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Unions of arcs from Fourier partial sums

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Abstract. Elementary complex analysis and Hilbert space methods show that a union of at most n arcs on the circle is uniquely determined by the nth Fourier partial sum of its characteristic function. The endpoints of the arcs can be recovered from the coefficients appearing in the partial sum by solving two polynomial equations.

Contents

We let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and for any subset E of $\mathbb{T}$ and integer k we write

$$\widehat{E}(k) = \frac{1}{2\pi}\int_E e^{-ikt}\,dt$$

for the kth Fourier coefficient of the characteristic function χ_E of E. As bounded functions with the same sequence of Fourier coefficients agree almost everywhere, any subset E of $\mathbb{T}$ is determined up to a set of measure zero by the sequence $\widehat{E}(k)$. If E is known to have additional structure, the entire sequence may not be needed to recover E. Our present subject is a simple yet nontrivial illustration of this principle.

An *arc* is by definition a closed, connected, proper and nonempty subset of $\mathbb{T}$. We declare $\mathbb{T}$ along with the empty set to be a "union of 0 arcs."

Theorem 1. *If n is a nonnegative integer and E_1 and E_2 are unions of at most n arcs satisfying*

$$\widehat{E_1}(k) = \widehat{E_2}(k), \qquad 0 \le k \le n, \tag{1}$$

then $E_1 = E_2$.

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Thus a set E that is known to be a union of at most n arcs can be recovered *completely* from the nth Fourier partial sum of χ_E, regardless of any quantitative sense in which this partial sum fails to approximate χ_E. This stands in slight contrast to the well-known defects of Fourier partial sum approximation of functions with jump discontinuities, such as the Gibbs phenomenon (see, e.g., [4, Chapter 17]). Significantly, the property of the Fourier basis expressed by Theorem 1 is not shared by other orthonormal systems of functions on $\mathbb{T}$ (see §3).

Our proof of Theorem 1 exploits a connection between unions of arcs and certain rational functions — the *Blaschke products*, whose properties we recall in §1. Each Blaschke product has a nonnegative integer *order*. In §2 we construct an injection $E \mapsto b_E$ from the set of finite unions of arcs to the set of Blaschke products with the property that if E is a union of at most n arcs, then b_E has order at most n. This map has the property that if E_1 and E_2 satisfy (1), then b_{E_1} and b_{E_2} have the same nth order Taylor polynomial at 0. To prove Theorem 1 it then suffices to observe, as we do in §3, that a Blaschke product of order at most n is determined by its nth order Taylor polynomial.

With Theorem 1 in hand, one may ask how to recover E from a partial list of Fourier coefficients in an explicit fashion. This is the subject of §4, where we present an algorithm for testing whether or not a given tuple of complex numbers takes the form $(\widehat{E}(k))_{k=0}^{n}$ for a union E of at most n arcs, and for finding the endpoints of these arcs in terms of the Fourier coefficients in this case.

Perhaps because of its elementary nature, we have not found Theorem 1 explicitly stated in the literature, though it is known, and the literature abounds with theorems on the reconstruction of a function from partial knowledge of its Fourier transform. In [6], for example, it is shown that a function on $\mathbb{T}$ that is piecewise constant on a partition of $\mathbb{T}$ into m arcs may be recovered from its mth Fourier partial sum. (Note that Theorem 1 concludes slightly more from a stronger hypothesis.) And our method is by no means the only route to Theorem 1. It is possible to give a purely algebraic proof exploiting the fact that for $1 \le k \le n$ the numbers $\widehat{E}(k)$ are polynomials in the endpoints of the arcs of E.

We are indebted to Donald Sarason for many valuable discussions, and to Mihalis Kolountzakis for drawing our attention to [6].

1. Blaschke products

Definition. A (finite) Blaschke product is a function of the form

$$b(z) = \lambda \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z} \tag{2}$$

for some nonnegative integer n, some $\lambda \in \mathbb{T}$, and some $a_1, \ldots, a_n \in \mathbb{D}$. The nonnegative integer n is called the *order* of the Blaschke product.

