

# Cells and constructible representations in type $B$

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ABSTRACT. We examine the partition of a finite Coxeter group of type  $B$  into cells determined by a weight function  $L$ . The main objective of these notes is to reconcile Lusztig’s description of constructible representations in this setting with conjectured combinatorial descriptions of cells.

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## 1. Introduction

Consider a finite Coxeter group  $W$  together with a weight function  $L : W \rightarrow \mathbb{Z}$ , as in [15]. Every weight function is specified by its values on

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Received November 28, 2007.

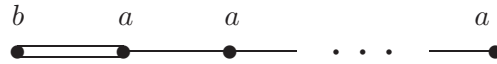
*Mathematics Subject Classification.* 20C08, 05E10.

*Key words and phrases.* Unequal parameter Iwahori–Hecke algebra, domino tableaux.

the simple reflections in  $W$  and defines an Iwahori–Hecke algebra  $\mathcal{H}$  by explicit generators and relations. Furthermore, following Lusztig, a weight function determines a partition  $W$  into left, right, and two-sided cells, each one of which carries representations of  $\mathcal{H}$  and  $W$  [15]. Their role in the representation theory of reductive algebraic groups over finite or  $p$ -adic fields is described in Chapter 0 of [15]. Cells also arise in the study of rational Cherednik algebras and the Calogero–Moser space (see [11] and [10]).

Left cell representations of  $W$  are intimately related to its constructible representations; that is, the minimal class of representations of  $W$  which contains the trivial representation and is closed under truncated induction and tensoring with sign. In fact, left-cell and constructible representations coincide when  $L$  is the length function on  $W$  (see [14]). With the additional stipulation that the conjectures (P1)–(P15) of [15] hold, M. Geck has shown this to be true for general weight functions as well ([6]).

Left cells are well understood for dihedral groups and Coxeter groups of type  $F_4$ . We focus our attention on the remaining case of Coxeter groups of type  $B_n$ . The weight function is then specified by two integer parameters  $a$  and  $b$ :



Given  $a, b \neq 0$ , we may assume both are positive by [9](5.4.1), and write  $s = \frac{b}{a}$  for their quotient. Parameterizations of the left, right, and two-sided cells of  $W$  have been obtained by Garfinkle [5] in the equal parameter case  $s = 1$ , by Lusztig [13] and Bonnafé, Geck, Iancu, and Lam [3] for  $s = \frac{1}{2}$  and  $s = \frac{3}{2}$ , and Bonnafé–Iancu [2] and Bonnafé [1] in the asymptotic case  $s > n - 1$ . Furthermore, a description for the remaining values of  $s$  has been conjectured by Bonnafé, Geck, Iancu, and Lam in [3]. On the other hand, constructible representations of  $W$  were already described by Lusztig for all values of  $s$  by relying on conjectures (P1)–(P15) of [15].

The above parametrizations of cells in type  $B_n$  can be stated in terms of families of standard domino tableaux of arbitrary rank. Reconciling this description of cells with Lusztig’s parametrization of constructible representations is therefore a natural question and is the main purpose of this paper. We focus our attention on the case  $s \in \mathbb{N}$ , excluding the cases for which the left cell and constructible representations are conjectured to be irreducible. In this setting, our main result shows the consistency of Lusztig’s conjectures (P1)–(P15) with the conjectural descriptions of cells. As corollaries to this work, we amend the original conjectural description of two-sided Kazhdan–Lusztig cells in [3], and examine under what circumstances Lusztig’s notion of special representation can exist in the unequal parameter case.

The paper is organized as follows. Section 2 defines cells in unequal parameter Hecke algebras and summarizes the requisite combinatorics. In Section 3, we examine the conjectural combinatorial description of cells in

Weyl groups of type  $B_n$  and its consequences. Finally, Section 4 connects this work with constructible representations.

## 2. Definitions and preliminaries

We begin by defining Kazhdan–Lusztig cells for unequal parameter Hecke algebras. The rest of the section is devoted to the combinatorics pertinent to their conjectured parametrization in type  $B_n$ . Our main goal is to describe certain properties of cycles in a domino tableau, first on the level of partitions, and then on the level of symbols.

**2.1. Kazhdan–Lusztig cells.** Let  $(W, S)$  be a Coxeter system with weight function  $L : W \rightarrow \mathbb{Z}$  which takes positive values on all  $s \in S$ . Define  $\mathcal{H}$  to be the generic Iwahori–Hecke algebra over  $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$  with parameters  $\{v_s \mid s \in S\}$ , where  $v_w = v^{L(w)}$  for all  $w \in W$ . The algebra  $\mathcal{H}$  is free over  $\mathcal{A}$  and has a basis  $\{T_w \mid w \in W\}$  in terms of which multiplication takes the form

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \text{ and} \\ T_{sw} + (v_s - v_s^{-1})T_w & \text{if } \ell(sw) < \ell(w). \end{cases}$$

for  $s \in S$  and  $w \in W$ . As in [15](5.2), it is possible to construct a Kazhdan–Lusztig basis of  $\mathcal{H}$  which we denote by  $\{C_w \mid w \in W\}$ . For  $x, y \in W$  and some  $h_{xyz} \in \mathcal{A}$ , multiplication in  $\mathcal{H}$  takes the form

$$C_x C_y = \sum_{z \in W} h_{xyz} C_z.$$

**Definition 2.1.** ([15](8.1)) Fix  $(W, S)$  a Coxeter system with a weight function  $L$ .

- (1) We will say  $w' \leq_{\mathcal{L}} w$  if there exists  $s \in S$  for which  $C_{w'}$  appears with a nonzero coefficient in  $C_s C_w$  and reuse the same notation  $\leq_{\mathcal{L}}$  for the transitive closure of this binary relation. The equivalence relation associated with the preorder  $\leq_{\mathcal{L}}$  will be denoted by  $\sim_{\mathcal{L}}$  and its equivalence classes will be called *Kazhdan–Lusztig left cells* of  $W$ .
- (2) We will say  $w' \leq_{\mathcal{R}} w$  iff  $w'^{-1} \leq_{\mathcal{L}} w^{-1}$ , write  $\sim_{\mathcal{R}}$  for the corresponding equivalence relation and call its equivalence classes the *Kazhdan–Lusztig right cells* of  $W$ .
- (3) Finally, we define  $\leq_{\mathcal{LR}}$  as the pre-order generated by  $\leq_{\mathcal{L}}$  and  $\leq_{\mathcal{R}}$ , write  $\sim_{\mathcal{LR}}$  for the corresponding equivalence relation and call its equivalence classes the *Kazhdan–Lusztig two-sided cells* of  $W$ .

