M(C^*(G))$. Consider the map $\delta_0 : \mathscr{A} \to M(C^*(G, \mathscr{A}) \otimes C^*(G))$ defined by $\delta_0(a_s) = \iota(a_s) \otimes u(s)$. This clearly gives a $*$-homomorphism of $\mathscr{A}$, and nondegeneracy of δ_0 follows directly from nondegeneracy of ι. That δ_0 is strictly continuous follows from strict continuity of $\iota : \mathscr{A} \to M(C^*(G, \mathscr{A}))$ and $u : G \to M(C^*(G))$. To see this, let $a_{s_i} \to a_s$ in $\mathscr{A}$, and let $x \in C^*(G, \mathscr{A}) \otimes C^*(G)$. Since $C^*(G, \mathscr{A})$ embeds nondegenerately in $M(C^*(G, \mathscr{A}) \otimes C^*(G))$ via $b \mapsto b \otimes 1$, by the Hewitt–Cohen factorization theorem we can write $x = (b \otimes 1)y$ for some $b \in C^*(G, \mathscr{A})$ and $y \in C^*(G, \mathscr{A}) \otimes C^*(G)$. Since $\iota(a_{s_i})b \to \iota(a_s)b$ in norm, we have $\iota(a_{s_i})b \otimes 1 \to \iota(a_s)b \otimes 1$ in norm in $M(C^*(G, \mathscr{A}) \otimes C^*(G))$. Since the map $u : G \to M(C^*(G))$ is strictly continuous, and since $a_{s_i} \to a_s$ implies $s_i \to s$ in G, we have $(1 \otimes u(s_i))y \to (1 \otimes u(s))y$ in norm in $C^*(G, \mathscr{A}) \otimes C^*(G)$. Since multiplication is norm continuous,

$$\big(\iota(a_{s_i}) \otimes u(s_i)\big)x = \big(\iota(a_{s_i}) \otimes u(s_i)\big)(b \otimes 1)y = \big(\iota(a_{s_i})b \otimes 1\big)\big(1 \otimes u(s_i)\big)y$$

converges in norm to

$$\big(\iota(a_s)b \otimes 1\big)\big(1 \otimes u(s)\big)y = \big(\iota(a_s) \otimes u(s)\big)x.$$

Thus Lemma 1.3 gives a unique nondegenerate $*$-homomorphism $\delta : C^*(G,\mathscr{A}) \to M(C^*(G,\mathscr{A}) \otimes C^*(G))$ such that $\delta \circ \iota = \delta_0$, and by (1.7) we have

$$\delta(f) = \int_G \iota(f(s)) \otimes u(s)\, ds \quad \text{for } f \in \Gamma_c(G;\mathscr{A}).$$

To see that δ is injective, let $1_G : G \to \mathbb{C}$ be the constant function with value 1, and regard 1_G as an element of the Fourier–Stieltjes algebra $B(G) = C^*(G)^*$. Then for $f \in \Gamma_c(G;\mathscr{A})$ equation (1.8) and strict continuity of the slice map give

$$(\mathrm{id} \otimes 1_G)(\delta(f)) = \int (\mathrm{id} \otimes 1_G)(\iota(f(s)) \otimes u(s))\, ds = \int \iota(f(s))\, ds = f.$$

Thus $(\mathrm{id} \otimes 1_G) \circ \delta = \mathrm{id}_{C^*(G,\mathscr{A})}$ by continuity and density, so δ is injective.

Now if $a_s \in A_s$, then

$$\begin{aligned}(\delta \otimes \mathrm{id}) \circ \delta_0(a_s)) &= (\delta \otimes \mathrm{id})(\iota(a_s) \otimes u(s)) = \iota(a_s) \otimes u(s) \otimes u(s) \\ &= (\mathrm{id} \otimes \delta_G)(\iota(a_s) \otimes u(s)) = (\mathrm{id} \otimes \delta_G) \circ \delta_0(a_s).\end{aligned}$$

Thus the coaction identity (1.2) follows from uniqueness in Lemma 1.3 together with the usual manipulations with vector valued integrals as justified, for example, in [21, Lemma C.11].

Finally, for the nondegeneracy condition (1.3), we elaborate on the argument sketched in the paragraph preceding [7, Lemma 1.3]. Consider the map $\zeta_0 : \mathscr{A} \times G \to M(C^*(G,\mathscr{A}) \otimes C^*(G))$ defined by $\zeta_0(a_s, t) = \iota(a_s) \otimes u(t)$, where $\mathscr{A} \times G$ is the Fell bundle over $G \times G$ defined in Section 2. Arguing as for δ_0 shows that ζ_0 is a strictly continuous nondegenerate $*$-homomorphism, and so Lemma 1.3 gives a nondegenerate $*$-homomorphism $\zeta : C^*(G \times G, \mathscr{A} \times G) \to M(C^*(G,\mathscr{A}) \otimes C^*(G))$ such that $\zeta \circ \iota = \zeta_0$.

In particular, using (1.7) and (1.8) we have, for $f \in \Gamma_c(G;\mathscr{A})$ and $g \in C_c(G)$,

$$\begin{aligned}\zeta(f \boxtimes g) &= \int_{G\times G} \zeta_0\big((f \boxtimes g)(s,t)\big)\, d(s,t) \\ &= \int_G \int_G \zeta_0(f(s)g(t), t)\, ds\, dt \\ &= \int_G \iota(f(s))\, ds \otimes \int_G g(t)u(t)\, dt \\ &= f \otimes g,\end{aligned}$$

which implies that ζ maps $C^*(G \times G, \mathscr{A} \times G)$ onto (and into) $C^*(G,\mathscr{A}) \otimes C^*(G)$.

Similarly, if $f \star g$ is the element of $\Gamma_c(G \times G; \mathscr{A} \times G)$ defined in Lemma 2.2, then for $a \otimes b \in C^*(G,\mathscr{A}) \otimes C^*(G)$ we have

$$\zeta(f \star g)(a \otimes b) = \int_{G\times G} \zeta_0(f \star g)(s,t)(a \otimes b)\, d(s,t)$$

$$= \int_G \int_G \zeta_0(f(s)g(s^{-1}t), t)(a \otimes b)\,dt\,ds$$
$$= \int_G \int_G \iota(f(s))a \otimes g(s^{-1}t)u(t)b\,dt\,ds$$

which, after $t \mapsto st$, is

$$= \int_G \int_G \iota(f(s))a \otimes g(t)u(st)b\,dt\,ds$$
$$= \int_G \int_G \big(\iota(f(s)) \otimes u(s)\big)\big(a \otimes g(t)u(t)b\big)\,dt\,ds$$
$$= \Big(\int_G \delta_0(f(s))\,ds\Big)\Big(\int_G 1 \otimes g(t)u(t)\,dt\Big)(a \otimes b)$$
$$= \delta(f)(1 \otimes g)(a \otimes b).$$

Thus, the multiplier $\delta(f)(1 \otimes g)$ of $C^*(G, \mathscr{A}) \otimes C^*(G)$ coincides with the image $\zeta(f \star g)$. The set of sections of the form $f \star g$ was shown in Lemma 2.2 to have dense span in $\Gamma_c(G \times G; \mathscr{A} \times G)$, so the images $\zeta(f \star g)$ have dense span in $C^*(G, \mathscr{A}) \otimes C^*(G)$. It follows that δ satisfies the nondegeneracy condition (1.3). □

Remark 3.2. It is clear from the above proof that saturation of the Fell bundle $\mathscr{A} \to G$ is not necessary for Proposition 3.1.

Remark 3.3. Not every coaction is isomorphic to one constructed from a Fell bundle as in Proposition 3.1 [13, Example 2.3(6)]. For *abelian* G, in [6, Theorem 11.14] Exel effectively characterizes which coactions do arise from Fell bundles (modulo the correspondence between coactions of G and actions of the Pontryagin dual group $\widehat{G}$).

Proposition 3.4. *Let $\mathscr{A}$ be a separable Fell bundle over a group G, and let δ be the coaction of G on $C^*(G, \mathscr{A})$ described in Proposition 3.1. Further let $\pi_0 : \mathscr{A} \to M(B)$ be a strictly continuous nondegenerate $*$-homomorphism, with integrated form $\pi : C^*(G, \mathscr{A}) \to M(B)$, and let $\mu : C_0(G) \to M(B)$ be a nondegenerate homomorphism. Then the pair (π, μ) is a covariant homomorphism of $(C^*(G, \mathscr{A}), G, \delta)$ if and only if*

$$\pi_0(a_s)\mu(f) = \mu \circ \mathrm{lt}_s(f)\pi_0(a_s) \quad \text{for } s \in G,\ a_s \in A_s \text{ and } f \in C_0(G), \tag{3.2}$$

where lt *is the action of G on $C_0(G)$ by left translation:* $\mathrm{lt}_s(f)(t) = f(s^{-1}t)$.

Proof. First assume that (π, μ) is covariant. Because $B(G) \cap C_0(G)$ is dense in $C_0(G)$, it suffices to verify (3.2) for $f \in B(G)$. So fix $f \in B(G)$, and put $g = \mathrm{lt}_s(f) \in B(G)$. By [4, Proposition A.34], we have

$$(\mathrm{id}_B \otimes g)\big((\mu \otimes \mathrm{id})(w_G)\big) = \mu(g)$$

where $\mathrm{id} \otimes g : M(C_0(G) \otimes C^*(G)) \to M(C_0(G))$ denotes the slice map. Then

$$\mu \circ \mathrm{lt}_s(f)\pi_0(a_s) = \mu(g)\pi_0(a_s) = (\mathrm{id}_B \otimes g)\big((\mu \otimes \mathrm{id})(w_G)\big)\pi_0(a_s)$$

which, by [4, Lemma A.30], is

$$= (\mathrm{id}_B \otimes g)\big((\mu \otimes \mathrm{id})(w_G)(\pi(\iota(a_s)) \otimes 1)\big)$$

which, by the covariance condition (1.4), is

$$\begin{aligned} &= (\mathrm{id}_B \otimes g)\big((\pi \otimes \mathrm{id})(\delta(\iota(a_s)))(\mu \otimes \mathrm{id})(w_G)\big) \\ &= (\mathrm{id}_B \otimes g)\big((\pi(\iota(a_s)) \otimes u(s))(\mu \otimes \mathrm{id})(w_G)\big) \\ &= (\mathrm{id}_B \otimes g)\big((\pi_0(a_s) \otimes 1)(\mu \otimes \mathrm{id})\big((1 \otimes u(s))w_G\big)\big) \end{aligned}$$

which, after applying [4, Lemma A.30] and writing $(\mathrm{lt}_{s^{-1}} \otimes \mathrm{id})(w_G)$ for the multiplier $r \mapsto u(sr)$, is

$$= \pi_0(a_s)(\mathrm{id}_B \otimes g)\big((\mu \otimes \mathrm{id})\big((\mathrm{lt}_{s^{-1}} \otimes \mathrm{id})(w_G)\big)\big)$$

which, since $(\mu \otimes \mathrm{id}) \circ (\mathrm{lt}_{s^{-1}} \otimes \mathrm{id}) = \mu \circ \mathrm{lt}_{s^{-1}} \otimes \mathrm{id}$ as a nondegenerate homomorphism of $C_0(G) \otimes C^*(G)$ into $M(B) \otimes C^*(G) \subseteq M(B \otimes C^*(G))$, is

$$= \pi_0(a_s)(\mathrm{id}_B \otimes g)\big((\mu \circ \mathrm{lt}_{s^{-1}} \otimes \mathrm{id})(w_G)\big)$$

which, by [4, Proposition A.34], is

$$\begin{aligned} &= \pi_0(a_s)\mu \circ \mathrm{lt}_{s^{-1}}(g) \\ &= \pi_0(a_s)\mu(f). \end{aligned}$$

Conversely, the above computation can be rearranged to show that, if (3.2) holds, then

$$\begin{aligned} &(\mathrm{id}_B \otimes g)\big((\mu \otimes \mathrm{id})(w_G)(\pi(\iota(a_s)) \otimes 1)\big) \\ &\qquad = (\mathrm{id}_B \otimes g)\big((\pi \otimes \mathrm{id})(\delta(\iota(a_s)))(\mu \otimes \mathrm{id})(w_G)\big) \end{aligned}$$

for every $g \in B(G)$. Since slicing by elements of $B(G)$ separates points in $M(C^*(G, \mathscr{A}) \otimes C^*(G))$, it follows that the covariance condition (1.4) holds for every a of the form $\iota(a_s)$, which then implies (by Lemma 1.3) that it holds for every element of $C^*(G, \mathscr{A})$. □

We include the following proposition since it might be useful elsewhere, although we will not need it in the present paper.

Proposition 3.5. *If α is an action of a group G on a C^*-algebra B, and $\mathscr{A} \to G$ is the associated semidirect-product Fell bundle, then the isomorphism*

$$B \rtimes_\alpha G \cong C^*(G, \mathscr{A})$$

carries the dual coaction $\hat{\alpha}$ to the coaction δ of G on $C^*(G, \mathscr{A})$ described in Proposition 3.1.

Proof. We recall that the isomorphism $\theta : B \rtimes_\alpha G \to C^*(G, \mathscr{A})$ is characterized on generators by

$$\theta(i_B(b)i_G(f)) = \int_G f(s)\iota(b,s)\,ds \quad \text{for } b \in B \text{ and } f \in C_c(G)$$

(which follows from [12, §VIII.5.7]). Thus,

$$\begin{aligned}\delta \circ \theta(i_B(b)i_G(f)) &= \int_G f(s)\delta(\iota(b,s))\,ds \\ &= \int_G f(s)\iota(b,s) \otimes s\,ds \\ &= \int_G f(s)(\theta \otimes \mathrm{id})(i_B(b)i_G(s) \otimes s)\,ds \\ &= \int_G f(s)(\theta \otimes \mathrm{id}) \circ \hat{\alpha}(i_B(b)i_G(s))\,ds \\ &= (\theta \otimes \mathrm{id}) \circ \hat{\alpha}(i_B(b)i_G(f)).\end{aligned}$$

□

4. Transformation bundles

Having defined a coaction δ on the C^*-algebra $C^*(G, \mathscr{A})$ of a Fell bundle over a group, an obvious next step is to consider the corresponding crossed product. In the next section, we will show that $C^*(G, \mathscr{A}) \rtimes_\delta G$ is isomorphic to the C^*-algebra of a Fell bundle over a groupoid. The purpose of this short section is to describe that groupoid and Fell bundle.