If $n = 0$ we interpret the empty product as 1. The domain of a Blaschke product is either $\mathbb{T}$, $\mathbb{D}$, or the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$, depending on context. A Blaschke product is evidently a rational function that maps $\mathbb{T}$ to itself and has no poles in $\mathbb{D}$ (it suffices to check the case $n = 1$). It is well known that these properties characterize the Blaschke products.

Proposition 1. *If a rational function r maps $\mathbb{T}$ to itself and has no poles in $\mathbb{D}$, then it is a Blaschke product of order equal to the number n of zeros of r in $\mathbb{D}$, counted according to multiplicity.*

Proof. We induct on n. If $n = 0$, then $r = q^{-1}$ for some polynomial q; write $q(z) = \sum_{k=0}^{m} q_k z^k$ with $q_m \neq 0$. As $q(\mathbb{T}) \subseteq \mathbb{T}$ we have

$$q(z)^{-1} = \overline{q(z)} = \overline{q((\overline{z})^{-1})} = \sum_{k=0}^{m} \overline{q_k} z^{-k} = \frac{\sum_{k=0}^{m} \overline{q_k} z^{m-k}}{z^m}, \qquad z \in \mathbb{T},$$

so this holds for all nonzero $z \in \mathbb{D}$. As q has no zeros in $\mathbb{D}$, the extreme right hand side has no pole at 0; thus $m = 0$ and q is constant as desired.

If r has $n+1$ zeros in $\mathbb{D}$, choose one, a, and note that $r(z) \cdot (\frac{z-a}{1-\overline{a}z})^{-1}$ has n zeros in $\mathbb{D}$ and maps $\mathbb{T}$ to itself. □

Definition. If b is a Blaschke product, we let $U_b = \{z \in \mathbb{T} : \operatorname{Im} b(z) \geq 0\}$.

If the zeros of a Blaschke product are $a_1, \ldots, a_n$, we calculate from (2)

$$\frac{zb'(z)}{b(z)} = \sum_{j=1}^{n} \frac{1 - |a_j|^2}{|z - a_j|^2} > 0, \qquad z \in \mathbb{T},$$

so the argument of $b(e^{it})$ is strictly increasing in t. The argument principle implies that $b(e^{it})$ travels n times counterclockwise around $\mathbb{T}$ as t runs from 0 to 2π.

Corollary 1. *A Blaschke product b has order n if and only if U_b is a disjoint union of n arcs.*

This is the main reason we include $\mathbb{T}$ as a "union of 0 arcs."

2. Blaschke products from unions of arcs

Let $S = \{z \in \mathbb{C} : 0 \leq 2 \operatorname{Re} z \leq 1\}$ and let ϕ denote the function

$$\phi(z) = \frac{\exp(2\pi i(z - 1/4)) - 1}{\exp(2\pi i(z - 1/4)) + 1}.$$

It is easy to show (see, e.g., [2, §III.3]) that ϕ maps S bijectively onto $\overline{\mathbb{D}} \setminus \{\pm 1\}$, that ϕ restricts to an analytic bijection of the interior of S with $\mathbb{D}$, that ϕ maps the right boundary line of S onto $\{z \in \mathbb{T} : \operatorname{Im} z > 0\}$, and that ϕ maps the left boundary line of S onto $\{z \in \mathbb{T} : \operatorname{Im} z < 0\}$.

Proposition 2. *If E is a disjoint union of $n \geq 0$ arcs and h_E is given by*

$$h_E(z) = \frac{1}{2}\widehat{E}(0) + \sum_{k=1}^{\infty} \widehat{E}(k) z^k, \qquad z \in \mathbb{D}, \tag{3}$$

then h_E is an analytic map of $\mathbb{D}$ into S, and the function $\mathbb{D} \to \overline{\mathbb{D}}$ given by

$$b_E = \phi \circ h_E$$

extends uniquely to a Blaschke product $\overline{\mathbb{D}} \to \overline{\mathbb{D}}$ of order n satisfying $U_{b_E} = E$.

Using the formulas for ϕ and h_E one can show without much work that b_E is a rational function; the work in proving Proposition 2 is to establish that b_E has the mapping properties of Proposition 1, and hence is a Blaschke product, and to prove that $U_{b_E} = E$.