Each Kazhdan–Lusztig cell carries a representation of the Iwahori–Hecke algebra  $\mathcal{H}$ . We reconstruct the definition of [15](8.3). If  $\mathfrak{C}$  is a Kazhdan–Lusztig left cell and  $w \in \mathfrak{C}$ , then

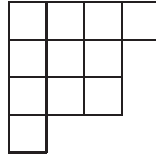
$$[\mathfrak{C}]_{\mathcal{A}} = \bigoplus_{w' \leq_{\mathcal{L}} w} \mathcal{A} C_{w'} \Big/ \bigoplus_{w' \leq_{\mathcal{L}} w, w' \notin \mathfrak{C}} \mathcal{A} C_{w'},$$

is a quotient of two left ideals in  $\mathcal{H}$  and therefore is a left  $\mathcal{H}$ -module. The set  $[\mathfrak{C}]_{\mathcal{A}}$  does not depend on the specific choice of  $w \in \mathfrak{C}$ , is free over  $\mathcal{A}$ , and has a basis  $\{e_w \mid w \in \mathfrak{C}\}$  indexed by elements of  $\mathfrak{C}$  where  $e_w$  is the image of  $C_w$  in the above quotient. Elements of  $\mathcal{H}$  act on  $[\mathfrak{C}]_{\mathcal{A}}$  via  $C_x e_y = \sum_{z \in \mathfrak{C}} h_{xyz} e_z$  for  $x \in W$  and  $y \in \mathfrak{C}$ . Finally, by restricting to scalars,  $[\mathfrak{C}]_{\mathcal{A}}$  gives rise to a  $W$ -module which we denote by  $[\mathfrak{C}]$ . The situation is similar for right cells. If  $\mathfrak{D}$  is a Kazhdan–Lusztig two-sided cell and  $w \in \mathfrak{D}$ , then

$$[\mathfrak{D}]_{\mathcal{A}} = \bigoplus_{w' \leq_{\mathcal{LR}} w} \mathcal{A}C_{w'} / \bigoplus_{w' \leq_{\mathcal{LR}} w, w' \notin \mathfrak{D}} \mathcal{A}C_{w'},$$

is a quotient of two two-sided ideals of  $\mathcal{H}$  and therefore is a  $\mathcal{H}$ -bimodule. The set  $[\mathfrak{D}]_{\mathcal{A}}$  does not depend on the specific choice of  $w \in \mathfrak{D}$ , is free over  $\mathcal{A}$ , and has a basis  $\{e_w \mid w \in \mathfrak{D}\}$  indexed by elements of  $\mathfrak{D}$  where  $e_w$  is the image of  $C_w$  in the above quotient. Again by restricting to scalars,  $[\mathfrak{D}]_{\mathcal{A}}$  gives rise to a  $W$ -module which we denote by  $[\mathfrak{D}]$ .

**2.2. Partitions.** Let  $p = (p_1, p_2, \dots, p_k)$  be a partition of  $m$  with the convention  $p_1 \geq p_2 \geq \dots \geq p_k > 0$ . We will routinely identify a partition  $p$  with its Young diagram  $Y_p$ , or a left-justified array of boxes whose lengths decrease from top to bottom. Thus the partition  $(4, 3, 3, 1) = (4, 3^2, 1)$  will correspond to the Young diagram



If the Young diagram of a partition can be tiled by dominos, we will say that the underlying partition is of *rank zero*. In general, suppose that we can remove a domino from a Young diagram in such a way that what remains is another Young diagram justified at the same row and column. Repeating this process starting with a partition  $p$  will eventually terminate, and the remainder will be a Young diagram of a partition  $(r, r - 1, r - 2, \dots, 1)$  for some  $r \geq 0$ . We will write  $p \in \mathcal{P}_r$  and say that  $p$  is of rank  $r$ . The rank of a partition is unique; each partition of  $m$  belongs to  $\mathcal{P}_r$  for exactly one value of  $r$ . The *core* of  $p$  is the triangular partition  $(r, r - 1, r - 2, \dots, 1)$ .

Let  $s_{ij}$  denote the square in row  $i$  and column  $j$  of a Young diagram and extend this notion somewhat by letting  $i$  and  $j$  take values that may not describe squares in the Young diagram. We define two sets of squares related to the Young diagram of a partition  $p$ .

**Definition 2.2.** For a partition  $p \in \mathcal{P}_r$  and its corresponding Young diagram  $Y_p$ , consider squares  $s_{ij}$  such that  $i + j \equiv r \pmod{2}$ ,  $i + j > r + 1$ , and either the addition of  $s_{ij}$  to  $Y_p$  or the removal of  $s_{ij}$  from  $Y_p$  yields another Young diagram. Among these, we will say

- $s_{ij} \in \mathcal{C}(p)$  iff  $i$  is odd.
- $s_{ij} \in \mathcal{H}(p)$  iff  $i$  is even.

We will write  $\mathcal{HC}(p)$  for the union  $\mathcal{C}(p) \cup \mathcal{H}(p)$ . Furthermore, we define sets  $\mathcal{HC}^*(p)$ ,  $\mathcal{C}^*(p)$ , and  $\mathcal{H}^*(p)$  exactly as above, but requiring  $i + j > r + 2$  instead. We will say  $s_{ij} \in \mathcal{HC}(p)$  is *filled* if it lies in  $Y_p$  itself, otherwise, we will say it is *empty*. Finally, let  $\gamma_p = |\{s_{ij} \in Y_p \mid i + j = r + 2\}|$  and  $\kappa_p$  be the number of filled squares in  $\mathcal{HC}(p)$ .

**Fact 2.1.** We have the following easy consequences:

- (1) The element of  $\mathcal{HC}(p)$  with the smallest row number lies in  $\mathcal{C}(p)$ .
- (2) Elements of  $\mathcal{C}(p)$  and  $\mathcal{H}(p)$  alternate with increasing row number.
- (3)  $|\mathcal{H}(p)| \leq |\mathcal{C}(p)|$ .
- (4) Both  $\gamma_p = r + 1$  as well as  $\kappa_p \neq 0$  imply that  $\mathcal{HC}(p) = \mathcal{HC}^*(p)$ .

**Example 2.3.** In the Young diagram of the rank 2 partition  $p = (4, 3^2, 1)$ , these sets are  $\mathcal{C}(p) = \mathcal{C}^*(p) = \{s_{15}, s_{33}, s_{51}\}$  and  $\mathcal{H}(p) = \mathcal{H}^*(p) = \{s_{24}, s_{42}\}$ .

**Definition 2.4.** Let  $s_{ij}, s_{kl}$ , and  $s_{mn} \in \mathcal{HC}(p)$ . We will say that  $s_{mn}$  lies between  $s_{ij}$  and  $s_{kl}$  iff  $m$  is between  $i$  and  $k$  and  $n$  is between  $j$  and  $l$  (where  $m$  is between  $i$  and  $k$  iff  $i \leq m \leq k$  or  $i \geq m \geq k$ ). We will say that two squares of  $\mathcal{HC}(p)$  are *adjacent in  $\mathcal{HC}(p)$*  if no other square of  $\mathcal{HC}(p)$  lies between them.

**Remark 2.5.** The set  $\mathcal{HC}(p)$  coincides with the union of the sets of corners and holes of  $p$  as defined for  $r = 0$  and  $1$  in [4]. When  $r = 0$ ,  $\mathcal{C}(p)$  is the set of corners and  $\mathcal{H}(p)$  is the set of holes; however, our definitions diverge from Garfinkle’s for  $r = 1$ . This is not unexpected: when  $r = 1$ , the left cells identified in Conjecture 4.1 are not the ones studied by Garfinkle. Their parametrization depends on a different choice of variable squares, see Remark 2.7.

**Remark 2.6.** The set  $\mathcal{HC}^*(p)$  is precisely the union of *addable* and *removable* squares as defined in [11]. Following [11], the *heart* of  $p$  will be the partition obtained from  $p$  by removing all filled squares of  $\mathcal{HC}^*(p)$ .