Let G be a locally compact group, and let $G \times_{\mathrm{lt}} G$ denote the transformation groupoid associated to the action lt of G on itself by left translation, with multiplication and inverse

$$(s, tr)(t, r) = (st, r) \quad \text{and} \quad (s,t)^{-1} = (s^{-1}, st) \quad \text{for } s, t, r \in G.$$

Note that the unit space is $(G \times_{\mathrm{lt}} G)^0 = \{e\} \times G$, and the range and source maps are given by

$$r(s,t) = (e, st) \quad \text{and} \quad s(s,t) = (e,t).$$

It it not hard to check that we get a left Haar system on $G \times_{\mathrm{lt}} G$ via

$$\int_{G \times_{\mathrm{lt}} G} f(u,v)\,d\lambda^{r(s,t)}(u,v) = \int_G f(u, u^{-1}st)\,du \quad \text{for } f \in C_c(G \times_{\mathrm{lt}} G).$$

Now let $\mathscr{A} \to G$ be a Fell bundle over the locally compact group G. The map $\varphi : (s,t) \mapsto s$ is a groupoid homomorphism of $G \times_{\mathrm{lt}} G$ onto the group G. The pull-back Fell bundle $\varphi^*\mathscr{A}$ (see Lemma 1.1) will be called the *transformation Fell bundle* $\mathscr{A} \times_{\mathrm{lt}} G \to G \times_{\mathrm{lt}} G$. We will use the bijection $(a_s, (s,t)) \mapsto (a_s, t)$ to identify the total space of $\mathscr{A} \times_{\mathrm{lt}} G$ with the Cartesian product $\mathscr{A} \times G$. Then the multiplication is

$$(a_s, tr)(b_t, r) = (a_s b_t, r) \quad \text{for } s, t, r \in G,\ a_s \in A_s \text{ and } b_t \in A_t,$$

and the involution is

$$(a_s, t)^* = (a_s^*, st).$$

For future reference, the convolution in $\Gamma_c(G \times_{\mathrm{lt}} G; \mathscr{A} \times_{\mathrm{lt}} G)$ is given by

$$(h * k)(s,t) = \int_G h(u, u^{-1}st)k(u^{-1}s, t)\, du \tag{4.1}$$

and the involution by

$$h^*(s,t) = h\big((s,t)^{-1}\big)^* = h(s^{-1}, st)^*. \tag{4.2}$$

Note that every $h \in \Gamma_c(G \times_{\mathrm{lt}} G; \mathscr{A} \times_{\mathrm{lt}} G)$ is of the form

$$h(s,t) = \big(h_1(s,t), t\big)$$

for a continuous function $h_1 : G \times_{\mathrm{lt}} G \to \mathscr{A}$ with $h_1(s,t) \in A_s$.

5. Coaction crossed product

Our purpose in this section is to prove the following:

Theorem 5.1. *Let $\mathscr{A}$ be a separable Fell bundle over a group G, and let δ be the associated coaction on $C^*(G, \mathscr{A})$ described in Proposition 3.1. If $q : \mathscr{A} \times_{\mathrm{lt}} G \to G \times_{\mathrm{lt}} G$ is the transformation Fell bundle constructed in the preceding section, then there is an isomorphism*

$$\theta : C^*(G, \mathscr{A}) \rtimes_\delta G \to C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G)$$

such that

$$\theta\big(j_{C^*(G,\mathscr{A})}(f) j_G(g)\big) = (\Delta^{\frac{1}{2}} f) \boxtimes g \quad \textit{for } f \in \Gamma_c(G; \mathscr{A}) \textit{ and } g \in C_c(G), \tag{5.1}$$

where $(\Delta^{\frac{1}{2}} f) \boxtimes g \in \Gamma_c(G \times_{\mathrm{lt}} G; \mathscr{A} \times_{\mathrm{lt}} G)$ is defined by

$$((\Delta^{\frac{1}{2}} f) \boxtimes g)(s,t) = (\Delta(s)^{\frac{1}{2}} f(s) g(t), t).$$

Remark 5.2. For G discrete, this is a special case of [5, Corollary 2.8].

Proof. We will obtain θ as the integrated form of a covariant homomorphism $(\theta_{\mathscr{A}}, \theta_G)$ of $(C^*(G, \mathscr{A}), G, \delta)$ into $M(C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G))$ such that

$$\theta_{\mathscr{A}}(f)\theta_G(g) = (\Delta^{\frac{1}{2}} f) \boxtimes g \in \Gamma_c(G \times_{\mathrm{lt}} G; \mathscr{A} \times_{\mathrm{lt}} G) \tag{5.2}$$

for $f \in \Gamma_c(G; \mathscr{A})$ and $g \in C_c(G)$. It will follow that $\theta = \theta_{\mathscr{A}} \rtimes \theta_G$ maps $A \rtimes_\delta G$ into $C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G)$, satisfies (5.1), and is surjective because

$$\{\, f \boxtimes g : f \in \Gamma_c(G; \mathscr{A}), g \in C_c(G) \,\}$$

has inductive-limit-dense span in $\Gamma_c(G \times_{\mathrm{lt}} G; \mathscr{A} \times_{\mathrm{lt}} G)$. We will show that θ is injective by finding a representation Π of $C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G)$ such that $\Pi \circ \theta$ is a faithful regular representation of $C^*(G, \mathscr{A}) \rtimes_\delta G$.

We will obtain $\theta_{\mathscr{A}} : C^*(G, \mathscr{A}) \to M\big(C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G)\big)$ as the integrated form of a $*$-homomorphism $\theta_0^{\mathscr{A}} : \mathscr{A} \to M\big(C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G)\big)$

(as in Lemma 1.3). Given $a_s \in A_s$, we define an operator $\theta_0^{\mathscr{A}}(a_s)$ on $\Gamma_c(G \times_{\mathrm{lt}} G; \mathscr{A} \times_{\mathrm{lt}} G)$ by[6]

$$(\theta_0^{\mathscr{A}}(a_s)h)(t,r) = \big(a_s h_1(s^{-1}t,r)\Delta(s)^{\frac{1}{2}}, r\big). \tag{5.3}$$

Then it is straightforward to verify that $\theta_0^{\mathscr{A}}(a_s)\theta_0^{\mathscr{A}}(a_t) = \theta_0^{\mathscr{A}}(a_s a_t)$. Moreover, if $h, k \in \Gamma_c(G \times_{\mathrm{lt}} G; \mathscr{A} \times_{\mathrm{lt}} G)$, we have

$$\big\langle \theta_0^{\mathscr{A}}(a_s)h \,,\, k\big\rangle(t,r) = \big((\theta_0^{\mathscr{A}}(a_s)h)^* * k\big)(t,r)$$

which, in view of the formula for convolution given by (4.1), is

$$= \int_G (\theta_0^{\mathscr{A}}(a_s)h)^*(u, u^{-1}tr)\, k(u^{-1}t, r)\, du$$

which, using the formula for the involution given by (4.2), is

$$\begin{aligned}
&= \int_G (\theta_0^{\mathscr{A}}(a_s)h)(u^{-1}, tr)^*\, k(u^{-1}t, r)\, du \\
&= \int_G \big(a_s h_1(s^{-1}u^{-1}, tr)\Delta(s)^{\frac{1}{2}}, tr\big)^* \, \big(k_1(u^{-1}t, r), r\big)\, du \\
&= \int_G \big(h_1(s^{-1}u^{-1}, tr)^* a_s^* \Delta(s)^{\frac{1}{2}}, u^{-1}tr\big) \, \big(k_1(u^{-1}t, r), r\big)\, du \\
&= \int_G \big(h_1(s^{-1}u^{-1}, tr)^* a_s^* k_1(u^{-1}t, r), r\big)\, \Delta(s)^{\frac{1}{2}} du
\end{aligned}$$

which, after sending $u \mapsto us^{-1}$, is

$$\begin{aligned}
&= \int_G \big(h_1(u^{-1}, tr)^* a_s^* k_1(su^{-1}t, r), r\big)\, \Delta(s)^{-\frac{1}{2}} du \\
&= \int_G \big(h_1(u^{-1}, tr)^*, u^{-1}tr\big) \big(a_s^* k_1(su^{-1}t, r)\Delta(s)^{-\frac{1}{2}}, r\big)\, du \\
&= \int_G \big(h_1(u^{-1}, tr), tr\big)^* \big(a_s^* k_1(su^{-1}t, r)\Delta(s)^{-\frac{1}{2}}, r\big)\, du \\
&= \int_G h(u^{-1}, tr)^* (\theta_0^{\mathscr{A}}(a_s^*)k)(u^{-1}t, r)\, du \\
&= \int_G h^*(u, u^{-1}tr) (\theta_0^{\mathscr{A}}(a_s^*)k)(u^{-1}t, r)\, du \\
&= \big(h^* * (\theta_0^{\mathscr{A}}(a_s^*)k)\big)(t,r) \\
&= \big\langle h \,,\, \theta_0^{\mathscr{A}}(a_s^*)k\big\rangle(t,r).
\end{aligned}$$

[6]The operator $\theta_0^{\mathscr{A}}(a_s)$ defined in (5.3) is analogous to $\iota(a_s)$ defined in Lemma 1.2. The modular function appearing in its definition is required to make $\theta_0^{\mathscr{A}}$ $*$-preserving. We need it here because there is no modular function in the involution in $\Gamma_c(G \times_{\mathrm{lt}} G; \mathscr{A} \times_{\mathrm{lt}} G)$.

If we choose $b_e \in \tilde{A}_e$ such that $\|a_s\|^2 1_{A_e} - a_s^* a_s = b_e^* b_e$, then since $\theta_0^{\mathscr{A}}$ makes sense and is multiplicative on $\tilde{A}_e$, and since the preceding computation certainly holds for $b_e \in \tilde{A}_e$, we see that

$$\begin{aligned}\|a_s\|^2\langle h\,,\,h\rangle - \langle \theta_0^{\mathscr{A}}(a_s)h\,,\,\theta_0^{\mathscr{A}}(a_s)h\rangle &= \langle \theta_0^{\mathscr{A}}(\|a_s\|^2 1_{A_e} - a_s^* a_s)h\,,\,h\rangle \\ &= \langle \theta_0^{\mathscr{A}}(b_e)h\,,\,\theta_0^{\mathscr{A}}(b_e)h\rangle \geq 0\end{aligned}$$

for all $h \in \Gamma_c(G \times_{\mathrm{lt}} G; \mathscr{A} \times_{\mathrm{lt}} G)$. Thus $\theta_0^{\mathscr{A}}(a_s)$ extends to a bounded adjointable operator on $C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G)$ and we get a $*$-homomorphism $\theta_0^{\mathscr{A}} : \mathscr{A} \to M(C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G))$.

We need to show that $\theta_0^{\mathscr{A}}$ is strictly continuous and nondegenerate. For nondegeneracy, let $\{e_i\}$ be an approximate identity in A_e. It suffices to show that if $h \in \Gamma_c(G \times_{\mathrm{lt}} G; \mathscr{A} \times_{\mathrm{lt}} G)$ then $\theta_0^{\mathscr{A}}(e_i)h \to h$ in the inductive limit topology.[7] Notice that

$$\theta_0^{\mathscr{A}}(e_i)h(r,t) = (e_i h_1(r,t), t).$$

Since each A_r is an $A_e - A_e$-imprimitivity bimodule, $e_i h_1(r,t) \to h_1(r,t)$ for any $(r,t) \in G \times_{\mathrm{lt}} G$. Fix $\varepsilon > 0$. Since $a \mapsto \|a\|$ is continuous on $\mathscr{A}$, we can cover $\operatorname{supp} h_1$ with open sets $V_1, \dots, V_n$ and find $a_j \in A_e$ such that

$$\|a_j h_1(r,t) - h_1(r,t)\| < \frac{\varepsilon}{3} \quad \text{for all } (r,t) \in V_j.$$

Let $\{\,\varphi_j\,\} \subseteq C_c^+(G \times_{\mathrm{lt}} G)$ be such that $\operatorname{supp} \varphi_j \subseteq V_j$ and $\sum_j \varphi_j(r,t) \leq 1$ for all (r,t), with equality for $(r,t) \in \operatorname{supp} h_1$. Define $a \in C_c(G \times_{\mathrm{lt}} G, A_e)$ by

$$a(r,t) = \sum_j \varphi_j(r,t) a_j.$$

Then

$$\|a(r,t)h_1(r,t) - h_1(r,t)\| < \frac{\varepsilon}{3} \quad \text{for all } (r,t).$$

Clearly, there is an i_0 such that $i \geq i_0$ implies that

$$\|e_i a(r,t) - a(r,t)\| < \frac{\varepsilon}{3(\|h_1\|_\infty + 1)} \quad \text{for all } (r,t).$$

Since $\|e_i\| \leq 1$ for all i, we see that $i \geq i_0$ implies

$$\begin{aligned}&\|\theta_0^{\mathscr{A}}(e_i)h(r,t) - h(r,t)\| \\ &\qquad = \|e_i h_1(r,t) - h_1(r,t)\| \\ &\qquad \leq \|e_i h_1(r,t) - e_i a(r,t)h_1(r,t)\| + \|e_1 a(r,t)h_1(r,t) - a(r,t)h_1(r,t)\| \\ &\qquad\quad + \|a(r,t)h_1(r,t) - h_1(r,t)\| \\ &\qquad \leq 2\|h_1(r,t) - a(r,t)h_1(r,t)\| + \|e_i a(r,t) - a(r,t)\|\|h_1\|_\infty \\ &\qquad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

[7] This could be proved using [16, Lemma 8.1]. However, the proof of that lemma given in [16] is incorrect. Fortunately, it can be fixed along the same lines as presented here.

Therefore $\theta_0^{\mathscr{A}}(e_i)h \to h$ uniformly, so since $\operatorname{supp}\theta_0^{\mathscr{A}}(e_i)h = \operatorname{supp} h$ for all i, we have $\theta_0^{\mathscr{A}}(e_i)h \to h$ in the inductive limit topology, as desired.

Finally, for strict continuity we note that our separability assumption on $p : \mathscr{A} \to G$ guarantees that $\mathscr{A}$ is second countable [11, Proposition II.13.21]. Thus, it suffices to show that $\theta_0^{\mathscr{A}}$ takes convergent sequences to strictly convergent sequences.

So suppose $\{ a_i \}$ is a sequence converging to a in $\mathscr{A}$. Let $s = p(a)$, and for each i, let $s_i = p(a_i)$; so $s_i \to s$ in G. Since $\{ a_i \}$ must lie in a norm-bounded subset of $\mathscr{A}$, the image $\{ \theta_0^{\mathscr{A}}(a_i) \}$ is a bounded sequence in $M(C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G))$. Thus it suffices to show that $\theta_0^{\mathscr{A}}(a_i) \to \theta_0^{\mathscr{A}}(a)$ $*$-strongly [21, Proposition C.7]. Since $a_i^* \to a^*$ and $\theta_0^{\mathscr{A}}$ is $*$-preserving, it suffices to show strong convergence. Since $\{ \theta_0^{\mathscr{A}}(a_i) \}$ is bounded, it suffices to show that $\theta_0^{\mathscr{A}}(a_i)h \to \theta_0^{\mathscr{A}}(a)h$ in the inductive limit topology for each $h \in \Gamma_c(G \times_{\mathrm{lt}} G; \mathscr{A} \times_{\mathrm{lt}} G)$.

We can replace $\{a_i\}$ by a subsequence (keeping the same notation) such that the s_i's lie in a fixed compact neighborhood of s. Then the supports of the $\theta_0^{\mathscr{A}}(a_i)h$'s all lie in a fixed compact set, so it suffices to show that $\theta_0^{\mathscr{A}}(a_i)h \to \theta_0^{\mathscr{A}}(a)h$ uniformly. If not, then there are (r_i, t_i), all lying in a compact subset of $G \times_{\mathrm{lt}} G$, and an $\varepsilon > 0$ such that

$$\|\theta_0^{\mathscr{A}}(a_i)h(r_i, t_i) - \theta_0^{\mathscr{A}}(a)h(r_i, t_i)\| \geq \varepsilon. \tag{5.4}$$

Of course, we can pass to a subsequence, relabel, and assume that $(r_i, t_i) \to (r, t)$. But the left-hand side of (5.4) equals

$$\|a_i h_1(s_i^{-1} r_i, t_i) - a h_1(s^{-1} r_i, t_i)\|. \tag{5.5}$$

Since $(a_i, h_1(s_i^{-1} r_i, t_i))$ and $(a, h_1(s^{-1} r_i, t_i))$ both converge to $(a, h_1(s^{-1}r, t))$ in $\mathscr{A} \times \mathscr{A}$, and since multiplication is continuous from $\mathscr{A} \times \mathscr{A} \to \mathscr{A}$, it follows that $a_i h_1(s_i^{-1} r_i, t_i) - a h_1(s^{-1} r_i, t_i)$ tends to 0_{A_r} in $\mathscr{A}$. Therefore, (5.5) tends to zero, and this contradicts (5.4). Thus $\theta_0^{\mathscr{A}}$ is strictly continuous.