To motivate the argument, let us work nonrigorously for a moment. Formally we have the series expansion

$$\chi_E(z) = \sum_{k \in \mathbb{Z}} \widehat{E}(k) z^k, \qquad z \in \mathbb{T}, \tag{4}$$

and formal manipulation of the series (3) with $z \in \mathbb{T}$ then shows that

$$\chi_E(z) = h_E(z) + \overline{h_E(z)} = 2\operatorname{Re} h_E(z), \qquad z \in \mathbb{T}.$$

As χ_E is $\{0,1\}$ valued on $\mathbb{T}$, the maximum principle for harmonic functions then implies that h_E maps $\mathbb{D}$ into S, so $b_E = \phi \circ h_E$ maps $\overline{\mathbb{D}}$ into $\overline{\mathbb{D}}$ and sends the circle to itself. By Proposition 1 it follows that b_E is a Blaschke product; the equality $U_{b_E} = E$ comes from the mapping properties of ϕ on the boundary of S.

What makes this argument nonrigorous is that the series (4) does not converge for all $z \in \mathbb{T}$, and to equate χ_E with $2\operatorname{Re} h_E$ is to ignore the distinction between a discontinuous real valued function on $\mathbb{T}$ and a harmonic function on $\mathbb{D}$. To fill in these gaps, we need to use the actual connection between $2\operatorname{Re} h_E$ and χ_E — the former is the Poisson integral of the latter.

Proof. It is easily checked that (3) does define an analytic function on $\mathbb{D}$, e.g., because $\sum_{k=1}^{\infty} |\widehat{E}(k)|^2$ is convergent. One can then verify the identity

$$2h_E(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1 + ze^{-is}}{1 - ze^{-is}} \chi_E(e^{is})\, ds, \qquad z \in \mathbb{D}.$$

(Fix z, expand $\frac{1}{1-ze^{-is}}$ as a power series in z and interchange the sum and the integral.) Taking real parts it follows that for any $r \in [0,1)$ and any t

$$2\operatorname{Re} h_E(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} P_r(t-s)\chi_E(e^{is})\, ds, \tag{5}$$

where

$$P_r(t) = \operatorname{Re}\left(\frac{1 + re^{it}}{1 - re^{it}}\right)$$

is the *Poisson kernel*. It is elementary (see, e.g., [2, §X.2]) that for $r \in [0,1)$ the function P_r is nonnegative and satisfies $\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta)\, d\theta = 1$; thus (5) implies that $2\operatorname{Re} h_E(z) \in [0,1]$ for all $z \in \mathbb{D}$, and h_E maps $\mathbb{D}$ into S.

As r increases to 1, the P_r converge uniformly to the zero function on the complement of any neighborhood of 0 (see, e.g., [2, §X.2]). From (5) we conclude

$$\lim_{r\uparrow 1} 2\operatorname{Re} h_E(rz) = \chi_E(z) \tag{6}$$

at any $z \in \mathbb{T}$ at which χ_E is continuous. We conclude that for any such z the limit $\lim_{r\uparrow 1}(\phi \circ h_E)(rz)$ exists and is in $\mathbb{T}$.

We claim that $\phi \circ h_E$ is a rational function. In the case $n = 0$ this is clear. Otherwise, from the definition of ϕ it suffices to show that $\exp(2\pi i h_E)$ is a rational function, and for this it suffices to treat the case $n = 1$. In this case there are real numbers $a < b$ with $b - a < 2\pi$ satisfying $E = \{e^{it} : t \in [a,b]\}$, and $\widehat{E}(k) = \frac{\exp(-ikb) - \exp(-ika)}{-2\pi ik}$ for all $k > 0$. Let log denote the analytic logarithm defined on $\mathbb{C} \setminus \{z \in \mathbb{C} : z \leq 0\}$ that is real on the positive real axis and recall that $\log(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}$ for all $z \in \mathbb{D}$. A comparison of power series shows

$$h_E(z) = \frac{b-a}{4\pi} + \frac{1}{2\pi i}\Big(\log(1 - e^{-ib}z) - \log(1 - e^{-ia}z)\Big), \qquad z \in \mathbb{D},$$

so $\exp(2\pi i h_E) = \exp(i\frac{b-a}{2})\frac{1-e^{-ib}z}{1-e^{-ia}z}$ is rational.