**2.3. Tableaux.** Consider a partition  $p$ . A *domino tableau of shape  $p$*  is a Young diagram of  $p$  whose squares are labeled by a set  $M$  of nonnegative integers in such a way that every positive integer labels exactly two adjacent squares and all labels increase weakly along both rows and columns. A domino tableau is *standard* if  $M \cap \mathbb{N} = \{1, \dots, n\}$  for some  $n$ , and of *rank  $r$*  if  $p \in \mathcal{P}_r$  and  $0$  labels the square  $s_{ij}$  iff  $i + j < r + 2$ . We will write  $\text{SDT}_r(p)$  for the family of all standard domino tableaux of rank  $r$  and shape  $p$  and  $\text{SDT}_r(n)$  for the family of all standard domino tableaux of rank  $r$  which contain exactly  $n$  dominos.

The moving-through operation on a domino tableau defines another domino tableau whose labels agree on a certain subset of its squares. In a domino tableau of rank  $r$ , we will say that the square  $s_{ij}$  in row  $i$  and column  $j$  is *variable* iff  $i + j \equiv r \pmod{2}$ ; otherwise, we will say it is *fixed*.

**Remark 2.7.** Our choice of fixed and variable squares coincides with that of [4] when  $r = 0$  but not when  $r = 1$ . As mentioned in Remark 2.5, in this latter case the left cells identified in Conjecture 4.1 are not the ones studied by Garfinkle and their description relies on the above assignment of fixed and variable squares.

Consider a domino  $D = D(k, T)$  with label  $k \in \mathbb{N}$  in a domino tableau  $T$  of rank  $r$  and write  $\text{supp } D(k, T)$  for the set of its underlying squares. Then  $\text{supp } D(k, T)$  contains a fixed and a variable square. Suppose that we wanted to create another domino tableau by changing the label of the variable square of  $D$  in a way that preserved the labels of all fixed squares of  $T$  while perturbing  $T$  minimally. As in [4], this leads to the notion of a cycle in a domino tableau; we define it presently.

**Definition 2.8.** Suppose that  $\text{supp } D(k, T) = \{s_{ij}, s_{i+1, j}\}$  or  $\{s_{i, j-1}, s_{ij}\}$  and the square  $s_{ij}$  is fixed. Define  $D'(k)$  to be a domino labeled by the integer  $k$  with  $\text{supp } D'(k, T)$  equal to

$$\begin{aligned} &\{s_{ij}, s_{i-1, j}\} \text{ if } k < \text{label } s_{i-1, j+1}, \text{ and} \\ &\{s_{ij}, s_{i, j+1}\} \text{ if } k > \text{label } s_{i-1, j+1}. \end{aligned}$$

Alternately, suppose that  $\text{supp } D(k, T) = \{s_{ij}, s_{i-1, j}\}$  or  $\{s_{i, j+1}, s_{ij}\}$  and the square  $s_{ij}$  is fixed. Define  $\text{supp } D'(k, T)$  to be

$$\begin{aligned} &\{s_{ij}, s_{i, j-1}\} \text{ if } k < \text{label } s_{i+1, j-1}, \text{ and} \\ &\{s_{ij}, s_{i+1, j}\} \text{ if } k > \text{label } s_{i+1, j-1}. \end{aligned}$$

**Definition 2.9.** The cycle  $c = c(k, T)$  through  $k$  in a domino tableau  $T$  of rank  $r$  is a union of labels of dominos in  $T$  defined by the condition that  $l \in c$  if either  $l = k$ , or either  $\text{supp } D(l, T) \cap \text{supp } D'(m, T) \neq \emptyset$  or  $\text{supp } D'(l, T) \cap \text{supp } D(m, T) \neq \emptyset$  for some  $D(m, T) \in c$ .

We will refer to the set of dominos with labels in a cycle  $c$  as the cycle  $c$  itself. For a domino tableau  $T$  of rank  $r$  and a cycle  $c$  in  $T$ , define  $MT(T, c)$  by replacing every domino  $D(l, T) \in c$  by the corresponding domino  $D'(l, T)$ . It follows that  $MT(T, c)$  is domino tableau, and in general, the shape of  $MT(T, c)$  will either equal the shape of  $T$ , or one square will be removed (or added to the core) and one will be added [4](1.5.27). A cycle  $c$  is called *closed* in the former case and *open* in the latter. We will write  $OC(T)$  for the set of open cycles in  $T$ . For  $c \in OC(T)$ , we will write  $S_b(c)$  for the square that is either removed from the shape of  $T$  or added to the core of  $T$  by moving through  $c$ . Similarly, we will write  $S_f(c)$  for the square that is added to the shape of  $T$ . Note that  $S_b(c)$  and  $S_f(c)$  are always variable squares. Consistent with Garfinkle's notation in [5], we will write  $OC^*(T)$  for the set of noncore open cycles in  $T$ , that is, cycles for which both  $S_b(c)$  and  $S_f(c)$  lie in  $\mathcal{HC}^*(p)$  with  $p = \text{shape } T$ . For a cycle in  $OC^*(T)$ ,  $S_b(c) \in \mathcal{C}^*(p)$  and  $S_f(c) \in \mathcal{H}^*(p)$ , or  $S_b(c) \in \mathcal{H}^*(p)$  and  $S_f(c) \in \mathcal{C}^*(p)$ .

Let  $\mathcal{C}$  be a set of cycles in a domino tableau  $T$  of rank  $r$ . According to [4](1.5.29), moving through disjoint cycles in a domino tableau are independent operations, allowing us to unambiguously write  $MT(T, \mathcal{C})$  for the domino tableau obtained by simultaneously moving-through all of the cycles in the set  $\mathcal{C}$ . If  $\mathcal{C} \subset OC^*(T)$ , then  $MT(T, \mathcal{C})$  is another domino tableau of rank  $r$ ,  $\mathcal{C} \subset OC^*(MT(T, \mathcal{C}))$ , and  $MT(MT(T, \mathcal{C}), \mathcal{C}) = T$ . If  $\mathcal{C} = OC(T) \setminus OC^*(T)$ , then  $MT(T, \mathcal{C})$  can be interpreted as a domino tableau of rank  $r + 1$  and comes endowed with new sets of fixed and variable squares and consequently, cycles.

**Definition 2.10.** For a standard domino tableau of rank  $r$ , we define the cycle structure set of  $T$  as the set of ordered pairs  $cs(T)$  consisting of the beginning and final squares of every cycle in  $T$  and  $cs^*(T)$  as the restriction of this set to noncore open cycles. That is:

$$cs(T) = \{(S_b(c), S_f(c)) \mid c \in OC(T)\}, \text{ and}$$

$$cs^*(T) = \{(S_b(c), S_f(c)) \mid c \in OC^*(T)\}.$$

Finally, write  $\tilde{cs}(T)$  and  $\tilde{cs}^*(T)$  for the sets obtained from the above by changing their underlying ordered pairs into unordered pairs.

We would like a similar notion for partitions that does not directly rely on an underlying tableau. First note that if  $p = \text{shape } T$ , then  $\mathcal{HC}(p)$  consists exactly of the  $\kappa_p$  beginning and  $\kappa_p$  final squares of noncore open cycles of  $T$ , the  $\gamma_p$  final squares of core open cycles of  $T$ , and the  $r + 1 - \gamma_p$  empty squares adjacent to the core of  $Y_p$ ; consequently, we have  $|\mathcal{HC}(p)| = 2\kappa_p + r + 1$ .