Having dealt with $\theta_0^{\mathscr{A}}$, we turn to the definition of θ_G. For $f \in C_0(G)$ and $h \in \Gamma_c(G \times_{\mathrm{lt}} G; \mathscr{A} \times_{\mathrm{lt}} G)$ define

$$(\theta_G(f)h)(s, t) = (f(st)h_1(s, t), t). \tag{5.6}$$

We note that (5.6) makes perfectly good sense for $f \in C_0(G)^\sim$, and then $\theta_G(fg) = \theta_G(f)\theta_G(g)$ for $f, g \in C_0(G)^\sim$. Another computation shows that

$$\langle \theta_G(f)h \,,\, k \rangle = \langle h \,,\, \theta_G(\overline{f})k \rangle$$

for all such f. Writing $\|f\|_\infty^2 - \overline{f}f = \overline{g}g$ for some $g \in C_0(G)^\sim$, we thus have

$$\|f\|_\infty^2 \langle h \,,\, h \rangle - \langle \theta_G(f)h \,,\, \theta_G(f)h \rangle = \langle \theta_G(g)h \,,\, \theta_G(g)h \rangle \geq 0$$

for all h. Therefore $\theta_G(f)$ is bounded and we get a $*$-homomorphism of $C_0(G)$ into $M\big(C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \rtimes_{\mathrm{lt} G})\big)$.[8]

[8] In fact, modulo the obvious identification of G with $(G \times_{\mathrm{lt}} G)^{(0)}$, θ_G is just the natural map of $C_0(\mathcal{G}^{(0)})$ into the multiplier algebra of the C^*-algebra $C^*(\mathcal{G}, \mathscr{B})$ of a Fell bundle over a groupoid $\mathcal{G}$.

We let $\theta_{\mathscr{A}}$ be the integrated form of $\theta_0^{\mathscr{A}}$ (see Lemma 1.3). To see that $(\theta_{\mathscr{A}}, \theta_G)$ is covariant, we will use Proposition 3.4. For $a_s \in A_s$, $f \in C_c(G)$, $h \in \Gamma_c(G \times G; \mathscr{A} \times G)$, and $r, t \in G$ we have

$$\begin{aligned}\big(\theta_0^{\mathscr{A}}(a_s)\theta_G(f)h\big)(r,t) &= \big(a_s(\theta_G(f)h)_1(s^{-1}r,t)\Delta(s)^{\frac{1}{2}}, r\big)\\ &= \big(a_s f(s^{-1}rt)h_1(s^{-1}r,t)\Delta(s)^{\frac{1}{2}}, r\big)\\ &= \big(\mathrm{lt}_s(f)(rt)a_s h_1(s^{-1}r,t)\Delta(s)^{\frac{1}{2}}, r\big)\\ &= \big(\mathrm{lt}_s(f)(rt)(\theta_0^{\mathscr{A}}(a_s)h)_1(r,t), r\big)\\ &= \big(\theta_G \circ \mathrm{lt}_s(f)\theta_0^{\mathscr{A}}(a_s)h\big)(r,t).\end{aligned}$$

To verify (5.2), for $h \in \Gamma_c(G \times_{\mathrm{lt}} G; \mathscr{A} \times_{\mathrm{lt}} G)$ we have

$$\begin{aligned}\big(\theta_{\mathscr{A}}(f)\theta_G(g)h\big)(s,t) &= \int \big(\theta_0^{\mathscr{A}}(f(r))\theta_G(g)h\big)(s,t)\,dr\\ &= \int \big(f(r)(\theta_G(g)h)_1(r^{-1}s,t)\Delta(r)^{\frac{1}{2}}, t\big)\,dr\\ &= \int \big(f(r)g(r^{-1}st)h_1(r^{-1}s,t)\Delta(r)^{\frac{1}{2}}, t\big)\,dr\\ &= \int \big(f(r)\Delta(r)^{\frac{1}{2}}g(r^{-1}st), r^{-1}st\big)\big(h_1(r^{-1}s,t),t\big)\,dr\\ &= \int \big((\Delta^{\frac{1}{2}}f)\boxtimes g\big)(r, r^{-1}st)h(r^{-1}s,t)\,dr\\ &= \big(((\Delta^{\frac{1}{2}}f)\boxtimes g\big) * h\big)(s,t).\end{aligned}$$

As outlined at the start of the proof, it follows from the above that the integrated form $\theta = \theta_{\mathscr{A}} \rtimes \theta_G$ maps $A \rtimes_\delta G$ onto $C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G)$. To show that θ is faithful, we will now construct a representation Π of $C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G)$ such that $\Pi \circ \theta$ is the regular representation $\Lambda = (\pi_{\mathscr{A}} \otimes \lambda) \circ \delta \rtimes (1 \otimes M)$ associated to a faithful representation $\pi_{\mathscr{A}}$ of $C^*(G, \mathscr{A})$. This will suffice since Λ is faithful by [4, Remark A.43(3)].

So let $\pi_{\mathscr{A}}$ be a faithful nondegenerate representation of $C^*(G, \mathscr{A})$ on a Hilbert space $\mathcal{H}$. Of course, $\pi_{\mathscr{A}}$ is the integrated form of a representation $\pi_0^{\mathscr{A}}$ of $\mathscr{A}$, by Lemma 1.3. For $h \in \Gamma_c(G \times_{\mathrm{lt}} G; \mathscr{A} \times_{\mathrm{lt}} G)$ and $\xi \in C_c(G, \mathcal{H}) \subseteq \mathcal{H} \otimes L^2(G)$, define $\Pi_0(h)\xi : G \to \mathcal{H}$ by

$$\big(\Pi_0(h)\xi\big)(t) = \int_G \pi_0^{\mathscr{A}}\big(h_1(s, s^{-1}t)\big)\xi(s^{-1}t)\Delta(s)^{-\frac{1}{2}}\,ds; \tag{5.7}$$

the integrand is in $C_c(G \times G, \mathcal{H})$, so (5.7) does define a vector in $\mathcal{H}$, and $\Pi_0(h)\xi \in C_c(G, \mathcal{H})$. It follows that (5.7) defines a linear operator $\Pi_0(h)$ on the dense subspace $C_c(G, \mathcal{H})$ of $\mathcal{H} \otimes L^2(G)$.

By [16, Theorem 4.13], to show that Π_0 extends to a representation

$$\Pi : C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G) \to B(\mathcal{H} \otimes L^2(G)),$$

it suffices to show that Π_0 is a *pre-representation* of $\mathscr{A} \times_{\mathrm{lt}} G$ on $C_c(G,\mathcal{H})$. Recall from [16, Definition 4.1] that to say that Π_0 is a pre-representation means that $\Pi_0 : \Gamma_c(G \times_{\mathrm{lt}} G; \mathscr{A} \times_{\mathrm{lt}} G) \to \mathrm{Lin}(C_c(G,\mathcal{H}))$ (where $\mathrm{Lin}(C_c(G,\mathcal{H}))$ denotes the algebra of all linear operators on the vector space $C_c(G,\mathcal{H})$) is an algebra homomorphism such that for all $\xi, \eta \in C_c(G,\mathcal{H})$:

(i) $h \mapsto \langle \Pi_0(h)\xi \,,\, \eta \rangle$ is continuous in the inductive limit topology;

(ii) $\langle \Pi_0(h)\xi \,,\, \eta \rangle = \langle \xi \,,\, \Pi_0(h^*)\eta \rangle$; and

(iii) $\Pi_0(\Gamma_c(G \times_{\mathrm{lt}} G; \mathscr{A} \times_{\mathrm{lt}} G))C_c(G,\mathcal{H})$ has dense span in $\mathcal{H} \otimes L^2(G)$.

Π_0 is obviously linear; we verify that it is multiplicative: for

$$f, g \in \Gamma_c(G \times_{\mathrm{lt}} G; \mathscr{A} \times_{\mathrm{lt}} G) \quad \text{and} \quad \xi \in C_c(G,\mathcal{H})$$

we have

$$\begin{aligned}
&\big(\Pi_0(f * g)\xi\big)(t)\\
&\quad = \int_G \pi_0^{\mathscr{A}}\big((f*g)_1(s, s^{-1}t)\big)\xi(s^{-1}t)\Delta(s)^{-\frac{1}{2}}\,ds\\
&\quad = \int_G\int_G \pi_0^{\mathscr{A}}\big(f_1(r, r^{-1}t)g_1(r^{-1}s, s^{-1}t)\big)\xi(s^{-1}t)\Delta(s)^{-\frac{1}{2}}\,ds\,dr
\end{aligned}$$

which, after $s \mapsto rs$, is

$$\begin{aligned}
&\quad = \int_G\int_G \pi_0^{\mathscr{A}}\big(f_1(r, r^{-1}t)g_1(s, s^{-1}r^{-1}t)\big)\xi(s^{-1}r^{-1}t)\Delta(rs)^{-\frac{1}{2}}\,ds\,dr\\
&\quad = \int_G \pi_0^{\mathscr{A}}\big(f_1(r, r^{-1}t)\big)\\
&\qquad\quad \cdot \left(\int_G \pi_0^{\mathscr{A}}\big(g_1(s, s^{-1}r^{-1}t)\big)\xi(s^{-1}r^{-1}t)\Delta(s)^{-\frac{1}{2}}\,ds\right)\Delta(r)^{-\frac{1}{2}}\,dr\\
&\quad = \big(\Pi_0(f)\Pi_0(g)\xi\big)(t).
\end{aligned}$$

For (i), it suffices to show that if $K \subseteq G \times G$ is compact and $\{h_n\}$ is a sequence converging uniformly to 0 in $\Gamma_K(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G)$ then

$$\langle \Pi_0(h_n)\xi \,,\, \eta \rangle \to 0 \quad \text{for all } \xi, \eta \in C_c(G,\mathcal{H}).$$

We have

$$\begin{aligned}
\langle \Pi_0(h_n)\xi \,,\, \eta \rangle &= \int_G \big\langle \big(\Pi_0(h_n)\xi\big)(t) \,,\, \eta(t) \big\rangle\,dt\\
&= \int_G\int_G \big\langle \pi_0^{\mathscr{A}}(h_n(s, s^{-1}t))\xi(s^{-1}t) \,,\, \eta(t) \big\rangle\,ds\,dt,
\end{aligned}$$

which converges to 0 since the integrands converge uniformly to 0 and the integration is over a compact set.

For (ii) we have

$$\langle \Pi_0(h)\xi \,,\, \eta \rangle = \int_G \big\langle \big(\Pi_0(h)\xi\big)(t) \,,\, \eta(t) \big\rangle\,dt$$

$$= \int_G \int_G \bigl\langle \pi_0^{\mathscr{A}}(h_1(s, s^{-1}t))\xi(s^{-1}t)\ ,\ \eta(t)\bigr\rangle \Delta(s)^{-\frac{1}{2}}\, dt\, ds$$
$$= \int_G \int_G \bigl\langle \xi(s^{-1}t)\ ,\ \pi_0^{\mathscr{A}}(h_1(s, s^{-1}t)^*)\eta(t)\bigr\rangle \Delta(s)^{-\frac{1}{2}}\, dt\, ds$$

which, after $t \mapsto st$, is

$$= \int_G \int_G \bigl\langle \xi(t), \pi_0^{\mathscr{A}}(h_1(s, t)^*)\eta(st)\bigr\rangle \Delta(s)^{-\frac{1}{2}}\, dt\, ds$$

which, after $s \mapsto s^{-1}$, is

$$= \int_G \int_G \bigl\langle \xi(t)\ ,\ \pi_0^{\mathscr{A}}(h_1(s^{-1}, t)^*)\eta(s^{-1}t)\bigr\rangle \Delta(s)^{-\frac{1}{2}}\, dt\, ds$$
$$= \int_G \int_G \bigl\langle \xi(t)\ ,\ \pi_0^{\mathscr{A}}((h^*)_1(s, s^{-1}t))\eta(s^{-1}t)\bigr\rangle \Delta(s)^{-\frac{1}{2}}\, ds\, dt$$
$$= \int_G \bigl\langle \xi(t)\ ,\ \bigl(\Pi_0(h^*)\eta\bigr)(t)\bigr\rangle\, dt$$
$$= \bigl\langle \xi\ ,\ \Pi_0(h^*)\eta\bigr\rangle.$$

For (iii), it suffices to show that for $f \in \Gamma_c(G; \mathscr{A})$ and $g \in C_c(G)$ we have

$$\Pi_0((\Delta^{\frac{1}{2}} f) \boxtimes g) = (\pi_{\mathscr{A}} \otimes \lambda) \circ \delta(f)(1 \otimes M_g),$$

because the ranges of the operators on the right-hand side have dense span in $\mathcal{H} \otimes L^2(G)$ since the regular representation of $A \times_\delta G$ is nondegenerate. For $\xi \in C_c(G, \mathcal{H})$ we have

$$\Bigl(\Pi_0\bigl((\Delta^{\frac{1}{2}} f) \boxtimes g\bigr)\xi\Bigr)(t) = \int_G \pi_0^{\mathscr{A}}\bigl((f \boxtimes g)_1(s, s^{-1}t)\bigr)\xi(s^{-1}t)\Delta(s)^{-\frac{1}{2}}\, ds$$
$$= \int_G \pi_0^{\mathscr{A}}(f(s))g(s^{-1}t)\xi(s^{-1}t)\, ds$$
$$= \int_G \pi_0^{\mathscr{A}}(f(s))(M_g\xi)(s^{-1}t)\, ds$$
$$= \int_G \pi_0^{\mathscr{A}}(f(s))(\lambda_s M_g\xi)(t)\, ds$$
$$= \int_G \bigl(\bigl(\pi_0^{\mathscr{A}}(f(s)) \otimes \lambda_s M_g\bigr)\xi\bigr)(t)\, ds$$
$$= \int_G \bigl(\bigl(\pi_{\mathscr{A}}(\iota(f(s))) \otimes \lambda_s M_g\bigr)\xi\bigr)(t)\, ds$$
$$= \int_G \bigl(\bigl(\pi_{\mathscr{A}} \otimes \lambda\bigr)\bigl(\iota(f(s)) \otimes u(s)\bigr)\bigl(1 \otimes M_g\bigr)\xi\bigr)(t)\, ds$$
$$= \int_G \bigl((\pi_{\mathscr{A}} \otimes \lambda) \circ \delta(\iota(f(s)))(1 \otimes M_g)\xi\bigr)(t)\, ds$$
$$= \bigl((\pi_{\mathscr{A}} \otimes \lambda) \circ \delta(f)(1 \otimes M_g)\xi\bigr)(t).$$

As we explained above, we now can conclude that Π_0 extends uniquely to a nondegenerate representation Π of $C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G)$, and then the above calculation verifies that $\Pi \circ \theta$ agrees with the regular representation $\Lambda = (\pi_{\mathscr{A}} \otimes \lambda) \circ \delta \times (1 \otimes M)$ on the generators $j_A(f) j_G(g)$ for $f \in \Gamma_c(G; \mathscr{A})$ and $g \in C_c(G)$. Hence $\Pi \circ \theta = \Lambda$ on all of $C^*(G, \mathscr{A}) \times_\delta G$ by linearity, continuity, and density. □

6. Semidirect-product bundles

To prove our main theorem in Section 8, we are going to need to build a Fell bundle over groupoid arising as a semidirect product. In this section, we give the construction of this *semidirect-product Fell bundle*. We will investigate the structure of the corresponding Fell bundle C^*-algebra in Section 7.