At this point we know that $b_E = \phi \circ h_E$ is a rational function mapping $\mathbb{D}$ into itself. From (6) we deduce that b_E maps $\mathbb{T}$ into itself, so b_E is a Blaschke product by Proposition 1. The equality $U_{b_E} = E$ then follows from (6). The order of b_E is n by Corollary 1. □

If E_1 and E_2 are two unions of arcs related by (1), it is clear from the definition that h_{E_1} and h_{E_2} have the same nth order Taylor polynomial at 0. As ϕ is analytic at 0, the same is true of b_{E_1} and b_{E_2}.

Corollary 2. *If $n \geq 0$ and E_1 and E_2 are each unions of at most n arcs satisfying*

$$\widehat{E_1}(k) = \widehat{E_2}(k), \qquad 0 \leq k \leq n, \tag{7}$$

then there are Blaschke products b_1 and b_2, each of order at most n, satisfying $E_j = U_{b_j}$ for $j = 1,2$ and

$$\widehat{b_1}(k) = \widehat{b_2}(k), \qquad 0 \leq k \leq n. \tag{8}$$

3. Blaschke products from Toeplitz matrices

Fix a positive integer n for the remainder of this section. Our goal is to show that Blaschke products b_1 and b_2 having order at most n and satisfying

(8) must be equal. Let L^2 denote the space of square-integrable functions $\mathbb{T} \to \mathbb{C}$, with inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\overline{g(e^{it})}\,dt, \qquad f, g \in L^2.$$

(We identify two functions if they agree almost everywhere.)

For $0 \le k \le n$ we let ζ^k denote the function $\mathbb{T} \to \mathbb{C}$ given by $z \mapsto z^k$. It is immediate that $\{\zeta^k : 0 \le k \le n\}$ is an orthonormal subset of L^2. We denote its span, the space of analytic polynomials of degree at most n, by P; we let $\pi : L^2 \to P$ denote the orthogonal projection.

Definition. If $f : \mathbb{T} \to \mathbb{C}$ is bounded, $T_f : P \to P$ denotes the linear map given by

$$T_f\xi = \pi(f\xi), \qquad \xi \in P.$$

Here $f\xi$ is the pointwise product of f and ξ.

If we let $\|T_f\|$ denote the norm of T_f regarded as a linear operator on P and write $\|f\|_\infty = \sup_{z\in\mathbb{T}} |f(z)|$, it is clear that

$$\|T_f\| \le \|f\|_\infty$$

for any bounded f. It is also clear that for any such f

$$\langle T_f\zeta^k, \zeta^j\rangle = \widehat{f}(j-k), \qquad 0 \le j, k \le n,$$

so the matrix of T_f with respect to the orthonormal basis $\{\zeta^k : 0 \le k \le n\}$ is constant along its diagonals (it is a *Toeplitz matrix*).

If f is a Blaschke product, then f is analytic on $\overline{\mathbb{D}}$, so the matrix of T_f is lower triangular with first column $(\widehat{f}(k))_{k=0}^n$. Our hypothesis (8) is thus that $T_{b_1} = T_{b_2}$, and to deduce that $b_1 = b_2$ it suffices to show how to recover a Blaschke product b of order at most n from the operator T_b it induces on P.

Lemma 1. *If b is a Blaschke product of order at most n, then $\|T_b\| = 1$, and for any nonzero $r \in P$ satisfying $\|T_b r\| = \|r\|$ one has $T_b r = br$.*

Proof. There are nonzero polynomials p and q, each of degree at most n, satisfying $b = p/q$. Clearly $T_b q = p$, and as b maps $\mathbb{T}$ to itself, we have $|p(z)| = |q(z)|$ for all $z \in \mathbb{T}$, so $\|p\| = \|q\|$. We deduce that $\|T_b q\| = \|q\|$ and thus $\|T_b\| \ge 1$; since also $\|T_b\| \le \|b\|_\infty = 1$, we conclude $\|T_b\| = 1$.