**Definition 2.11.** Consider  $p \in \mathcal{P}_r$ . A *cycle structure set*  $\sigma$  for  $p$ , or alternately, for  $\mathcal{HC}(p)$ , is a pairing of squares in  $\mathcal{H}^*(p)$  with squares in  $\mathcal{C}^*(p)$  for which:

- (1) Exactly  $\gamma_p$  squares remain unpaired.
- (2) Every square  $c \in \mathcal{H}^*(p)$  which lies between  $a$  and  $b$  for a pair  $\{a, b\} \in \sigma$  must be paired with another square which lies between  $a$  and  $b$ .

**Example 2.12.** The partition  $(4, 3^2, 1)$  of rank  $r = 2$  admits exactly four cycle structure sets:  $\{\{s_{15}, s_{24}\}\}$ ,  $\{\{s_{24}, s_{33}\}\}$ ,  $\{\{s_{33}, s_{42}\}\}$ , and  $\{\{s_{42}, s_{51}\}\}$ .

Note that a cycle structure set for  $p$  contains exactly  $\kappa_p$  pairs. Cycle structure sets for tableaux and partitions are closely related. Given a standard domino tableau  $T$  of rank  $r$ , the set  $\sigma = \tilde{cs}^*(T)$  is a cycle structure set for the partition  $p = \text{shape}(T) \in \mathcal{P}_r$  by elementary properties of open cycles. Conversely, as detailed in the following proposition, a cycle structure set for an arbitrary partition  $p$  always arises as a cycle structure set for some domino tableau.

**Proposition 2.13.** *If  $p \in \mathcal{P}_r$  and  $\sigma$  is a cycle structure set for  $p$ , then there exists a standard domino tableau  $T$  of rank  $r$  with  $\tilde{cs}^*(T) = \sigma$ .*



**Proof.** We proceed by induction on the number of pairs in  $\sigma$ . If  $\sigma$  is empty, then any  $T \in \text{SDT}_r(p)$  suffices. Otherwise, consider a pair  $\{s, s'\} \in \sigma$ . By Definition 2.11(2),  $s$  and  $s'$  can be chosen in such a way that they are adjacent in  $\mathcal{HC}(p)$ . Lemma 3.4 of [11] implies that there exists a standard domino tableau  $T'$  of rank  $r$  and a noncore open cycle  $c$  in  $T'$  with  $S_b(c) = s$  and  $S_f(c) = s'$  whose dominos form a rim ribbon  $R$  of  $T'$ . If  $R$  contains  $t$  dominos, then by the proof of [11](3.4),  $T'$  can be chosen in such a way that the dominos in  $T' \setminus R$  are labeled by elements of the set  $\{1, 2, \dots, n-t\}$ . Now by Definition 2.11(2),  $\sigma \setminus \{s, s'\}$  is either empty or contains a pair  $\{s'', s'''\}$  adjacent in  $\mathcal{HC}(\text{shape}(T' \setminus R))$  and the proposition follows by induction.  $\square$

**Proposition 2.14.** *If  $p \in \mathcal{P}_r$  and  $\mathcal{S}$  is a subset of  $\mathcal{HC}^*(p)$  consisting of  $\kappa_p$  elements, then there exists a cycle structure set  $\sigma$  for  $p$  where each pair in  $\sigma$  contains exactly one element of  $\mathcal{S}$ .*

**Proof.** When  $|\mathcal{S}| = \kappa_p = 0$ , the only cycle structure set for  $p$  is empty. Assuming that  $|\mathcal{S}| = \kappa_p \geq 1$ , Fact 2.1 implies  $\mathcal{HC}(p) = \mathcal{HC}^*(p)$ . Therefore, we can find a pair  $\{s, t\}$  of squares adjacent in  $\mathcal{HC}(p)$  with  $s \in \mathcal{S}$  and  $t \notin \mathcal{S}$ . Again by Fact 2.1,  $\mathcal{C}(p)$  and  $\mathcal{H}(p)$  alternate with increasing row number and consequently one of  $s$  and  $t$  must lie in  $\mathcal{C}^*(p)$  and the other in  $\mathcal{H}^*(p)$ . Working recursively, this pair can be extended to a pairing of elements of  $\mathcal{H}^*(p)$  with elements of  $\mathcal{C}^*(p)$  which satisfies the properties of a cycle structure set.  $\square$

**2.4. Symbols.** A symbol of defect  $s$  is an array of numbers of the form

$$\Lambda = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_{N+s} \\ & \mu_1 & \mu_2 & \dots & \mu_N \end{pmatrix}$$

where  $\{\lambda_i\}$  and  $\{\mu_i\}$  are, perhaps empty, strictly increasing sequences of nonnegative integers. Define an equivalence relation on the set of symbols of defect  $s$  by letting  $\Lambda$  be equivalent to the symbol

$$\Lambda' = \begin{pmatrix} 0 & \lambda_1 + 1 & \lambda_2 + 2 & \dots & \lambda_{N+s} + N + s \\ & 0 & \mu_1 + 1 & \dots & \mu_N + N \end{pmatrix}.$$

We will write  $\text{Sym}_s$  for the set of equivalence classes of symbols of defect  $s$ . It is possible to define a map from partitions to symbols via the following procedure. Given a partition  $p = (p_1, p_2, \dots, p_k)$ , form an extended partition  $p^\sharp = (p_1, p_2, \dots, p_{k'})$  by adding an additional zero term to  $p$  if the rank of  $p$  has the same parity as  $k$ . The set  $\{p_i + k' - i\}_{i=1}^{k'}$  can be divided into odd and even parts  $\{2\mu_i + 1\}_{i=1}^N$  and  $\{2\lambda_i\}_{i=1}^{N+s}$  from which the symbol  $\Lambda_p$  corresponding to  $p$  can be constructed by arranging the  $\lambda_i$  and  $\mu_i$  as above. We will write  $\tilde{p}_i$  for the entry of  $\Lambda_p$  determined from the part  $p_i$  of  $p^\sharp$ .

Let  $\mathcal{P}^2$  be the set of ordered pairs of partitions, and write  $\mathcal{P}^2(n)$  for the subset of  $\mathcal{P}^2$  where the sum of parts of both partitions sum to  $n$ . Given a symbol of defect  $s$ , it is also possible to construct an ordered pair of partitions. With  $\Lambda$  as above, let  $d_\Lambda = \{\lambda_i - i + 1\}_{i=1}^{N+s}$  and  $f_\Lambda = \{\mu_i - i + 1\}_{i=1}^N$ . The following is an immediate consequence of [12](2.7).



**Theorem 2.15.** *The maps  $p \mapsto \Lambda_p$  and  $\Lambda \mapsto (d_\Lambda, f_\Lambda)$  define bijections*

$$\mathcal{P}_r \rightarrow \text{Sym}_{r+1} \rightarrow \mathcal{P}^2$$

*for all values of  $r$ . Furthermore, the composition of these two maps yields a bijection between  $\mathcal{P}_r(n)$  and  $\mathcal{P}^2(n)$ .*

Since the set  $\mathcal{P}^2(n)$  parameterizes the irreducible representations of the Weyl group  $W_n$  of type  $B_n$ , the above can be used to identify irreducible  $W_n$ -modules with symbols of fixed defect, as in [15], or with partitions of fixed rank. It is the latter interpretation that we employ in the following sections. We will write  $[p]$  and  $[\Lambda]$  for the irreducible  $W_n$ -modules associated to the partition  $p$  and the symbol  $\Lambda$  in this manner.

We would like to understand the correspondence of Theorem 2.15 in slightly greater detail. For a symbol  $\Lambda$ , we write  $Z_1(\Lambda)$  for the set of entries that appear once among its rows, and  $Z_2(\Lambda)$  for the set of entries that appear twice.