To begin, let $\mathcal{G}$ be a locally compact Hausdorff groupoid with Haar system $\{ \lambda^u \}_{u \in \mathcal{G}^{(0)}}$, and let G be a second countable locally compact group. An *action* of G on $\mathcal{G}$ is a homomorphism $\beta : G \to \operatorname{Aut} \mathcal{G}$ such that $(x, t) \mapsto \beta_t(x)$ is continuous from $\mathcal{G} \times G$ to $\mathcal{G}$. (Note that automorphisms of a groupoid do not necessarily fix the unit space pointwise.) Given an action β of G on $\mathcal{G}$, the *semidirect-product groupoid* $\mathcal{G} \times_\beta G$ comprises the Cartesian product $\mathcal{G} \times G$ with multiplication

$$(x, t)(y, s) = (x \beta_t(y), ts)$$

whenever $s(x) = \beta_t(r(y))$ and inverse $(x, t)^{-1} = (\beta_{t^{-1}}(x^{-1}), t^{-1})$ ([22, Definition I.1.7]). Note that we have $(\mathcal{G} \times_\beta G)^0 = \mathcal{G}^0 \times \{e\}$, with

$$r(x, t) = (r(x), e) \quad \text{and} \quad s(x, t) = (\beta_t^{-1}(s(x)), e).$$

Also note that $C_c(\mathcal{G}) \odot C_c(G)$ is inductive-limit dense in $C_c(\mathcal{G} \times_\beta G)$.

Now suppose $p : \mathscr{B} \to \mathcal{G}$ is a separable Fell bundle over $\mathcal{G}$. An *action* of G on $\mathscr{B}$ is a homomorphism $\alpha : G \to \operatorname{Aut} \mathscr{B}$ such that $(b, t) \mapsto \alpha_t(b)$ is continuous from $\mathscr{B} \times G \to \mathscr{B}$, together with an *associated action* β of G on $\mathcal{G}$ such that $p\big(\alpha_t(b)\big) = \beta_t\big(p(b)\big)$ for all $t \in G$ and $b \in \mathscr{B}$.

Remark 6.1. The compatibility of α and β allows us to write down, for each $t \in G$, an automorphism $\overline{\alpha}_t$ of $\Gamma_c(\mathcal{G}; \mathscr{B})$ given by

$$\overline{\alpha}_t(f)(x) = \alpha_t\big(f(\beta_t^{-1}(x)\big). \tag{6.1}$$

Since $\overline{\alpha}_t$ is clearly continuous from the inductive limit topology to the norm topology, it follows from Proposition 1.8 that $\overline{\alpha}_t$ extends to an automorphism of $C^*(\mathcal{G}, \mathscr{B})$. Similarly, $t \mapsto \overline{\alpha}_t(f)$ is continuous from G into $C^*(\mathcal{G}, \mathscr{B})$, so we obtain an action $\overline{\alpha}$ of G on $C^*(\mathcal{G}, \mathscr{B})$.

Proposition 6.2. *Let α be an action of G on a Fell bundle $p : \mathscr{A} \to \mathcal{G}$, with associated action β of G on $\mathcal{G}$. Then the Banach bundle $q : \mathscr{B} \times_\alpha G \to \mathcal{G} \times_\beta G$ with total space $\mathscr{B} \times G$ and bundle projection $q(b, t) = (p(b), t)$ becomes a Fell bundle when equipped with the multiplication given by*

$$(b_x, t)(c_y, s) = (b_x \alpha_t(c_y), ts) \quad \textit{whenever } s(x) = r(\beta_t(y))$$

and the involution given by

$$(b_x, t)^* = (\alpha_{t^{-1}}(b_x)^*, t^{-1}).$$

We refer to a Fell bundle which arises from a group action as in Proposition 6.2 as a *semidirect-product Fell bundle*.

Sketch of Proof. For convenience, we'll write $C_{(x,t)}$ for the fibre of $\mathscr{B} \times_\alpha G$ over (x, t). Verifying the axioms that $\mathscr{B} \times_\alpha G$ is a Fell bundle is routine with the possible exception of seeing that $C_{(x,t)}$ is a $C_{(r(x),e)} - C_{(\beta_t^{-1}(s(x)),e)}$-imprimitivity bimodule via the operations inherited from $\mathscr{B} \times_\alpha G$. However, $C_{(x,t)}$ is naturally identified with B_x, and the latter is given to be a $B_{r(x)} - B_{s(x)}$-imprimitivity bimodule with respect to the operations inherited from $\mathscr{B}$. Furthermore, α_t restricts to a C^*-algebra isomorphism of $B_{\beta_t^{-1}(s(x))}$ onto $B_{s(x)}$. Therefore B_x is naturally a $B_{r(x)} - B_{\beta_t^{-1}(s(x))}$-imprimitivity bimodule. The right action is given by $x \cdot b = x\alpha_t(b)$ and the right inner product is given by

$$\langle x, y\rangle_{B_{\beta_t^{-1}(s(x))}} = \alpha_t^{-1}\big(\langle x\ , y\rangle_{B_{s(x)}}\big).$$

Now it is a simple matter to see that the given operations in $\mathscr{B} \times_\alpha G$ induce the same structure on $C_{(x,t)}$ as does the identification of $C_{(x,t)}$ with B_x. $\square$

In order to have a Haar system on a semidirect-product groupoid $\mathcal{G} \times_\beta G$, we will need β to be compatible with the Haar system on $\mathcal{G}$ in the following sense.

Definition 6.3. An action $\beta : G \to \operatorname{Aut} \mathcal{G}$ is *invariant* if for all $u \in \mathcal{G}^{(0)}$, $f \in C_c(\mathcal{G})$, and $t \in G$ we have

$$\int_{\mathcal{G}} f(\beta_t(y))\, d\lambda^u(y) = \int_{\mathcal{G}} f(y)\, d\lambda^{\beta_t(u)}(y),$$

i.e., β_t transforms the measure on $r^{-1}(u)$ to the measure on $r^{-1}(\beta_t(u))$. If $\alpha : G \to \operatorname{Aut} \mathscr{B}$ is an action on a Fell bundle $\mathscr{B} \to \mathcal{G}$ with associated action $\beta : G \to \operatorname{Aut} \mathcal{G}$, we say α is *invariant* if β is.

Proposition 6.4. *Let $\beta : G \to \operatorname{Aut} \mathcal{G}$ be an invariant action on a groupoid $\mathcal{G}$ with Haar system $\{\lambda^u\}_{u\in\mathcal{G}^{(0)}}$. Then*

$$d\lambda^{(u,e)}(y, s) = d\lambda^u(y)\, ds$$

is a Haar system on $\mathcal{G} \times_\beta G$.

Proof. The left-invariance property we need is that for $h \in C_c(\mathcal{G} \times_\beta G)$ and $(x, t) \in \mathcal{G} \times_\beta G$ we have

$$\int_{\mathcal{G}\times G} h\big((x, t)(y, s)\big)\, d\lambda^{s(x,t)}(y, s) = \int_{\mathcal{G}\times G} h(y, s)\, d\lambda^{r(x,t)}(y, s),$$

and it suffices to take $h = f \otimes g$, where $f \in G_c(\mathcal{G})$ and $g \in C_c(G)$. Fix $x \in \mathcal{G}$ with $s(x) = v$ and $r(x) = u$. In the left-hand integral we must have

$$(r(y), e) = r(y, s) = s(x, t) = (\beta_t^{-1}(s(x)), e) = (\beta_t^{-1}(v), e),$$

and in the right-hand integral we must have

$$(r(y), e) = (r(x), e) = (u, e).$$

Since

$$(x, t)(y, s) = (x\beta_t(y), ts),$$

we must show that

$$\int_{\mathcal{G}} \int_G f(x\beta_t(y))g(ts)\, ds\, d\lambda^{\beta_t^{-1}(v)}(y) = \int_{\mathcal{G}} \int_G f(y)g(s)\, ds\, d\lambda^u(y).$$

We have

$$\begin{aligned}\int_{\mathcal{G}} \int_G f(x\beta_t(y))g(ts)\, ds\, d\lambda^{\beta_t^{-1}(v)}(y) &= \int_{\mathcal{G}} f(x\beta_t(y)) \int_G g(ts)\, ds\, d\lambda^{\beta_t^{-1}(v)}(y)\\ &= \int_{\mathcal{G}} f(x\beta_t(y))\, d\lambda^{\beta_t^{-1}(v)}(y) \int_G g(s)\, ds\end{aligned}$$

and similarly

$$\int_{\mathcal{G}} \int_G f(y)g(s)\, ds\, d\lambda^u(y) = \int_{\mathcal{G}} f(y)\, d\lambda^u(y) \int_G g(s)\, ds,$$

so it remains to verify

$$\int_{\mathcal{G}} f(x\beta_t(y))\, d\lambda^{\beta_t^{-1}(v)}(y) = \int_{\mathcal{G}} f(y)\, d\lambda^u(y).$$

But invariance of the action β gives

$$\int_{\mathcal{G}} f(x\beta_t(y))\, d\lambda^{\beta_t^{-1}(v)}(y) = \int_{\mathcal{G}} f(xy)\, d\lambda^v(y),$$

which equals $\int_{\mathcal{G}} f(y)\, d\lambda^u(y)$ because λ is a Haar system. □

For reference, we record the formula for convolution in $C_c(\mathcal{G} \times_\beta G)$:

$$(h * k)(x, t) = \int_{\mathcal{G}} \int_G h(y, s)k\big(\beta_s^{-1}(y^{-1}x), s^{-1}t\big)\, ds\, d\lambda^{r(x)}(y).$$

Thus in $\Gamma_c(\mathcal{G} \times_\beta G; \mathscr{B} \times_\alpha G)$ the convolution is given by

$$\begin{aligned}(h * k)(x, t) &= \int_{\mathcal{G}} \int_G h(y, s)k\big(\beta_s^{-1}(y^{-1}x), s^{-1}t\big)\, ds\, d\lambda^{r(x)}(y)\\ &= \int_{\mathcal{G}} \int_G (h_1(y, s), s)\big(k_1\big(\beta_s^{-1}(y^{-1}x), s^{-1}t\big), s^{-1}t\big)\, ds\, d\lambda^{r(x)}(y)\\ &= \int_{\mathcal{G}} \int_G \big(h_1(y, s)\alpha_s\big(k_1\big(\beta_s^{-1}(y^{-1}x), s^{-1}t\big)\big), t\big)\, ds\, d\lambda^{r(x)}(y).\end{aligned}$$

As with product bundles (see Section 2), each section $h \in \Gamma_c(\mathcal{G} \times_\beta G; \mathscr{B} \times_\alpha G)$ is of the form

$$h(x,t) = (h_1(x,t), t),$$

where $h_1 \in C_c(\mathcal{G} \times_\beta G, \mathscr{B})$ satisfies $h_1(x,t) \in B_x$. So in particular

$$(h * k)_1(x,t) = \int_{\mathcal{G}} \int_G h_1(y,s) \alpha_s\big(k_1\big(\beta_s^{-1}(y^{-1}x), s^{-1}t\big)\big)\, ds\, d\lambda^{r(x)}(y). \tag{6.2}$$

The involution in $\Gamma_c(\mathcal{G} \times_\beta G; \mathscr{B} \times_\alpha G)$ is given by

$$\begin{aligned} h^*(x,t) &= h\big((x,t)^{-1}\big)^* = h\big(\beta_t^{-1}(x^{-1}), t^{-1}\big)^* = \big(h_1\big(\beta_t^{-1}(x^{-1}), t^{-1}\big), t^{-1}\big)^* \\ &= \big(\alpha_t\big(h_1\big(\beta_t^{-1}(x^{-1}), t^{-1}\big)^*\big), t\big), \end{aligned}$$

so in particular

$$h_1^*(x,t) = \alpha_t\big(h_1\big(\beta_t^{-1}(x^{-1}), t^{-1}\big)^*\big).$$

7. Action crossed product

We now relate the C^*-algebra of a semidirect-product bundle to the crossed product.

Theorem 7.1. *Let $p : \mathscr{B} \to \mathcal{G}$ be a separable Fell bundle over a locally compact Hausdorff groupoid with Haar system $\{\lambda^u\}_{u \in G^{(0)}}$, and let $\alpha : G \to \operatorname{Aut} \mathscr{B}$ be an action of a second countable locally compact group G on $\mathscr{B}$ with an invariant associated action β of G on $\mathcal{G}$. Let*

$$q : \mathscr{B} \times_\alpha G \to \mathcal{G} \times_\beta G$$

denote the associated semidirect-product Fell bundle over the semidirect-product groupoid as defined in Section 6, and let $\overline{\alpha} : G \to \operatorname{Aut} C^*(\mathcal{G}, \mathscr{B})$ denote the concomitant action described in Remark 6.1. Then there is a unique isomorphism

$$\sigma : C^*(\mathcal{G}, \mathscr{B}) \rtimes_{\overline{\alpha}} G \longrightarrow C^*(\mathcal{G} \times_\beta G, \mathscr{B} \times_\alpha G)$$

such that if $f \in \Gamma_c(\mathcal{G}; \mathscr{B})$ and $g \in C_c(G)$ then $\sigma(i_{\mathscr{B}}(f) i_G(g))$ is the continuous compactly supported section of $\mathscr{B} \times_\alpha G$ given by

$$\sigma\big(i_{\mathscr{B}}(f) i_G(g)\big)(x,t) = \big(f(x) g(t) \Delta(t)^{\frac{1}{2}}, t\big). \tag{7.1}$$

Proof. Uniqueness is immediate from density. For existence, we will obtain σ as the integrated form of a covariant homomorphism $(\sigma_{\mathscr{B}}, \sigma_G)$ of $(C^*(\mathcal{G}, \mathscr{B}), G, \overline{\alpha})$ into $M(C^*(\mathcal{G} \times_\beta G, \mathscr{B} \times_\alpha G))$ such that

$$\sigma_{\mathscr{B}}(f) \sigma_G(g) = f \boxtimes (\Delta^{\frac{1}{2}} g) \in \Gamma_c(\mathcal{G} \times_\beta G; \mathscr{B} \times_\alpha G) \tag{7.2}$$

for $f \in \Gamma_c(\mathcal{G}; \mathscr{B})$ and $g \in C_c(G)$. It will follow that σ maps $C^*(\mathcal{G}, \mathscr{B}) \rtimes_{\overline{\alpha}} G$ into $C^*(\mathcal{G} \times_\beta G, \mathscr{B} \times_\alpha G)$, satisfies (7.1), and is surjective because the sections in (7.2) have inductive-limit-dense span in $\Gamma_c(\mathcal{G} \times_\beta G; \mathscr{B} \times_\alpha G)$.

To define $\sigma_{\mathscr{B}}$, we appeal to Proposition 1.7, viewing $C^*(\mathcal{G} \times_\beta G, \mathscr{B} \times_\alpha G)$ as a right Hilbert module over itself, with dense subspace $\Gamma_c(\mathcal{G} \times_\beta G; \mathscr{B} \times_\alpha G)$.