If $r \in P$ satisfies $\|T_b r\| = \|r\|$ we have

$$\|r\|^2 = \|T_b r\|^2 = \|\pi(br)\|^2 \le \|br\|^2 = \int_0^{2\pi} |b(e^{it})|^2 |r(e^{it})|^2\,dt = \|r\|^2,$$

from which $\|\pi(br)\| = \|br\|$ and thus $\pi(br) = br$ as desired. $\square$

Remark 1. The argument of Lemma 1 can be modified to show that if f is bounded and analytic on $\overline{\mathbb{D}}$ and $\|f\|_\infty = 1$, then $\|T_f\| \le 1$ with equality if and only if f is a Blaschke product of order at most n. With more work, one can prove the Caratheodory–Fejer theorem: every lower triangular

$(n+1)\times(n+1)$ Toeplitz M satisfying $\|M\|=1$ is of the form T_f for such an f.

We are now in a position to give a short proof of Theorem 1.

Proof of Theorem 1. By Corollary 2 there are Blaschke products b_1 and b_2 of order at most n satisfying $U_{b_j}=E_j$ for $j=1,2$ and $\widehat{b_1}(k)=\widehat{b_2}(k)$ for $0\le k\le n$. This second fact implies that $T_{b_1}=T_{b_2}$. By Lemma 1 there is nonzero $q\in P$ satisfying $\|T_{b_1}q\|=\|T_{b_2}q\|=\|q\|$ and

$$b_1=\frac{T_{b_1}q}{q}=\frac{T_{b_2}q}{q}=b_2,$$

so $E_1=U_{b_1}=U_{b_2}=E_2$. □

As the Fourier coefficients of a bounded function are coefficients with respect to an orthonormal basis of the Hilbert space L^2, one might wonder if Theorem 1 is a special case of a simpler result about arbitrary orthonormal bases of L^2. This is not the case. There are, for example, orthonormal bases B for L^2 with the property that for every finite subset $F\subseteq B$, there is an arc A with the property that every element of F is constant on A. (The basis $(e^{2\pi it}\mapsto f(t))_{f\in H}$, where H is the *Haar basis* of $L^2[0,1]$ constructed in [3, §III.1], has this property.) In this situation, if $E\subseteq A$ and $E'\subseteq A$ are any two unions of arcs with the same total measure, one will have $\langle\chi_E,f\rangle=\langle\chi_{E'},f\rangle$ for all $f\in F$: *any* finite collection of coefficients with respect to B must fail to distiguish infinitely many unions of n arcs from one another.

4. An algorithm

Let $\mathcal{F}$ denote the map sending a union of at most n arcs E to the tuple $(\widehat{E}(k))_{k=0}^n$ in $\mathbb{C}^{n+1}$. Suppose $c=(c_k)_{k=0}^n$ is given, and we desire to know whether or not c in the range of $\mathcal{F}$. The arguments of the previous sections give us the following procedure. (We use the orthonormal basis of §3 to identify linear operators on P with $(n+1)\times(n+1)$ matrices.)

(1) Calculate the nth Taylor polynomial at 0 for $\phi(\frac{c_0}{2}+\sum_{k=1}^n c_k z^k)$, and make its coefficients the first column of a lower-triangular Toeplitz matrix M.

(2) Evaluate $\|M\|$.
If $\|M\|\ne 1$, then c is not in the range of $\mathcal{F}$.

(3) Otherwise $\|M\|=1$ and by the Caratheodory–Fejer theorem (see Remark 1) there is a unique Blaschke product f of order at most n satisfying $M=T_f$. Find $F=U_f$ (e.g., by solving $f(z)=\pm1$ to get the endpoints of the arcs) and calculate the coefficients of the nth order Taylor polynomial at 0 for b_F.
If these coefficients are the first column of M then $b_F=f$ and $c=\mathcal{F}(F)$; otherwise c is not in the range of $\mathcal{F}$.

Remark 2. The third step of the algorithm is necessary as the map $E \mapsto b_E$ from unions of n arcs to Blaschke products of order n is not surjective. One can check, for example, that of the Blaschke products $b_t(z) = \frac{z^n - t}{1 - tz^n}$ for real $|t| < 1$, all of which satisfy $U_{b_t} = U_{b_0}$, only b_0 is in the range of $E \mapsto b_E$.