**Lemma 2.16.** *The set  $Z_1(\Lambda_p)$  of single entries of the symbol  $\Lambda_p$  can be identified with the parts of  $p^\sharp$  whose rows in the Young diagram  $Y_p$  end in a, perhaps empty, square of  $\mathcal{HC}(p)$ . This establishes a bijective map  $Z_1(\Lambda_p) \leftrightarrow \mathcal{HC}(p)$ .*

**Proof.** We first show that elements of  $Z_2(\Lambda_p)$  arise from the rows of  $Y_p$  which do not terminate in a square of  $\mathcal{HC}(p)$ . Note that  $z \in Z_2(\Lambda_p)$  implies  $z = \tilde{p}_i = \tilde{p}_{i+1}$  for some  $i$ , and furthermore  $p_i$  must equal  $p_{i+1}$ . A parity argument shows that if the row of  $p_i$  in  $Y_p$  ends in a fixed square,  $\tilde{p}_i$  will differ from  $\tilde{p}_{i+1}$ . Hence every  $z \in Z_2(\Lambda_p)$  must correspond to a pair of consecutive equal parts  $p_i, p_{i+1}$  of  $p^\sharp$  where the row of  $p_i$  in  $Y_p$  ends in a variable square. It is easy to check that  $\tilde{p}_i = \tilde{p}_{i+1}$  for such a pair. Hence elements of  $Z_2(\Lambda_p)$  correspond to pairs of consecutive equal rows of  $Y_p$ , the first of which ends in a variable square. Such pairs yield precisely the rows of  $Y_p$  which do not terminate in a square of  $\mathcal{HC}(p)$  and the lemma follows.  $\square$

### 3. Combinatorial cells

This section examines equivalence relations on the Weyl group of type  $B_n$  defined via a Robinson–Schensted algorithm and standard domino tableaux. As stated more explicitly in the next section, the equivalence classes which they define are expected to coincide with unequal parameter Kazhdan–Lusztig cells in type  $B_n$ .

**3.1. Robinson–Schensted algorithms.** The Weyl group  $W_n$  of type  $B_n$  is the group of permutations of the set  $\{\pm 1, \pm 2, \dots, \pm n\}$  which commute with the involution  $i \mapsto -i$ . Generalized Robinson–Schensted maps  $G_r : W_n \rightarrow \text{SDT}_r(n) \times \text{SDT}_r(n)$  defined in [4] and [22] construct bijections between elements of the Weyl group of type  $B_n$  and same-shape pairs of standard domino tableaux of rank  $r$  for each nonnegative integer  $r$ . We will

write  $G_r(w) = (S_r(w), T_r(w))$  for the image of an element  $w$  and refer to the components of the ordered pair as the rank  $r$  left and right tableaux of  $w$ .

There is a natural description of the relationship between the bijections  $G_r$  for differing  $r$  described in terms of the moving though map for open cycles, see [18]. We also point out that for  $r$  sufficiently large,  $G_r$  recovers another generalization of the Robinson–Schensted algorithm for hyperoctahedral groups defined in [21] and [17]. See [19] for a more detailed description.

### 3.2. Combinatorial left, right, and two-sided cells.

**Definition 3.1.** Consider  $x, y \in W_n$  of type  $B_n$  and fix a nonnegative integer  $r$ . We will say:

- (1)  $x \approx_{\mathcal{L}} y$  if their right tableaux of rank  $r$  are related by moving through some set of noncore open cycles, that is, iff  $T_r(x) = MT(T_r(y), \mathcal{C})$  for some  $\mathcal{C} \subset OC^*(T_r(y))$ .
- (2)  $x \approx_{\mathcal{R}} y$  iff  $x^{-1} \approx_{\mathcal{L}} y^{-1}$ .
- (3) Write  $x \approx_{\mathcal{LR}} y$  for the relation generated by  $\approx_{\mathcal{L}}$  and  $\approx_{\mathcal{R}}$ .

Defined in this way, we will call the equivalence classes of  $\approx_{\mathcal{L}}$ ,  $\approx_{\mathcal{R}}$ , and  $\approx_{\mathcal{LR}}$  in  $W_n$  *reducible combinatorial left, right, and two-sided cells of rank  $r$* . Within this paper, we will generally omit the adjective “reducible” of this definition. Although we suppress it in the notation, the cells depend on the choice of parameter  $r$ . By [22](4.2), the map  $w \mapsto w^{-1}$  on  $W_n$  carries combinatorial left cells to combinatorial right cells and preserves combinatorial two-sided cells. As seen in Figures 1 and 2, combinatorial cells do not behave simply with respect to a change in  $r$ , although it is possible to describe a precise relationship [18]. When  $r > n - 2$ , the situation is somewhat simpler. There are no noncore open cycles, implying both, that combinatorial left cells are determined simply by right tableaux, and by the main result of [18], that for these values of  $r$ , all combinatorial cells are actually independent of  $r$ .

The main result of [19] shows that combinatorial left cells admit the following alternate description. A similar characterization holds for combinatorial right cells.

**Theorem 3.2** ([19]). *Combinatorial left cells in the Weyl group of type  $B_n$  are generated by the equivalence relations of having the same right tableau in either rank  $r$  or rank  $r + 1$ .*

**Proposition 3.3.** *Consider  $x, y \in W_n$ , fix a nonnegative integer  $r$ , and let  $p$  and  $p'$  be the shapes of  $T_r(x)$  and  $T_r(y)$  respectively. Then  $x \approx_{\mathcal{LR}} y$  iff  $\mathcal{HC}(p) = \mathcal{HC}(p')$  and  $p$  and  $p'$  differ only in the choice of filled squares in  $\mathcal{HC}(p)$ .*

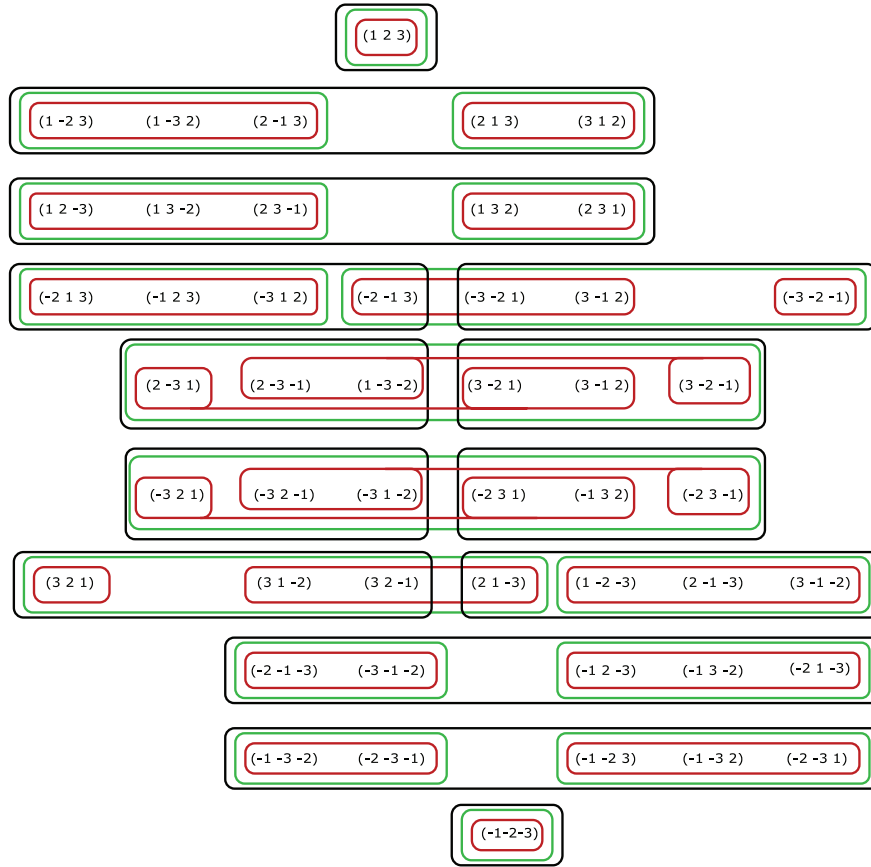


FIGURE 1. Combinatorial left cells in  $W_3$ . Black represents cells for rank  $r = 0$ , green represents  $r = 1$ , and red represents  $r \geq 2$ . The cells are not successive refinements for increasing values of the partition rank parameter  $r$ .