For $f \in \Gamma_c(\mathcal{G};\mathscr{B})$ we define a linear operator $\sigma_{\mathscr{B}}(f)$ on $\Gamma_c(\mathcal{G}\times_\beta G;\mathscr{B}\times_\alpha G)$ by

$$\big(\sigma_{\mathscr{B}}(f)h\big)(y,t) = \int_{\mathcal{G}} \big(f(x)h_1(x^{-1}y,t),t\big)\,d\lambda^{r(y)}(x).$$

Seeing that $\sigma_{\mathscr{B}} : \Gamma_c(\mathcal{G};\mathscr{B}) \to \mathrm{Lin}(\Gamma_c(\mathcal{G}\times_\beta G;\mathscr{B}\times_\alpha G))$ is an algebra homomorphism is straightforward: for $f,g \in \Gamma_c(\mathcal{G};\mathscr{B})$ and $h \in \Gamma_c(\mathcal{G}\times_\beta G;\mathscr{B}\times_\alpha G)$ we have

$$\begin{aligned}\big(\sigma_{\mathscr{B}}(f)\sigma_{\mathscr{B}}(g)h\big)(y,t) &= \int_{\mathcal{G}} \big(f(x)\big(\sigma_{\mathscr{B}}(g)h\big)_1(x^{-1}y,t),t\big)\,d\lambda^{r(y)}(x)\\ &= \int_{\mathcal{G}}\int_{\mathcal{G}} \big(f(x)g(z)h_1(z^{-1}x^{-1}y,t),t\big)\,d\lambda^{s(x)}(z)\,d\lambda^{r(y)}(x),\end{aligned}$$

which, after using Fubini and sending $z \mapsto x^{-1}z$, is

$$\begin{aligned}&= \int_{\mathcal{G}}\int_{\mathcal{G}} \big(f(x)g(x^{-1}z)h_1(z^{-1}y,t),t\big)\,d\lambda^{r(y)}(x)\,d\lambda^{r(y)}(z)\\ &= \int_{\mathcal{G}} \big(f * g(z)h_1(z^{-1}y,t),t\big)\,d\lambda^{r(y)}(z)\\ &= \big(\sigma_{\mathscr{B}}(f*g)h\big)(y,t).\end{aligned}$$

Thus, it remains to verify that $\sigma_{\mathscr{B}}$ satisfies (i), (ii) and (iii) of Proposition 1.7.

To check (i), we compute as follows. For $h,k \in \Gamma_c(\mathcal{G}\times_\beta G;\mathscr{B}\times_\alpha G)$, we have

$$\begin{aligned}&\big\langle \sigma_{\mathscr{B}}(f)h\,,\,k\big\rangle_1(x,t) = \big((\sigma_{\mathscr{B}}(f)h)^* * k\big)_1(x,t)\\ &= \int_{\mathcal{G}}\int_G (\sigma_{\mathscr{B}}(f)h)^*_1(y,s)\alpha_s\big(k_1(\beta_s^{-1}(y^{-1}x),s^{-1}t)\big)\,ds\,d\lambda^{r(x)}(y)\\ &= \int_{\mathcal{G}}\int_G \alpha_s\big((\sigma_{\mathscr{B}}(f)h)_1(\beta_s^{-1}(y^{-1}),s^{-1})\big)^*\alpha_s\big(k_1(\beta_s^{-1}(y^{-1}x),s^{-1}t)\big)\,ds\,d\lambda^{r(x)}(y)\\ &= \int_{\mathcal{G}}\int_G\int_{\mathcal{G}} \alpha_s\big(f(z)h_1(z^{-1}\beta_s^{-1}(y^{-1}),s^{-1})\big)^*\alpha_s\big(k_1(\beta_s^{-1}(y^{-1}x),s^{-1}t)\big)\\ &\qquad\qquad d\lambda^{r(\beta_s^{-1}(y^{-1}))}(z)\,ds\,d\lambda^{r(x)}(y)\\ &= \int_{\mathcal{G}}\int_G\int_{\mathcal{G}} \alpha_s\Big(h_1(z^{-1}\beta_s^{-1}(y^{-1}),s^{-1})^*f(z)^*k_1(\beta_s^{-1}(y^{-1}x),s^{-1}t)\Big)\\ &\qquad\qquad d\lambda^{r(\beta_s^{-1}(y^{-1}))}(z)\,ds\,d\lambda^{r(x)}(y)\end{aligned}$$

which, after $z \mapsto \beta_s^{-1}(y^{-1})z$ for fixed y, is

$$\begin{aligned}&= \int_{\mathcal{G}}\int_G\int_{\mathcal{G}} \alpha_s\Big(h_1(z^{-1},s^{-1})^*f(\beta_s^{-1}(y^{-1})z)^*k_1(\beta_s^{-1}(y^{-1}x),s^{-1}t)\Big)\\ &\qquad\qquad d\lambda^{s(\beta_s^{-1}(y^{-1}))}(z)\,ds\,d\lambda^{r(x)}(y)\end{aligned}$$

which, by invariance of the action β (in the variable z), is

$$= \int_{\mathcal{G}} \int_G \int_{\mathcal{G}} \alpha_s\Big(h_1(\beta_s^{-1}(z^{-1}), s^{-1})^* f(\beta_s^{-1}(y^{-1}z))^* k_1(\beta_s^{-1}(y^{-1}x), s^{-1}t)\Big) \\ \lambda^{s(y^{-1})}(z)\, ds\, d\lambda^{r(x)}(y)$$

which, by Fubini, is

$$= \int_{\mathcal{G}} \int_G \int_{\mathcal{G}} \alpha_s\Big(h_1(\beta_s^{-1}(z^{-1}), s^{-1})^* f(\beta_s^{-1}(y^{-1}z))^* k_1(\beta_s^{-1}(y^{-1}x), s^{-1}t)\Big) \\ d\lambda^{r(x)}(y)\, ds\, \lambda^{r(x)}(z)$$

which, after $y \mapsto zy$ for fixed z, is

$$= \int_{\mathcal{G}} \int_G \int_{\mathcal{G}} \alpha_s\Big(h_1(\beta_s^{-1}(z^{-1}), s^{-1})^* f(\beta_s^{-1}(y^{-1}))^* k_1(\beta_s^{-1}(y^{-1}z^{-1}x), s^{-1}t)\Big) \\ d\lambda^{s(z)}(y)\, ds\, \lambda^{r(x)}(z)$$

which, by invariance of β (in y), is

$$\begin{aligned} &= \int_{\mathcal{G}} \int_G \int_{\mathcal{G}} \alpha_s\Big(h_1(\beta_s^{-1}(z^{-1}), s^{-1})^* f(y^{-1})^* k_1(y^{-1}\beta_s^{-1}(z^{-1}x), s^{-1}t)\Big) \\ &\qquad\qquad d\lambda^{s(\beta_s^{-1}(z))}(y)\, ds\, \lambda^{r(x)}(z) \\ &= \int_{\mathcal{G}} \int_G \int_{\mathcal{G}} \alpha_s\big(h_1(\beta_s^{-1}(z^{-1}), s^{-1})^*\big)\alpha_s\big(f^*(y)k_1(y^{-1}\beta_s^{-1}(z^{-1}x), s^{-1}t)\big) \\ &\qquad\qquad d\lambda^{r(\beta_s^{-1}(z^{-1}x))}(y)\, ds\, d\lambda^{r(x)}(z) \\ &= \int_{\mathcal{G}} \int_G h_1^*(z,s)\alpha_s\big((\sigma_{\mathscr{B}}(f^*)k)_1(\beta_s^{-1}(z^{-1}x), s^{-1}t)\big)\, ds\, d\lambda^{r(x)}(z) \\ &= \big(h^* * (\sigma_{\mathscr{B}}(f^*)k)\big)_1(x,t) \\ &= \big\langle h\,,\, \sigma_{\mathscr{B}}(f^*)k\big\rangle_1(x,t). \end{aligned}$$

To check the continuity condition (ii) of Proposition 1.7, it suffices to show that if $L \subseteq \mathcal{G}$ is compact and $f_i \to 0$ uniformly in $\Gamma_L(\mathcal{G}, \mathscr{B})$, then for each $h, k \in \Gamma_c(\mathcal{G} \times_\beta G; \mathscr{B} \times_\alpha G)$ there exists a compact set $K \subseteq \mathcal{G} \times_\beta G$ such that $\big\langle \sigma_{\mathscr{B}}(f_i)h\,,\, k\big\rangle \to 0$ uniformly in $\Gamma_K(\mathcal{G} \times_\beta G, \mathscr{B} \times_\alpha G)$. Using continuity of the action of G on $\mathcal{G}$, it is routine to verify that for any such h and k there exists a compact set K such that $\operatorname{supp}\big\langle \sigma_{\mathscr{B}}(f_i)h\,,\, k\big\rangle \subseteq K$ for every i. Then, to verify uniform convergence, we notice that for each i,

$$\|\big\langle \sigma_{\mathscr{B}}(f_i)h\,,\, k\big\rangle\|_\infty \le M\|f_i\|_\infty \|h\|_\infty \|k\|_\infty,$$

where $M = \sup_{u \in \mathcal{G}^{(0)}} \lambda^{(e,u)}(K)$.

For the nondegeneracy condition (iii) of Proposition 1.7, note that if $f, g \in \Gamma_c(\mathcal{G}; \mathscr{B})$ and $h \in C_c(G)$, then

$$\sigma_{\mathscr{B}}(f)(g \boxtimes h) = (f * g) \boxtimes h,$$

where $g \boxtimes h \in \Gamma_c(\mathcal{G} \times_\beta G; \mathscr{B} \times_\alpha G)$ is defined by $(g \boxtimes h)(x,t) = (g(x)h(t), t)$. Letting f run through an approximate identity $\{f_i\}$ for $\Gamma_c(\mathcal{G}; \mathscr{B})$ in the inductive limit topology (see [16, Proposition 6.10]), we have $f_i * g \to g$, hence $(f_i * g) \boxtimes h \to g \boxtimes h$, both nets converging in the inductive limit topology. Since such sections $g \boxtimes h$ have dense span in $\Gamma_c(\mathcal{G} \times_\beta G; \mathscr{B} \times_\alpha G)$, hence in $C^*(\mathcal{G} \times_\beta G, \mathscr{B} \times_\alpha G)$, nondegeneracy follows.

Now we conclude from Proposition 1.7 that $\sigma_{\mathscr{B}}$ extends to a nondegenerate $*$-homomorphism of $C^*(\mathcal{G}, \mathscr{B})$ into $M(C^*(\mathcal{G} \times_\beta G, \mathscr{B} \times_\alpha G))$, as required.

We now turn to σ_G. Fix $s \in G$, and for each $h \in \Gamma_c(\mathcal{G} \times_\beta G; \mathscr{B} \times_\alpha G)$, define $\sigma_G(s)h \in \Gamma_c(\mathcal{G} \times_\beta G; \mathscr{B} \times_\alpha G)$ by

$$\big(\sigma_G(s)h\big)(x,t) = \big(\alpha_s\big(h_1(\beta_s^{-1}(x), s^{-1}t)\big)\Delta(s)^{\frac{1}{2}}, t\big).$$

Then for $h, k \in \Gamma_c(\mathcal{G} \times_\beta G; \mathscr{B} \times_\alpha G)$ we have

$$\begin{aligned}
\big\langle \sigma_G(s)h\,,\,\sigma_G(s)k\big\rangle_1(x,t) &= \big((\sigma_G(s)h)^* * (\sigma_G(s)k)\big)_1(x,t)\\
&= \int_{\mathcal{G}}\int_G (\sigma_G(s)h)_1^*(y,r)\alpha_r\big((\sigma_G(s)k)_1(\beta_r^{-1}(y^{-1}x), r^{-1}t)\big)\,dr\,d\lambda^{r(x)}(y)\\
&= \int_{\mathcal{G}}\int_G \alpha_r\big((\sigma_G(s)h)_1(\beta_r^{-1}(y^{-1}), r^{-1})^*\big)\\
&\qquad\qquad \alpha_r\big((\sigma_G(s)k)_1(\beta_r^{-1}(y^{-1}x), r^{-1}t)\big)\,dr\,d\lambda^{r(x)}(y)\\
&= \int_{\mathcal{G}}\int_G \alpha_r\big(\alpha_s(h_1(\beta_s^{-1}(\beta_r^{-1}(y^{-1})), s^{-1}r^{-1})^*\Delta(s)^{\frac{1}{2}}\big)\\
&\qquad\qquad \alpha_r\big(\alpha_s(k_1(\beta_s^{-1}(\beta_r^{-1}(y^{-1}x)), s^{-1}r^{-1}t)\Delta(s)^{\frac{1}{2}}\big)\,dr\,d\lambda^{r(x)}(y)\\
&= \int_{\mathcal{G}}\int_G \alpha_{rs}\big(h_1(\beta_{rs}^{-1}(y^{-1}), (rs)^{-1})^*\big)\\
&\qquad\qquad \alpha_{rs}\big(k_1(\beta_{rs}^{-1}(y^{-1}x), (rs)^{-1}t)\big)\,\Delta(s)dr\,d\lambda^{r(x)}(y)
\end{aligned}$$

which, after $r \mapsto rs^{-1}$, is

$$\begin{aligned}
&= \int_{\mathcal{G}}\int_G \alpha_r\big(h_1(\beta_r^{-1}(y^{-1}), r^{-1})^*\big)\alpha_r\big(k_1(\beta_r^{-1}(y^{-1}x), r^{-1}t)\big)\,dr\,d\lambda^{r(x)}(y)\\
&= \int_{\mathcal{G}}\int_G h_1^*(y,r)\alpha_r\big(k_1(\beta_r^{-1}(y^{-1}x), r^{-1}t)\big)\,dr\,d\lambda^{r(x)}(y)\\
&= (h^* * k)_1(x,t) = \langle h\,,\,k\rangle_1(x,t).
\end{aligned}$$

Since we clearly have $\sigma_G(s)\sigma_G(t) = \sigma_G(st)$ and $\sigma_G(e)$ is the identity, it follows that $\sigma_G(s)$ defines a unitary in $M(C^*(\mathcal{G} \times_\beta G, \mathscr{B} \times_\alpha G))$.

To see the resulting homomorphism $\sigma_G : G \to M(C^*(\mathcal{G} \times_\beta G, \mathscr{B} \times_\alpha G))$ is strictly continuous, it suffices (by [21, Corollary C.8]) to show that if $s_i \to e$ in G and $h \in \Gamma_c(\mathcal{G} \times_\beta G; \mathscr{B} \times_\alpha G)$ then $\sigma_G(s_i)h \to h$ in the inductive limit topology. Without loss of generality all the s_i's are contained in some compact neighborhood V of e. Choose compact sets $K \subseteq \mathcal{G}$ and $L \subseteq G$ such that $\operatorname{supp} h \subseteq K \times L$. Then for each i we have

$$\operatorname{supp} \sigma_G(s_i)h \subseteq \beta_{V^{-1}}(K) \times V^{-1}L,$$

which is compact by continuity of the action β. The uniform continuity of h and continuity of the actions α and β guarantee that

$$\lim_i \alpha_{s_i}\big(h_1(\beta_{s_i^{-1}}(x), s_i^{-1}t)\big) = h_1(x,t)$$

uniformly in (x,t), so $\sigma_G(s_i)h \to h$ uniformly. Thus $\sigma_G(s_i)h \to h$ in the inductive limit topology.