If we know in advance that $c = \mathcal{F}(E)$ is in the range of $\mathcal{F}$, this algorithm can recover E from c in a somewhat explicit fashion. The matrix M constructed from c is T_{b_E}; Lemma 1 implies that if we choose a nonzero $q \in P$ satisfying $\|Mq\| = \|q\|$, we will have $b_E = \frac{Mq}{q}$. If q is chosen so as to have minimal degree, the polynomials Mq and q will have no nontrivial common factors. In this case the degree of q is the order of b_E, and the endpoints of the arcs of E — the solutions to $b_E(z) = 1$ and $b_E(z) = -1$ — are the roots of the polynomials $Mq - q$ and $Mq + q$. A computer has no difficulty carrying out this procedure to find the arcs of E to any given precision from the tuple $c = \mathcal{F}(E)$.

As this algorithm involves solving polynomial equations, we cannot expect symbolic formulas for these endpoints of the arcs of E in terms of the Fourier coefficients $\widehat{E}(k)$. Formulas for the polynomials $Mq \pm q$, however, can be obtained with some effort. The entries of M are polynomials in $\exp(2\pi i \widehat{E}(0))$, $\widehat{E}(1), \ldots, \widehat{E}(n)$ with complex coefficients. As M has norm 1, a vector q will satisfy $\|Mq\| = \|q\|$ if and only if q is an eigenvector for the self-adjoint matrix M^*M corresponding to the eigenvalue 1; we can find such a q by using Gaussian elimination, for example. As the entries of M^*M are polynomials in the entries of M and their complex conjugates, the coefficients of q and $Mq \pm q$ will be rational functions in $\exp(2\pi i \widehat{E}(0))$, $\widehat{E}(1), \ldots, \widehat{E}(n)$ and their complex conjugates. Cases may arise in computing $Mq \pm q$ symbolically: in row reducing the symbolic matrix $M^*M - I$, one needs to know whether or not certain functions of the matrix entries are zero — but explicit formulas can be obtained in every case.

We give one example. Suppose that E is a union of at most two arcs, with $\widehat{E}(0)$, $\widehat{E}(1)$, and $\widehat{E}(2)$ given. Write $E_0 = \exp(2\pi i \widehat{E}(0))$ and $E_k = -2\pi i k \widehat{E}(k)$ for $k = 1, 2$. Carrying out the above procedure, one finds that if both E_1 and the denominator of

$$a = \frac{E_2 \overline{E_1} + 2E_1 - E_1^2 \overline{E_1} - 2E_1 E_0}{E_1^2 E_0 + E_2 E_0 - E_2 + E_1^2},$$

are nonzero, then the starting points of the arcs of E are the solutions z of the equation

$$z^2 - az + \left(\frac{\overline{E_1} + (1 - E_0)a}{E_1 E_0} \right) = 0.$$

The endpoints of the arcs of E are given by a similar formula.

References

[1] Carathéodory, C; Fejér, L. Über den Zusammenhang der Extremen von harmonischen Funktionen mit ihren Koeffizienten und über den Picard-Laudau'schen Satz, *Rend. Circ. Mat. Palermo* **32** (1911), 218–239.

[2] Conway, J. B. Functions of one complex variable, second ed., Graduate Texts in Mathematics, vol. 11, *Springer-Verlag, New York*, 1978. MR503901 (80c:30003), Zbl 0277.30001.

[3] Haar, A. Zur Theorie der orthogonalen Funktionensysteme, *Math. Ann.* **69** (1910), no. 3, 331–371. MR1511592.

[4] Körner, T. W. Fourier analysis, second ed., *Cambridge University Press, Cambridge*, 1989. MR924154 (89f:42001), Zbl 0649.42001.

[5] Sarason, D. Generalized interpolation in H^∞, *Trans. Amer. Math. Soc.* **127** (1967), 179–203. MR0208383 (34 #8193), Zbl 0145.39303.

[6] Vetterli, Martin; Marziliano, Pina; Blu, Thierry. Sampling signals with finite rate of innovation. *IEEE Trans. Signal Process.* **50** (2002), no. 6, 1417–1428. MR1930786 (2003i:94019).

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