**Proof.** Suppose  $x \approx_{\mathcal{LR}} y$ . Note that if either  $x' \approx_{\mathcal{L}} x''$  or  $x' \approx_{\mathcal{R}} x''$ , then  $T_r(x')$  and  $T_r(x'')$  or  $S_r(x')$  and  $S_r(x'')$  differ by moving through a, perhaps empty, set of noncore open cycles. Since moving through noncore open cycles acts on the level of partitions by only changing which squares are filled in  $\mathcal{HC}(p)$ , the forward direction of the above follows.

For the other direction, first note that two elements whose tableaux are of the same shape are necessarily in the same combinatorial two-sided cell: if  $T$  and  $S$  are two standard dominos of the same shape, then

$$G_r^{-1}(X, T) \approx_{\mathcal{L}} G_r^{-1}(T, T) \approx_{\mathcal{R}} G_r^{-1}(S, T) \approx_{\mathcal{L}} G_r^{-1}(S, S) \approx_{\mathcal{R}} G_r^{-1}(Y, S)$$

for all  $X$  and  $Y$  of the same shape. The rest of the proof follows as in [11](3.5).  $\square$

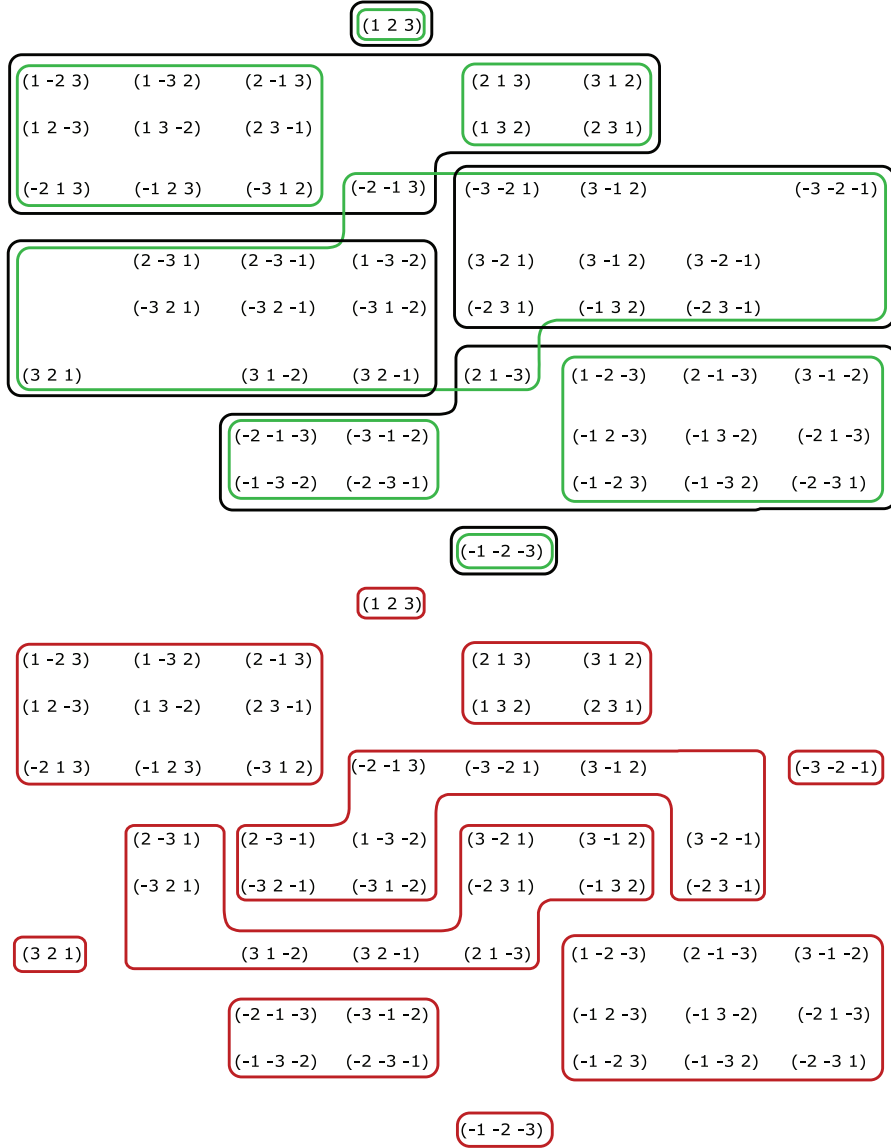


FIGURE 2. Combinatorial double cells in  $W_3$ . Black represents cells for rank  $r = 0$ , green represents  $r = 1$ , and red represents  $r \geq 2$ . The cells are not successive refinements for increasing values of the partition rank parameter  $r$ .

**3.3. Tableau shapes of elements within combinatorial cells.** We examine more closely the sets of partitions that appear among shapes of tableaux of elements in combinatorial cells. Fix a combinatorial left cell

$\mathfrak{C}$  and a combinatorial two-sided cell  $\mathfrak{D}$  of rank  $r$ . Write  $\pi(\mathfrak{C})$  and  $\pi(\mathfrak{D})$  for the sets of partitions that appear among tableaux shapes of their elements.

It is clear that the rank  $r$  right tableaux of the elements of  $\mathfrak{C}$  share a common cycle structure set and  $\pi(\mathfrak{C})$  consists exactly of those partitions derived from a choice of a filled square in each of its constituent pairs. If  $k_{\mathfrak{C}}$  is the number of noncore open cycles in the tableaux of the elements of  $\mathfrak{C}$ , then  $|\pi(\mathfrak{C})| = 2^{k_{\mathfrak{C}}}$ . The partitions in  $\pi(\mathfrak{D})$  can be determined via the following observations.

- (1) According to Proposition 3.3, the sets  $\mathcal{HC}(p)$ ,  $\mathcal{H}(p)$ , and  $\mathcal{C}(p)$  are constant among  $p \in \pi(\mathfrak{D})$ . We will emphasize this by writing  $\mathcal{HC}_{\mathfrak{D}}$ ,  $\mathcal{H}_{\mathfrak{D}}$ , and  $\mathcal{C}_{\mathfrak{D}}$  for these sets.
- (2) The number of filled squares in  $\mathcal{HC}_{\mathfrak{D}}$  is constant on  $\pi(\mathfrak{D})$ . We will denote this number by  $\kappa_{\mathfrak{D}}$ . It is equal to  $\kappa_p$  for any  $p \in \pi(\mathfrak{D})$ .