Now we verify that the pair $(\sigma_{\mathscr{B}}, \sigma_G)$ is covariant for $(C^*(\mathcal{G},\mathscr{B}), G, \overline{\alpha})$. If $f \in \Gamma_c(\mathcal{G};\mathscr{B})$ and $s \in G$, then for each $h \in \Gamma_c(\mathcal{G} \times_\beta G; \mathscr{B} \times_\alpha G)$ and $(y,t) \in \mathcal{G} \times_\beta G$, we have

$$\begin{aligned}
&\big(\sigma_G(s)\sigma_{\mathscr{B}}(f)h\big)_1(y,t)\\
&\qquad = \alpha_s\big((\sigma_{\mathscr{B}}(f)h)_1(\beta_s^{-1}(y), s^{-1}t)\big)\Delta(s)^{\frac12}\\
&\qquad = \int_{\mathcal{G}} \alpha_s\big(f(x)h_1(x^{-1}\beta_s^{-1}(y), s^{-1}t)\big)\Delta(s)^{\frac12}\, d\lambda^{r(\beta_s^{-1}(y))}(x)
\end{aligned}$$

which, by invariance of β, is

$$\begin{aligned}
&= \int_{\mathcal{G}} \alpha_s\big(f(\beta_s^{-1}(x))h_1(\beta_s^{-1}(x^{-1}y), s^{-1}t)\big)\Delta(s)^{\frac12}\, d\lambda^{r(y)}(x)\\
&= \int_{\mathcal{G}} \overline{\alpha}_s(f)(x)\big(\sigma_G(s)h\big)_1(x^{-1}y,t)\, d\lambda^{r(y)}(x)\\
&= \big(\sigma_{\mathscr{B}}(\overline{\alpha}_s(f))\sigma_G(s)h\big)_1(y,t).
\end{aligned}$$

Next we verify (7.2): for $h \in \Gamma_c(\mathcal{G} \times_\beta G; \mathscr{B} \times_\alpha G)$ and $(y,s) \in \mathcal{G} \times_\beta G$, we have

$$\begin{aligned}
&\big(\sigma_{\mathscr{B}}(f)\sigma_G(g)h\big)(y,s)\\
&= \left(\int_{\mathcal{G}} f(x)\big(\sigma_G(g)h\big)_1(x^{-1}y,s)\, d\lambda^{r(y)}(x), s\right)\\
&= \left(\int_{\mathcal{G}}\int_G f(x)g(t)\alpha_t\Big(h_1\big(\beta_t^{-1}(x^{-1}y), t^{-1}s\big)\Big)\Delta(t)^{\frac12}\, dt\, d\lambda^{r(y)}(x), s\right)\\
&= \int_{\mathcal{G}}\int_G \big(f(x)g(t)\alpha_t\big(h_1(\beta_t^{-1}(x^{-1}y), t^{-1}s)\big), s\big)\Delta(t)^{\frac12}\, dt\, d\lambda^{r(y)}(x)\\
&= \int_{\mathcal{G}}\int_G \big(f(x)g(t)\Delta(t)^{\frac12}, t\big)\big(h_1(\beta_t^{-1}(x^{-1}y), t^{-1}s), t^{-1}s\big)\, dt\, d\lambda^{r(y)}(x)\\
&= \int_{\mathcal{G}}\int_G \big(f \boxtimes (\Delta^{\frac12}g)\big)(x,t)h(\beta_t^{-1}(x^{-1}y), t^{-1}s)\, dt\, d\lambda^{r(y)}(x)\\
&= \big(\big(f \boxtimes (\Delta^{\frac12}g)\big) * h\big)(y,s).
\end{aligned}$$

As outlined at the start of the proof, it follows from the above that the integrated form $\sigma = \sigma_{\mathscr{B}} \rtimes \sigma_G$ maps $C^*(\mathcal{G},\mathscr{B}) \rtimes_{\overline{\alpha}} G$ onto $C^*(\mathcal{G} \times_\beta G, \mathscr{B} \times_\alpha G)$.

To show injectivity of σ, it suffices to find a left inverse. We will begin by constructing a $*$-homomorphism $\tau : \Gamma_c(\mathcal{G} \times_\beta G; \mathscr{B} \times_\alpha G) \to C_c(G, \Gamma_c(\mathcal{G}; \mathscr{B}))$ which is continuous for the inductive limit topologies on each algebra, where (of course) $\Gamma_c(\mathcal{G}; \mathscr{B})$ is also given the inductive limit topology. Then, the composition

$$\Gamma_c(\mathcal{G}\times_\beta G; \mathscr{B}\times_\alpha G) \xrightarrow{\tau} C_c(G, \Gamma_c(\mathcal{G}; \mathscr{B})) \to C_c(G, C^*(\mathcal{G}, \mathscr{B})) \to C^*(\mathcal{G}, \mathscr{B})\times_{\overline{\alpha}} G$$

will be continuous from the inductive limit topology to the C^*-norm topology, and hence, by Proposition 1.8, will extend to a homomorphism, which we will also denote by τ, of $C^*(\mathcal{G}\times_\beta G, \mathscr{B}\times_\alpha G)$ into $C^*(\mathcal{G}, \mathscr{B}) \rtimes_{\overline{\alpha}} G$. Finally, we will check that $\tau \circ \sigma = \mathrm{id}$ on generators, and this will suffice.

For $h \in \Gamma_c(\mathcal{G} \times_\beta G; \mathscr{B} \times_\alpha G)$ and $t \in G$, it is clear that the rule

$$x \mapsto h_1(x, t)\Delta(t)^{-\frac{1}{2}}$$

defines an element $\tau(h)(t)$ of $\Gamma_c(\mathcal{G}; \mathscr{B})$. The discussion in [11, II.15.19] shows that the map $t \mapsto h_1(\cdot, t)$ from G into $\Gamma_c(\mathcal{G}; \mathscr{B})$ is inductive-limit continuous, and it follows that $t \mapsto \tau(h)(t)$ defines an inductive-limit continuous map $\tau(h)$ from G to $\Gamma_c(\mathcal{G}; \mathscr{B})$. Since $\tau(h)$ obviously has compact support, we therefore have $\tau(h) \in C_c(G, \Gamma_c(\mathcal{G}; \mathscr{B}))$, with

$$\tau(h)(t)(x) = h_1(x, t)\Delta(t)^{-\frac{1}{2}} \quad \text{for } t \in G \text{ and } x \in \mathcal{G}.$$

Now the rule $h \mapsto \tau(h)$ gives a map τ with domain $\Gamma_c(\mathcal{G} \times_\beta G; \mathscr{B} \times_\alpha G)$ which is clearly linear. To show that τ is continuous for the inductive limit topologies, it suffices to show that if $K \subseteq \mathcal{G}$ and $L \subseteq G$ are compact and $\{h_i\}$ is a net in $\Gamma_{K\times L}(\mathscr{B} \times_\alpha G)$ converging uniformly to 0, then $\tau(h_i) \to 0$ in the inductive limit topology of $C_c(G, \Gamma_c(\mathcal{G}; \mathscr{B}))$. Since $\operatorname{supp} \tau(h_i) \subseteq L$ for all i, it suffices to show that $\tau(h_i) \to 0$ uniformly. But this is obvious, since $h_i \to 0$ uniformly.

Next we show that τ is a $*$-homomorphism. For $h, k \in \Gamma_c(\mathcal{G}\times_\beta G; \mathscr{B}\times_\alpha G)$ we can use the argument[9] of [24, Lemma 1.108] to conclude that $\tau(h) * \tau(k)$, which is *a priori* an element of $C_c(G, C^*(\mathcal{G}, \mathscr{B}))$, lies in $C_c(G, \Gamma_c(\mathcal{G}; \mathscr{B}))$ and that we can pass "evaluation at x" through the integral in the third line of the following computation:

$$\begin{aligned}
&\big(\tau(h) * \tau(k)\big)(t)(x) \\
&\quad = \left(\int_G \tau(h)(s) * \overline{\alpha}_s\big(\tau(k)(s^{-1}t)\big)\, ds\right)(x) \\
&\quad = \int_G \big(\tau(h)(s) * \overline{\alpha}_s\big(\tau(k)(s^{-1}t)\big)\big)(x)\, ds \\
&\quad = \int_G \int_{\mathcal{G}} \tau(h)(s)(y)\overline{\alpha}_s\big(\tau(k)(s^{-1}t)\big)(y^{-1}x)\, d\lambda^{r(x)}(y)\, ds
\end{aligned}$$

[9]Lemma 1.108 of [24] as stated does not apply to a section algebra $\Gamma_c(\mathcal{G}; \mathscr{B})$ sitting inside a bundle C^*-algebra $C^*(\mathcal{G}, \mathscr{B})$, but it is easy to see that the argument gives the conclusion we need here.

$$= \int_G \int_{\mathcal{G}} h_1(y,s)\Delta(s)^{-\frac{1}{2}} \alpha_s\big(\tau(k)(s^{-1}t)\big(\beta_s^{-1}(y^{-1}x)\big)\big)\, d\lambda^{r(x)}(y)\, ds$$
$$= \int_G \int_{\mathcal{G}} h_1(y,s)\Delta(s)^{-\frac{1}{2}} \alpha_s\big(k_1\big(\beta_s^{-1}(y^{-1}x), s^{-1}t\big)\big)\Delta(s^{-1}t)^{-\frac{1}{2}}\, d\lambda^{r(x)}(y)\, ds$$
$$= (h * k)_1(x,t)\Delta(t)^{-\frac{1}{2}}$$
$$= \tau(h*k)(t)(x),$$

so τ is multiplicative. For the involution, we have

$$\begin{aligned}
\tau(h)^*(t)(x) &= \overline{\alpha}_t\big(\tau(h)(t^{-1})^*\Delta(t^{-1})\big)(x) \\
&= \overline{\alpha}_t\big(\tau(h)(t^{-1})^*\big)(x)\Delta(t^{-1}) \\
&= \alpha_t\big(\tau(h)(t^{-1})^*\big(\beta_{t^{-1}}(x)\big)\big)\Delta(t^{-1}) \\
&= \alpha_t\big(\tau(h)(t^{-1})\big(\beta_{t^{-1}}(x^{-1})\big)^*\big)\Delta(t^{-1}) \\
&= \alpha_t\big(h_1\big(\beta_{t^{-1}}(x^{-1}), t^{-1}\big)\Delta(t)^{\frac{1}{2}}\big)^*\Delta(t^{-1}) \\
&= \alpha_t\big(h_1\big(\beta_{t^{-1}}(x^{-1}), t^{-1}\big)\big)^*\Delta(t)^{-\frac{1}{2}} \\
&= (h^*)_1(x,t)\Delta(t)^{-\frac{1}{2}} \\
&= \tau(h^*)(t)(x).
\end{aligned}$$

Finally, we check $\tau \circ \sigma = \mathrm{id}$ on generators of the form $i_{\mathscr{B}}(f)i_G(g)$ for $f \in \Gamma_c(\mathcal{G};\mathscr{B})$ and $g \in C_c(G)$:

$$\begin{aligned}
\tau \circ \sigma\big(i_{\mathscr{B}}(f)i_G(g)\big)(t)(x) &= \tau\big(f \boxtimes (\Delta^{\frac{1}{2}}g)\big)(t)(x) \\
&= \big((f \boxtimes (\Delta^{\frac{1}{2}}g)\big)_1(x,t)\Delta(t)^{-\frac{1}{2}} \\
&= f(x)g(t) \\
&= \big(i_B(f)i_G(g)\big)(t)(x).
\end{aligned}$$ □

8. The canonical surjection is injective

Since Fell bundle C^*-algebras are generalizations of crossed products by actions, our main result generalizes the fact ([3, Proposition 3.4]) that the dual coaction on a crossed product is always maximal:

Theorem 8.1. *Let $\mathscr{A}$ be a separable Fell bundle over a group G, and let δ be the associated coaction of G on $C^*(G,\mathscr{A})$ as in Proposition 3.1. Then the canonical surjection*

$$\Phi : C^*(G,\mathscr{A}) \rtimes_\delta G \rtimes_{\hat{\delta}} G \to C^*(G,\mathscr{A}) \otimes \mathcal{K}(L^2(G))$$

is an isomorphism; hence δ is maximal.

To do prove Theorem 8.1, we will factor Φ into three isomorphisms, each involving the C^*-algebra of a Fell bundle over a groupoid. These isomorphisms will be presented in Propositions 8.2–8.4. We will use the following notation for canonical maps related to the double-crossed product $C^*(G,\mathscr{A})\rtimes_\delta G\rtimes_{\hat\delta} G$:

$$\begin{aligned} k_{\mathscr{A}} &= i_{C^*(G,\mathscr{A})\rtimes_\delta G}\circ j_{C^*(G,\mathscr{A})} : C^*(G,\mathscr{A})\to M(C^*(G,\mathscr{A})\rtimes_\delta G\rtimes_{\hat\delta} G)\\ k_{C(G)} &= i_{C^*(G,\mathscr{A})\rtimes_\delta G}\circ j_G : C_0(G)\to M(C^*(G,\mathscr{A})\rtimes_\delta G\rtimes_{\hat\delta} G)\\ k_G &= i_G : G\to M(C^*(A,\mathcal{G})\rtimes_\delta G\rtimes_{\hat\delta} G). \end{aligned}$$

Note that the double-crossed product is densely spanned by products of the form

$$k_{\mathscr{A}}(f)k_{C(G)}(g)k_G(h)\quad \text{for } f\in\Gamma_c(G;\mathscr{A}) \text{ and } g,h\in C_c(G).$$

Our first isomorphism involves an iterated product Fell bundle. As in Section 4, let $\mathscr{A}\times_{\mathrm{lt}} G$ be the transformation Fell bundle over the transformation groupoid $G\times_{\mathrm{lt}} G$. The group G acts on both $G\times_{\mathrm{lt}} G$ and $\mathscr{A}\times_{\mathrm{lt}} G$ by right translation in the second coördinate:

$$(\mathrm{id}_G\times\mathrm{rt})_r(s,t)=(s,tr^{-1})\quad\text{and}\quad (\mathrm{id}_{\mathscr{A}}\times\mathrm{rt})_r(a_s,t)=(a_s,tr^{-1}).$$

Thus we get a semidirect-product Fell bundle

$$\mathscr{A}\times_{\mathrm{lt}} G\times_{\mathrm{id}_{\mathscr{A}}\times\mathrm{rt}} G;$$

for simplicity, we will denote the corresponding semidirect-product groupoid $(G\times_{\mathrm{lt}} G)\times_{\mathrm{id}_G\times\mathrm{rt}} G$ by $\mathcal{S}$.

The action of G on $G\times_{\mathrm{lt}} G$ is invariant in the sense of Definition 6.3, since for each $(e,u)\in(G\times_{\mathrm{lt}} G)^0=\{e\}\times G$, $f\in C_c(G\times_{\mathrm{lt}} G)$, and $r\in G$ we have

$$\begin{aligned} \int_{G\times_{\mathrm{lt}} G} f\big((\mathrm{id}_G\times\mathrm{rt})_r(s,t)\big)\,d\lambda^{(e,u)}(s,t) &= \int_G f((\mathrm{id}_G\times\mathrm{rt})_r(s,s^{-1}u)\,ds\\ &= \int_G f(s,s^{-1}ur^{-1})\,ds\\ &= \int_{G\times_{\mathrm{lt}} G} f(s,t)\,d\lambda^{(e,ur^{-1})}(s,t)\\ &= \int_{G\times_{\mathrm{lt}} G} f(s,t)\,d\lambda^{(\mathrm{id}_G\times\mathrm{rt})_r(e,u)}. \end{aligned}$$

Therefore Proposition 6.4 gives a Haar system on $\mathcal{S}$, so we can form the Fell-bundle C^*-algebra $C^*(\mathcal{S},\mathscr{A}\times_{\mathrm{lt}} G\times_{\mathrm{id}_{\mathscr{A}}\times\mathrm{rt}} G)$.

Proposition 8.2. *There is an isomorphism*

$$\Theta : C^*(G,\mathscr{A})\rtimes_\delta G\rtimes_{\hat\delta} G\to C^*(\mathcal{S},\mathscr{A}\times_{\mathrm{lt}} G\times_{\mathrm{id}_{\mathscr{A}}\times\mathrm{rt}} G)$$

such that, for $f\in\Gamma_c(G;\mathscr{A})$ and $g,h\in C_c(G)$, the image

$$\Theta(k_{\mathscr{A}}(f)k_{C(G)}(g)k_G(h))$$

is in $\Gamma_c(\mathcal{S};\mathscr{A}\times_{\mathrm{lt}} G\times_{\mathrm{id}_{\mathscr{A}}\times\mathrm{rt}} G)$, with

$$\Theta\big(k_{\mathscr{A}}(f)k_{C(G)}(g)k_G(h)\big)(r,s,t)=\big(f(r)g(s)h(t)\Delta(rt)^{\frac12},s,t\big). \tag{8.1}$$

Proof. Theorem 5.1 gives an isomorphism

$$\theta : C^*(G,\mathscr{A}) \rtimes_\delta G \to C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G)$$

such that

$$\theta(j_{C^*(G,\mathscr{A})}(f)j_G(g)) = (\Delta^{\frac{1}{2}} f) \boxtimes g$$

for $f \in \Gamma_c(G;\mathscr{A})$ and $g \in C_c(G)$. We want to parlay this into our isomorphism Θ. First, we verify that θ is equivariant for the dual action of G on $C^*(G,\mathscr{A}) \rtimes_\delta G$ and the action $(\mathrm{id}_\mathscr{A} \times \mathrm{rt})^{\bar{}}$ coming from the action of G on $\mathscr{A} \times_{\mathrm{lt}} G$. Note that for $h \in \Gamma_c(G \times_{\mathrm{lt}} G; \mathscr{A} \times_{\mathrm{lt}} G)$,

$$(\mathrm{id}_\mathscr{A} \times \mathrm{rt})^{\bar{}}{}_s(h)(t,r) = \big(h_1(t,rs), r\big).$$

Thus for $f \in \Gamma_c(G;\mathscr{A})$, $g \in C_c(G)$ and $s \in G$ we have

$$\begin{aligned}(\mathrm{id}_\mathscr{A} \times \mathrm{rt})^{\bar{}}{}_s \circ \theta\big(j_{C^*(G,\mathscr{A})}(f)j_G(g)\big)(t,r) &= (\mathrm{id}_\mathscr{A} \times \mathrm{rt})^{\bar{}}{}_s\big((\Delta^{\frac{1}{2}} f) \boxtimes g\big)(t,r)\\ &= \big(\Delta(t)^{\frac{1}{2}} f(t) g(rs), t\big)\\ &= \big(\Delta(t)^{\frac{1}{2}} f(t)\mathrm{rt}_s(g)(r), t\big)\\ &= \big((\Delta^{\frac{1}{2}} f) \boxtimes \mathrm{rt}_s(g)\big)(t,r),\end{aligned}$$

so that

$$\begin{aligned}(\mathrm{id}_\mathscr{A} \times \mathrm{rt})^{\bar{}}{}_s \circ \theta\big(j_{C^*(G,\mathscr{A})}(f)j_G(g)\big) &= \theta\big(j_{C^*(G,\mathscr{A})}(f)j_G(\mathrm{rt}_s(g))\big)\\ &= \theta\Big(\hat{\delta}_s\big(j_{C^*(G,\mathscr{A})}(f)j_G(g)\big)\Big)\\ &= \theta \circ \hat{\delta}_s\big(j_{C^*(G,\mathscr{A})}(f)j_G(g)\big).\end{aligned}$$

Therefore we have an isomorphism

$$\theta \rtimes G : C^*(G,\mathscr{A}) \rtimes_\delta G \rtimes_{\hat{\delta}} G \to C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G) \times_{(\mathrm{id}_\mathscr{A} \times \mathrm{rt})^{\bar{}}} G.$$

Now, Theorem 7.1 gives an isomorphism

$$\sigma : C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G) \rtimes_{(\mathrm{id}_\mathscr{A} \times \mathrm{rt})^{\bar{}}} G \to C^*(\mathcal{S}, \mathscr{A} \times_{\mathrm{lt}} G \times_{\mathrm{id}_\mathscr{A} \times \mathrm{rt}} G)$$

taking a generator $i_{C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G)}(k) i_G(h)$ for $k \in \Gamma_c(G \times_{\mathrm{lt}} G; \mathscr{A} \times_{\mathrm{lt}} G)$ and $h \in C_c(G)$ to the section of $\mathscr{A} \times_{\mathrm{lt}} G \times_{\mathrm{id}_\mathscr{A} \times \mathrm{rt}} G$ given by

$$\sigma\big(i_{C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G)}(k)\mathrm{id}_G(h)\big)(r,s,t) = \big(k(r,s)h(t)\Delta(t)^{\frac{1}{2}}, s, t\big).$$

We now define Θ to be $\sigma \circ (\theta \rtimes G)$, and it only remains to verify (8.1). We have

$$\begin{aligned}&\Theta\big(k_\mathscr{A}(f) k_{C(G)}(g) k_G(h)\big)\\ &\quad = \sigma \circ (\theta \times G)\big(i_{C^*(A,\mathcal{G}) \times_\delta G}\big(j_{C^*(G,\mathscr{A})}(f)j_G(g)\big) i_G(h)\big)\\ &\quad = \sigma\big((\theta \times G)\big(i_{C^*(A,\mathcal{G}) \times_\delta G}\big(j_{C^*(G,\mathscr{A})}(f)j_G(g)\big) i_G(h)\big)\big)\\ &\quad = \sigma\big(i_{C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G)} \circ \theta\big(j_{C^*(G,\mathscr{A})}(f)j_G(g)\big) i_G(h)\big)\\ &\quad = \sigma\big(i_{C^*(G \times_{\mathrm{lt}} G, \mathscr{A} \times_{\mathrm{lt}} G)}\big((\Delta^{\frac{1}{2}} f) \boxtimes g\big) i_G(h)\big),\end{aligned}$$

so

$$\Theta\big(k_{\mathscr{A}}(f)k_{C(G)}(g)k_G(h)\big)(r,s,t) = \Big(\big((\Delta^{\frac12}f)\boxtimes g\big)(r,s)h(t)\Delta(t)^{\frac12},t\Big)$$
$$= \big(f(r)g(s)h(t)\Delta(rt)^{\frac12},s,t\big). \qquad \square$$

For our second isomorphism, we let $\mathcal{E}$ denote the equivalence relation groupoid $G\times G$ on the set G, and we endow $\mathcal{E}$ with the Haar system $\lambda^{(s,s)} = \delta_s\times\lambda$, where δ_s is the point mass at s, and λ is Haar measure on G. We then form the Cartesian product Fell bundle $\mathscr{A}\times\mathcal{E}$ over the Cartesian product groupoid $G\times\mathcal{E}$, in analogy with the group case in Section 2.

Proposition 8.3. *There is an isomorphism*

$$\Psi : C^*(\mathcal{S},\mathscr{A}\times_{\mathrm{lt}} G\times_{\mathrm{id}_{\mathscr{A}}\times\mathrm{rt}} G)\to C^*(G\times\mathcal{E},\mathscr{A}\times\mathcal{E})$$

such that, for $f\in\Gamma_c(\mathcal{S};\mathscr{A}\times_{\mathrm{lt}} G\times_{\mathrm{id}_{\mathscr{A}}\times\mathrm{rt}} G)$*, the image*

$$\Psi(f)\in\Gamma_c(G\times\mathcal{E};\mathscr{A}\times\mathcal{E}),$$

with

$$\Psi(f)(r,s,t) = (f_1(r,r^{-1}s,s^{-1}rt),s,t). \tag{8.2}$$

Proof. First notice that the groupoids $\mathcal{S} = (G\times_{\mathrm{lt}} G)\times_{\mathrm{id}_{\mathscr{A}}\times\mathrm{rt}} G$ and $G\times\mathcal{E}$ are isomorphic via the homeomorphism $\psi:\mathcal{S}\to G\times\mathcal{E}$ given by $\psi(r,s,t) = (r,rs,st)$. Furthermore, the homeomorphism $\Psi_0 : \mathscr{A}\times_{\mathrm{lt}} G\times_{\mathrm{id}_{\mathscr{A}}\times\mathrm{rt}} G\to\mathscr{A}\times\mathcal{E}$ given by $\Psi_0(a_r,s,t) = (a_r,rs,st)$ is a bundle map which covers ψ and is an isometric isomorphism on each fibre. Routine computations show that Ψ_0 also preserves the multiplication and involution. Hence we can define a $*$-isomorphism $\Psi:\Gamma_c(\mathcal{S};\mathscr{A}\times_{\mathrm{lt}} G\times_{\mathrm{id}_{\mathscr{A}}\times\mathrm{rt}} G)\to\Gamma_c(G\times\mathcal{E};\mathscr{A}\times\mathcal{E})$ by

$$\Psi(f)(r,s,t) = \Psi_0(f(\psi^{-1}(r,s,t)) = (f_1(r,r^{-1}s,s^{-1}rt),s,t).$$

Because Ψ_0 is a homeomorphism, Ψ is homeomorphic for the inductive limit topologies; therefore Ψ extends to an isomorphism of the bundle C^*-algebras which satisfies (8.2). $\square$

Proposition 8.4. *There is an isomorphism*

$$\Upsilon : C^*(G\times\mathcal{E},\mathscr{A}\times\mathcal{E})\to C^*(G,\mathscr{A})\otimes\mathcal{K}\big(L^2(G)\big)$$

such that, for every faithful nondegenerate representation $\pi : C^*(G,\mathscr{A})\to B(\mathcal{H})$*,* $f\in\Gamma_c(G\times\mathcal{E};\mathscr{A}\times\mathcal{E})$*, and* $\xi\in C_c(G,\mathcal{H})$*, we have*

$$\big((\pi\otimes\mathrm{id})\circ\Upsilon(f)\xi\big)(s) = \int_G\int_G \pi_0\big(f_1(r,s,t)\big)\xi(t)\Delta(r)^{-\frac12}\,dr\,dt, \tag{8.3}$$

where $\pi_0 = \pi\circ\iota$ *as in Lemma 1.3.*

The proposition depends on the following lemma, which may be of general interest. As above, $\mathscr{A}\times\mathcal{G}$ denotes the Cartesian product bundle over the Cartesian product groupoid $G\times\mathcal{G}$.

Lemma 8.5. *Let $\mathcal{G}$ be a second countable locally compact groupoid such that $C^*(\mathcal{G})$ is nuclear. There exists an an isomorphism $\omega : C^*(G\times\mathcal{G}, \mathscr{A}\times\mathcal{G}) \to C^*(G,\mathscr{A})\otimes C^*(\mathcal{G})$ such that*

$$\omega(g\boxtimes h) = (\Delta^{-\frac{1}{2}}g)\otimes h \quad \textit{for } g\in\Gamma_c(G;\mathscr{A}) \textit{ and } h\in C_c(\mathcal{G}),$$

where $g\boxtimes h\in\Gamma_c(G\times\mathcal{G};\mathscr{A}\times\mathcal{G})$ is defined by $(g\boxtimes h)(s,x) = (g(s)h(x),x)$.

Proof. For $a_t\in A_t$, define a linear operator $\rho_0^{\mathscr{A}}(a_t)$ on $\Gamma_c(G\times\mathcal{G};\mathscr{A}\times\mathcal{G})$ by[10]

$$\big(\rho_0^{\mathscr{A}}(a_t)h\big)_1(s,x) = a_t h_1(t^{-1}s,x)\Delta(t)^{\frac{1}{2}}.$$

Then a computation shows that

$$\big\langle \rho_0^{\mathscr{A}}(a_t)h\,,\,k\big\rangle = \big\langle h\,,\,\rho_0^{\mathscr{A}}(a_t^*)k\big\rangle \quad \text{for } h,k\in\Gamma_c(G\times\mathcal{G};\mathscr{A}\times\mathcal{G}),$$

where we are viewing $C^*(G\times\mathcal{G},\mathscr{A}\times\mathcal{G})$ as a right Hilbert module over itself with dense subspace $\Gamma_c(G\times\mathcal{G};\mathscr{A}\times\mathcal{G})$. Just as in the proof of Lemma 1.2, it follows that $\rho_0^{\mathscr{A}}(a_t)$ is bounded as an operator on $C^*(G\times\mathcal{G},\mathscr{A}\times\mathcal{G})$ with adjoint $\rho_0^{\mathscr{A}}(a_t^*)$, and that the rule $a_t\mapsto\rho_0^{\mathscr{A}}(a_t)$ therefore extends to a $*$-homomorphism $\rho_0^{\mathscr{A}}$ of $\mathscr{A}$ into $M(C^*(G\times\mathcal{G},\mathscr{A}\times\mathcal{G}))$. The proof that $\rho_0^{\mathscr{A}}$ is nondegenerate and strictly continuous also closely parallels the proof in Lemma 1.2 and will be omitted. Using Lemma 1.3, we get a nondegenerate homomorphism $\rho_{\mathscr{A}} : C^*(G,\mathscr{A})\to M(C^*(G\times\mathcal{G},\mathscr{A}\times\mathcal{G}))$.

Similarly, for $g\in C_c(\mathcal{G})$ we define an operator $\rho_{\mathcal{G}}(g)$ on $\Gamma_c(G\times\mathcal{G};\mathscr{A}\times\mathcal{G})$ by

$$(\rho_{\mathcal{G}}(g)h)_1(s,x) = \int_{\mathcal{G}} g(y)h_1(s,y^{-1}x)\,d\lambda^{r(x)}(y).$$

Another computation shows that

$$\langle\rho_{\mathcal{G}}(g)h\,,\,k\rangle = \big\langle h\,,\,\rho_{\mathcal{G}}(g^*)k\big\rangle \quad \text{for } h,k\in\Gamma_c(G\times\mathcal{G};\mathscr{A}\times\mathcal{G}).$$

Thus condition (i) of Proposition 1.7 is satisfied, and condition (ii) is not hard to check. Condition (iii) follows from the existence of an approximate identity for $C_c(\mathcal{G})$ in the inductive limit topology (cf. [15, Corollary 2.11]). Hence, $\rho_{\mathcal{G}}$ extends to a nondegenerate homomorphism of $C^*(\mathcal{G})$ into $M(C^*(G\times\mathcal{G},\mathscr{A}\times\mathcal{G}))$ by Proposition 1.7.