From Proposition 3.3, partitions in  $\pi(\mathfrak{D})$  are determined by choices of  $\kappa_{\mathfrak{D}}$  filled squares among  $\mathcal{HC}_{\mathfrak{D}}$ . Consequently,  $\pi(\mathfrak{D})$  contains exactly  $\binom{|\mathcal{HC}_{\mathfrak{D}}|}{\kappa_{\mathfrak{D}}}$  partitions. The next proposition points out a fundamental difference in the relationship between the sets  $\pi(\mathfrak{C})$  and  $\pi(\mathfrak{D})$  in the two cases when  $r = 0$  or  $r > n - 2$ , and when  $0 < r \leq n - 2$ .

**Proposition 3.4.** *Consider a combinatorial two-sided cell  $\mathfrak{D}$ . The intersection*

$$I_{\mathfrak{D}} = \bigcap_{\mathfrak{C} \subset \mathfrak{D}} \pi(\mathfrak{C})$$

*is nonempty iff  $\kappa_{\mathfrak{D}} = 0$  or  $\kappa_{\mathfrak{D}} = |\mathcal{H}_{\mathfrak{D}}|$ , in which case it contains a unique partition. In particular, this occurs for all combinatorial two-sided cells of rank  $r = 0$  and  $r > n - 2$ .*

**Proof.** We first claim that if  $p \in I_{\mathfrak{D}}$ , then all filled squares of  $\mathcal{HC}(p)$  must lie in  $\mathcal{H}^*(p)$ . This is trivially true if  $\kappa_p = 0$ , so assume otherwise and note that by Fact 2.1,  $\mathcal{HC}(p) = \mathcal{HC}^*(p)$ . Suppose  $c \in \mathcal{C}(p) = \mathcal{C}^*(p)$ . Since  $|\mathcal{HC}(p)| = 2\kappa_p + r + 1$  and  $|\mathcal{H}(p)| \leq |\mathcal{C}(p)|$ , we can choose a set of  $\kappa_p$  elements of  $\mathcal{C}^*(p)$  which excludes  $c$ . By Proposition 2.14, there is a cycle structure set for  $p$  which leaves  $c$  unpaired and by Proposition 2.13, there is a standard domino tableau  $T$  with  $\tilde{c}s^*(T) = \sigma$ . If we let  $\mathfrak{C}$  be the combinatorial left cell associated to  $T$ , then no partition in  $\pi(\mathfrak{C})$  has the square  $c$  filled, and consequently all filled squares of  $\mathcal{HC}(p)$  must lie in  $\mathcal{H}^*(p)$ .

Armed with this observation, suppose first that  $\kappa_{\mathfrak{D}} = |\mathcal{H}_{\mathfrak{D}}| \neq 0$  so that  $\mathcal{HC}_{\mathfrak{D}} = \mathcal{HC}_{\mathfrak{D}}^*$ . Every cycle structure set on  $\mathcal{HC}_{\mathfrak{D}}$  must pair all of the  $\kappa_{\mathfrak{D}}$  squares within  $\mathcal{H}_{\mathfrak{D}}$ , implying that the partition with precisely all squares of  $\mathcal{H}_{\mathfrak{D}}$  filled lies in  $\pi(\mathfrak{C})$  for all cells  $\mathfrak{C} \subset \mathfrak{D}$ . Furthermore, this partition is unique in  $I_{\mathfrak{D}}$ : any partition in  $\pi(\mathfrak{D})$  must have exactly  $\kappa_{\mathfrak{D}}$  filled squares in  $\mathcal{HC}_{\mathfrak{D}}$  and, as observed above, for partitions in  $I_{\mathfrak{D}}$  these must lie in  $\mathcal{H}_{\mathfrak{D}}$ .

When  $\kappa_{\mathfrak{D}} < |\mathcal{H}_{\mathfrak{D}}|$ , Proposition 2.14 can be used to construct a cycle structure set  $\sigma$  on  $\mathcal{HC}_{\mathfrak{D}}$  which leaves an arbitrary  $h \in \mathcal{H}_{\mathfrak{D}}$  unpaired. We can

associate a combinatorial left cell  $\mathfrak{C}$  to  $\sigma$  as above and note that  $h$  is empty in every partition of  $\pi(\mathfrak{C})$ . Since  $h$  was arbitrary, any partition appearing in  $I_{\mathfrak{D}}$  must have all  $h \in \mathcal{H}_{\mathfrak{D}}$  empty. Hence unless  $\kappa_{\mathfrak{D}} = 0$ ,  $I_{\mathfrak{D}}$  is empty. When  $\kappa_{\mathfrak{D}} = 0$ ,  $\pi(\mathfrak{D})$  consists of a unique partition and  $|I_{\mathfrak{D}}| = 1$ . Finally, we note that if  $r = 0$ ,  $k_{\mathfrak{D}} = |\mathcal{H}_{\mathfrak{D}}|$  and if  $r > n - 2$ ,  $k_{\mathfrak{D}} = 0$ .  $\square$

**Remark 3.5.** In the case  $r = 0$ , the unique partition in  $I_{\mathfrak{D}}$  is called special and corresponds to Lusztig’s notion of special representation of  $W_n$  under the map defined by Theorem 2.15 (see [13]). A consequence of the above proposition is that similarly distinguished partitions do not exist for a range of values of  $r$ . When interpreted in terms of the conjectures describing the Kazhdan–Lusztig cells in type  $B_n$  stated in the next section, and combined with the results of [20], this precludes the existence of distinguished representations of  $W_n$  in the general unequal parameter case.

**Example 3.6.** Consider the partition  $p = (4, 3^2, 1)$  of rank 2. Elements of  $W_4$  whose tableaux have shape  $p$  lie in a combinatorial two-sided cell  $\mathfrak{D}$ . The set  $\mathcal{HC}_{\mathfrak{D}} = \mathcal{HC}_{\mathfrak{D}}^*$  equals  $\{s_{15}, s_{24}, s_{33}, s_{42}, s_{51}\}$ , with only the square  $s_{33}$  filled, hence  $\kappa_{\mathfrak{D}} = 1$ . Consequently, listing the partitions of  $\pi(\mathfrak{D})$  entails deciding which square of  $\mathcal{HC}_{\mathfrak{D}}$  is filled. The possible partitions are  $(5, 3, 2, 1)$ ,  $(4^2, 2, 1)$ ,  $(4, 3^2, 1)$ ,  $(4, 3, 2^2)$ , and  $(4, 3, 2, 1^2)$ .

The shapes of elements in combinatorial left cells contained in this combinatorial two-sided cell fall into the following four categories:

$$\begin{aligned} &\{(5, 3, 2, 1), (4^2, 2, 1)\}, && \{(4^2, 2, 1), (4, 3^2, 1)\}, \\ &\{(4, 3^2, 1), (4, 3, 2^2)\}, && \{(4, 3, 2^2)\}, (4, 3, 2, 1^2)\}, \end{aligned}$$

each corresponding to a choice of a cycle structure set on  $\mathcal{HC}_{\mathfrak{D}}$ . In particular, it is clear that no partition is common to all of these sets.