Clearly, $\rho_{\mathscr{A}}$ and $\rho_{\mathcal{G}}$ commute. Since $C^*(\mathcal{G})$ is nuclear, we obtain a homomorphism $\rho_{\mathscr{A}}\otimes\rho_{\mathcal{G}}$ of $C^*(G,\mathscr{A})\otimes C^*(\mathcal{G})$ into $M(C^*(G\times\mathcal{G},\mathscr{A}\times\mathcal{G}))$. If $g\in\Gamma_c(G;\mathscr{A})$, $h\in C_c(\mathcal{G})$ and $k\in\Gamma_c(G\times\mathcal{G};\mathscr{A}\times\mathcal{G})$, then an argument patterned after the proof of [24, Lemma 1.108] implies that $\rho_{\mathscr{A}}(g)\rho_{\mathcal{G}}(h)k$ is in $\Gamma_c(G\times\mathcal{G};\mathscr{A}\times\mathcal{G})$ and that evaluation at $(s,x)\in G\times\mathcal{G}$ "passes through the integral" in the second step in the next calculation:

[10]Although this construction almost exactly parallels that in Lemma 1.2, we need to insert a modular function here (compare with (1.5)) because, just as with (5.3) in the proof of Theorem 5.1, there is no modular function in the definition of the involution in the $*$-algebra associated to a Fell bundle over a groupoid.

$$\begin{aligned}
\big(\rho_{\mathscr{A}}(g)\rho_{\mathcal{G}}(h)k\big)_1(s,x)
&= \Big(\int_G \big(\rho_0^{\mathscr{A}}(g(t))(\rho_{\mathcal{G}}(h)k)\big)_1\,dt\Big)(s,x)\\
&= \int_G \rho_0^{\mathscr{A}}(g(t))\big(\rho_{\mathcal{G}}(h)k\big)_1(s,x)\,dt\\
&= \int_G g(t)\big(\rho_{\mathcal{G}}(h)k\big)_1(t^{-1}s,x)\Delta(t)^{\frac12}\,dt\\
&= \int_G\int_{\mathcal{G}} g(t)h(y)k_1(t^{-1}s,y^{-1}x)d\lambda^{r(x)}(y)\Delta(t)^{\frac12}\,dt\\
&= \int_{G\times\mathcal{G}} \big((\Delta^{\frac12}g)\boxtimes h\big)_1(t,y)k_1((t,y)^{-1}(s,x))\,d\lambda^{r(s,x)}(t,y)\\
&= \big(((\Delta^{\frac12}g)\boxtimes h)*k\big)_1(s,x).
\end{aligned}$$

Therefore

$$\rho_{\mathscr{A}}\otimes\rho_{\mathcal{G}}(g\otimes h) = (\Delta^{\frac12}g)\boxtimes h \quad \text{for } g\in\Gamma_c(G;\mathscr{A}) \text{ and } h\in C_c(\mathcal{G}). \tag{8.4}$$

Since such elements $(\Delta^{\frac12}g)\boxtimes h$ span a dense subspace of $\Gamma_c(G\times\mathcal{G};\mathscr{A}\times\mathcal{G})$, $\rho_{\mathscr{A}}\otimes\rho_{\mathcal{G}}$ maps $C^*(G,\mathscr{A})\otimes C^*(\mathcal{G})$ (into and) onto $C^*(G\times\mathcal{G},\mathscr{A}\times\mathcal{G})$.

Now fix a faithful nondegenerate representation π of $C^*(G,\mathscr{A})$ on a Hilbert space $\mathcal{H}$, and let $\pi_0 : \mathscr{A}\to B(\mathcal{H})$ be the nondegenerate representation whose integrated form is π (as in Lemma 1.3). Further let τ be a faithful nondegenerate representation of $C^*(\mathcal{G})$ on a Hilbert space $\mathcal{K}$. By the Disintegration Theorem ([23, Proposition 4.2] or [17, Theorem 7.8]), we can assume $\mathcal{K} = L^2(\mathcal{G}^{(0)}*\mathscr{V},\mu)$, where $\mathcal{G}^{(0)}*\mathscr{V}$ is a Borel Hilbert bundle and μ is a finite quasi-invariant Radon measure on $\mathcal{G}^{(0)}$, such that τ is the integrated form of a groupoid representation τ_0 of $\mathcal{G}$; thus

$$\big(\tau(h)\kappa\big)(u) = \int_{\mathcal{G}} h(x)\tau_0(x)\kappa(s(x))\Delta_{\mathcal{G}}(x)^{-\frac12}\,d\lambda^u(x) \quad \text{for } h\in C_c(\mathcal{G}), \tag{8.5}$$

where $\Delta_{\mathcal{G}}$ is the Radon–Nikodym derivative of ν^{-1} with respect to $\nu=\mu\circ\lambda$. Note that we can identify $(G\times\mathcal{G})^{(0)}$ with $\mathcal{G}^{(0)}$. Then we can form a Borel Hilbert bundle $\mathcal{G}^{(0)}*(\mathcal{H}\otimes\mathscr{V})$ such that $(\mathcal{H}\otimes\mathscr{V})(u)=\mathcal{H}\otimes V(u)$ and such that $L^2(\mathcal{G}^{(0)}*(\mathcal{H}\otimes\mathscr{V}),\mu)$ can be identified with $H\otimes L^2(\mathcal{G}^{(0)}*\mathscr{V},\mu)$. Then we can define a Borel $*$-functor (see [16, Definition 4.5]) Π from $\mathscr{A}\times\mathcal{G}$ to $\operatorname{End}(\mathcal{G}^{(0)}*(\mathcal{H}\otimes\mathscr{V}))$ by

$$\Pi(a,x) = \pi_0(a)\otimes\tau_0(x).$$

If μ_G is a left Haar measure on G, then we get a Haar system $\{\underline{\lambda}^u\}_{u\in\mathcal{G}^{(0)}}$ on $G\times\mathcal{G}$ via $\underline{\lambda}^u=\mu_G\times\lambda^u$. Notice that the Radon–Nikodym derivative of $\underline{\nu}^{-1}$ with respect to $\underline{\nu}:=\underline{\lambda}\circ\mu$ is given by $(s,x)\mapsto\Delta(s)\Delta_{\mathcal{G}}(x)$. Then [16, Proposition 4.10] implies that Π integrates up to a $*$-homomorphism $L:\Gamma_c(G\times\mathcal{G};\mathscr{A}\times\mathcal{G})\to B(\mathcal{H}\otimes L^2(\mathcal{G}^{(0)}*\mathscr{V},\mu))$ given by

$$(8.6)\quad L(f)(\eta\otimes\kappa)(u) = \int_G\int_{\mathcal{G}} \pi_0\big(f_1(t,x)\big)\eta\otimes\tau_0(x)\kappa(s(x))\Delta_{\mathcal{G}}(x)^{-\frac12}\Delta(t)^{-\frac12}\,d\lambda^u(x)\;dt$$

which extends to a representation of $C^*(G\times\mathcal{G},\mathscr{A}\times\mathcal{G})$.

Now, using (8.4), for $g\in\Gamma_c(G;\mathscr{A})$ and $h\in C_c(\mathcal{G})$ we have

$$\begin{aligned}
&L\big(\rho_{\mathscr{A}}\otimes\rho_{\mathcal{G}}(g\otimes h)\big)(\eta\otimes\kappa)(u)\\
&\quad= L\big((\Delta^{\frac12}g)\boxtimes h\big)(\eta\otimes\kappa)(u)\\
&\quad= \int_G\int_{\mathcal{G}}\pi_0\big(\Delta^{\frac12}(t)g(t)h(x)\big)\eta\otimes\tau_0(x)\kappa(s(x))\Delta_{\mathcal{G}}(x)^{-\frac12}\,d\lambda^u(x)\,\Delta(t)^{-\frac12}\,dt\\
&\quad= \Big(\int_G\pi_0(g(t))\eta\,dt\Big)\otimes\Big(\int_{\mathcal{G}}h(x)\tau_0(x)\kappa(s(x))\Delta_{\mathcal{G}}(x)^{-\frac12}\,d\lambda^u(x)\Big)\\
&\quad= \pi(g)\eta\otimes\big(\tau(h)\kappa\big)(u)\\
&\quad= (\pi\otimes\tau)(g\otimes h)(\eta\otimes\kappa)(u).
\end{aligned}$$

It follows that $L\circ(\rho_{\mathscr{A}}\otimes\rho_{\mathcal{G}})=\pi\otimes\tau$, and since the latter is a faithful representation of $C^*(G,\mathscr{A})\otimes C^*(\mathcal{G})$, it follows that $\rho_{\mathscr{A}}\otimes\rho_{\mathcal{G}}$ is faithful.

To complete the proof, we just let $\omega=(\rho_{\mathscr{A}}\otimes\rho_{\mathcal{G}})^{-1}$. Then ω is an isomorphism of $C^*(G\times\mathcal{G},\mathscr{A}\times\mathcal{G})$ onto $C^*(G,\mathscr{A})\otimes C^*(\mathcal{G})$ and satisfies

$$\omega(g\boxtimes h)=(\Delta^{-\frac12}g)\otimes h. \qquad\square$$

Proof of Proposition 8.4. Note that $C^*(\mathcal{E})=C^*(\mathcal{E},\lambda)\cong\mathcal{K}(L^2(G))$. In fact, since $\mathcal{E}$ is groupoid-equivalent to the trivial group, $C^*(\mathcal{E})$ is simple, so the representation $\tau: C^*(\mathcal{E})\to B(L^2(G))$ defined by

$$\big(\tau(h)\kappa\big)(s)=\int_G h(s,t)\kappa(t)\,dt\quad\text{for } h\in C_c(\mathcal{E})\text{ and }\kappa\in C_c(G)\subseteq L^2(G)$$

is an isomorphism onto $\mathcal{K}(L^2(G))$. In particular, $C^*(\mathcal{E})$ is nuclear, so by Lemma 8.5, we have an isomorphism

$$\Upsilon:=(\mathrm{id}\otimes\tau)\circ\omega: C^*(G\times\mathcal{E},\mathscr{A}\times\mathcal{E})\to C^*(G,\mathscr{A})\otimes\mathcal{K}\big(L^2(G)\big).$$

If we let $\mathcal{E}^{(0)}*\mathbb{C}$ be the trivial bundle $G\times\mathbb{C}$, then we can identify $L^2(G)$ with $L^2(\mathcal{E}^{(0)}*\mathbb{C},\lambda)$ in the obvious way. Notice also that λ is a quasi-invariant measure on $\mathcal{E}^{(0)}$ with $\Delta_{\mathcal{E}}\equiv 1$. Thus the representation τ is essentially presented as in (8.5). (The representation τ_0 acts on $(s,z)\in G\times\mathbb{C}$ by $\tau_0(t,s)(s,z)=(t,z)$.) Thus, in the current situation, (8.6) reduces to

$$\big(L(f)\xi\big)(s)=\int_G\int_G\pi_0\big(f_1(r,s,t)\big)\xi(t)\Delta(r)^{-\frac12}\,dr\,dt$$

for $\xi\in C_c(G,\mathcal{H})\subseteq\mathcal{H}\otimes L^2(G)$. Now (8.3) is easily verified using the observation (from the proof of Lemma 8.5) that $(\pi\otimes\tau)\circ\omega=L$. $\square$

Proof of Theorem 8.1. We need to show that Φ is injective, and to do this we will show that the diagram

$$\begin{array}{ccc} C^*(G,\mathscr{A})\times_\delta G\times_{\hat\delta} G & \xrightarrow[\cong]{\Theta} & C^*(\mathcal{S},\mathscr{A}\times_{\mathrm{lt}} G\times_{\mathrm{id}_{\mathscr{A}}\times\mathrm{rt}} G) \\ \Phi\big\downarrow & & \cong\big\downarrow\Psi \\ C^*(G,\mathscr{A})\otimes\mathcal{K}(L^2(G)) & \xleftarrow[\Upsilon]{\cong} & C^*(\Gamma\times\mathcal{E},\mathscr{A}\times\mathcal{E}) \end{array}$$

commutes, where Θ, Ψ, and Υ are the isomorphisms of Propositions 8.2, 8.3, and 8.4, respectively.

Let $\pi : C^*(G,\mathscr{A})\to B(\mathcal{H})$ be a faithful nondegenerate representation on a Hilbert space $\mathcal{H}$. Let $f\in\Gamma_c(G;\mathscr{A})$, $g,h,\kappa\in C_c(G)$, and $\eta\in\mathcal{H}$. Then, to show that the diagram commutes, the following computation suffices. Applying Proposition 8.4, we have

$$\begin{aligned}&\big((\pi\otimes\mathrm{id})\circ\Upsilon\circ\Psi\circ\Theta\big(k_{\mathscr{A}}(f)k_{C(G)}(g)k_G(h)\big)(\eta\otimes\kappa)\big)(s)\\ &=\int_G\int_G \pi_0\big(\Psi\circ\Theta\big(k_{\mathscr{A}}(f)k_{C(G)}(g)k_G(h)\big)_1(r,s,t)\big)(\eta\otimes\kappa)(t)\Delta(r)^{-\frac12}\,dr\,dt\end{aligned}$$

which, by Proposition 8.3, is

$$=\int_G\int_G \pi_0\big(\Theta\big(k_{\mathscr{A}}(f)k_{C(G)}(g)k_G(h)\big)_1(r,r^{-1}s,s^{-1}rt)\big)\eta\kappa(t)\Delta(r)^{-\frac12}\,dr\,dt$$

which, by Proposition 8.2, is

$$=\int_G\int_G \pi_0\big(f(r)\Delta(r)^{\frac12}g(r^{-1}s)h(s^{-1}rt)\Delta(s^{-1}rt)^{\frac12}\big)\eta\kappa(t)\Delta(r)^{-\frac12}\,dr\,dt$$

which, after using Fubini and sending $t\mapsto r^{-1}st$, is

$$=\int_G\int_G \pi(f(r))g(r^{-1}s)h(t)\Delta(t)^{\frac12}\eta\kappa(r^{-1}st)\,dt\,dr$$

which, since $\rho_t\kappa(r^{-1}s)=\kappa(r^{-1}st)\Delta(t)^{\frac12}$, is

$$\begin{aligned}&=\int_G\int_G \pi_0(f(r))\eta g(r^{-1}s)(\rho_t\kappa)(r^{-1}s)h(t)\,dt\,dr\\ &=\int_G \pi_0(f(r))\eta g(r^{-1}s)\big(\rho(h)\kappa\big)(r^{-1}s)\,dr\\ &=\int_G \pi_0(f(r))\eta\big(M_g\rho(h)\kappa\big)(r^{-1}s)\,dr\\ &=\int_G \pi_0(f(r))\eta\big(\lambda_rM_g\rho(h)\kappa\big)(s)\,dr\\ &=\int_G \big(\pi_0(f(r))\eta\otimes\lambda_rM_g\rho(h)\kappa\big)(s)\,dr\end{aligned}$$

$$
\begin{aligned}
&= \int_G \big(\pi_0(f(r)) \otimes \lambda_r M_g\rho(h)\big)(\eta\otimes\kappa)(s)\,dr\\
&= \int_G \big((\pi\otimes\lambda)(f(r)\otimes r)(1\otimes M_g\rho(h)\big)(\eta\otimes\kappa)(s)\,dr\\
&= (\pi\otimes \mathrm{id})\bigg(\int_G \big((\mathrm{id}\otimes\lambda)\circ\delta(f(r))(1\otimes M_g\rho(h)\big)(\eta\otimes\kappa)(s)\,dr\bigg)\\
&= (\pi\otimes\mathrm{id})(\mathrm{id}\otimes\lambda)\circ\delta(f)\big(1\otimes M_g\rho(h)\big)(s)\\
&= (\pi\otimes\mathrm{id})\circ\Phi\big(k_{\mathscr{A}}(f)k_{C(G)}(g)k_G(h)\big)(s). \qquad \square
\end{aligned}
$$

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