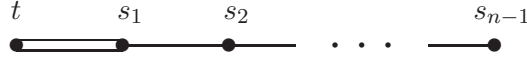
**Remark 3.7.** For every combinatorial left cell  $\mathfrak{C}$ ,  $\pi(\mathfrak{C})$  admits a natural structure of an elementary abelian 2-group. Since the right tableau of any element in  $\mathfrak{C} \subset \mathfrak{D}$  is of the form  $MT(T, \mathcal{C})$  for some  $C \subset OC^*(T)$ ,  $\pi(\mathfrak{C})$  is determined entirely by the positions of each of the  $\kappa_{\mathfrak{D}}$  noncore open cycles of  $T$ , and corresponds to the choices of a filled square within each pair of  $\tilde{c}s^*(T)$ . Because the moving-through operations on cycles in  $T$  are independent, a choice of a distinguished partition in  $\pi(\mathfrak{C})$  defines a natural structure of an elementary abelian 2-group of order  $2^{\kappa_{\mathfrak{D}}}$ . For  $r = 0$ , this is described in [16].

## 4. Kazhdan–Lusztig cells and constructible representations

We examine the relationship of Kazhdan–Lusztig cells and combinatorial cells in type  $B_n$  and reconcile Lusztig’s description of constructible representations with combinatorial cells. We restrict our attention to the case where the defining parameter  $s$  is an integer, focusing on the case when the

conjectured cells and constructible representations are not irreducible. The key will be the results of Section 2 relating partitions and symbols.

**4.1. Cells in type  $B_n$ .** We restrict the setting to the Weyl group of type  $B_n$  with generators as in the following diagram:



Suppose the weight function  $L$  is defined by  $L(t) = b$  and  $L(s_i) = a$  for all  $i$ . We will examine the case when  $\frac{b}{a} \in \mathbb{N}$ , and set  $s = \frac{b}{a}$ . The following is a conjecture of Bonnafé, Geck, Iancu, and Lam, and appears as Conjecture B in [3]:

**Conjecture 4.1** ([3]). *Consider a Weyl group of type  $B_n$  with a weight function  $L$  and parameter  $s$  defined as above. Kazhdan–Lusztig left, right, and two-sided cells for parameter  $s \in \mathbb{N}$  coincide with combinatorial left, right, and two-sided cells of rank  $s - 1$ .*

This conjecture is well-known to be true for  $s = 1$  by work of Garfinkle [5], and has been verified when  $s > n - 1$  by Bonnafé and Iancu, [2] and Bonnafé [1]. It has also been shown to hold for all values of  $s$  when  $n \leq 6$  (see [3]). The above is restated more explicitly as Conjecture D in [3]. It implicitly assumes the existence of a partition which is distinguished in the sense of Remark 3.5 within the partitions arising among tableaux of elements of a two-sided cell. However, in light of Proposition 3.4 such a partition does not exist in general and the characterization of Kazhdan–Lusztig two-sided cells in Conjecture D must be rephrased using the description of combinatorial two-sided cells of Proposition 3.3.

**4.2. Constructible representations.** The set of constructible representations  $\text{Con}(W)$  of a Weyl group  $W$  is the smallest class of representations which contains the trivial representation and is closed under truncated induction and tensoring with the sign representation, see [15](22.1). Although this is not clear from the notation, this set depends on the weight function chosen to define  $\mathcal{H}$ . For the results of this section, we assume that Lusztig’s conjectures (P1)–(P15) of [15](14.2) are true. Under this assumption, M. Geck has described the relationship between constructible representations and Kazhdan–Lusztig left cells:

**Proposition 4.2** ([6]). *Consider a finite Coxeter group  $W$  with a weight function  $L$  and let  $\mathfrak{C}$  be a Kazhdan–Lusztig left cell in  $W$  defined from  $L$ . Then:*

- (1)  $[\mathfrak{C}]$  is a constructible  $W$ -module.
- (2) Every constructible  $W$ -module can be obtained in this way.

Let us again restrict the setting to the Weyl group of type  $B_n$  with weight function  $L$  defining a parameter  $s$ . We begin our description of constructible



representations by first recalling the one of Lusztig [15](22.6). Let  $\Lambda$  be a symbol of defect  $s$  and let  $Z_1 = Z_1(\Lambda)$  and  $Z_2 = Z_2(\Lambda)$ . If  $Y \subset Z_1$ , define a new symbol

$$\Lambda_Y = \begin{pmatrix} Z_2 \sqcup Z_1 \setminus Y \\ Z_2 \sqcup Y \end{pmatrix}.$$

We would like to restrict the set of subsets  $Y$  for which this construction will be carried out.

**Definition 4.3.** An involution  $\iota : Z_1 \rightarrow Z_1$  is *admissible* iff:

- (1) It contains exactly  $s$  fixed points.
- (2) Whenever  $z' \in Z_1$  lies strictly between  $z$  and  $\iota(z)$  for any  $z \in Z_1$ , then  $z'$  is not a fixed point and  $\iota(z')$  lies strictly between  $z$  and  $z'$ .

Given an admissible involution  $\iota$ , define a set  $S_\iota$  consisting of subsets of  $Z_1$  by letting  $Y \in S_\iota$  iff it contains exactly one element from each orbit of  $\iota$ . Recalling the parametrization of  $W_n$ -modules by symbols of defect  $s$  from Section 2.4, define a  $W_n$ -module by

$$c(\Lambda, \iota) = \bigoplus_{Y \in S_\iota} [\Lambda_Y].$$

The modules  $c(\Lambda, \iota)$  and  $c(\Lambda', \iota')$  are equal iff  $\Lambda$  and  $\Lambda'$  have the set of entries and also  $\iota = \iota'$ .

**Proposition 4.4** ([15](22.23)). *Consider a symbol  $\Lambda$  together with an admissible involution  $\iota$ . Then:*

- (1)  $c(\Lambda, \iota)$  is a constructible  $W_n$ -module.
- (2) Every constructible  $W_n$ -module can be obtained in this way.

Now consider a partition  $p \in \mathcal{P}_{s-1}$ . If  $Y$  is a subset of  $\mathcal{HC}(p)$ , let  $p_Y$  be the partition obtained from the heart of  $p$  by filling exactly the squares of  $\mathcal{HC}(p)$  which correspond to  $Y$ . Given a cycle structure set  $\sigma$  for  $p$ , define a set  $S_\sigma$  consisting of subsets of  $\mathcal{HC}(p)$  by letting  $Y \in S_\sigma$  iff  $Y$  contains exactly one element from each pair in  $\sigma$ . Recalling the parametrization of  $W_n$ -modules by partitions of rank  $s - 1$  from Section 2.4, we define a  $W_n$ -module by

$$c(p, \sigma) = \bigoplus_{Y \in S_\sigma} [p_Y].$$

The modules  $c(p, \sigma)$  and  $c(p', \sigma')$  are equal iff  $p$  and  $p'$  have the same heart and  $\sigma = \sigma'$ . The  $W_n$ -modules obtained in this way are precisely the constructible ones.

**Theorem 4.5.** *Consider a partition  $p \in \mathcal{P}_{s-1}$  and a cycle structure set  $\sigma$  for  $p$ . Then:*

- (1)  $c(p, \sigma)$  is a constructible  $W_n$ -module.
- (2) Every constructible  $W_n$ -module can be obtained in this way.

**Proof.** Construct  $\Lambda_p$ , a symbol of rank  $s$ , as in Theorem 2.15. We first show that  $c(p, \sigma) = c(\Lambda_p, \iota)$  for some admissible involution  $\iota$ . When  $\mathcal{HC}(p) \supsetneq \mathcal{HC}^*(p)$ , we have  $|\mathcal{HC}(p)| = s$  and Definition 2.11(1) implies that the only cycle structure set  $\sigma$  on  $p$  is trivial. Hence  $c(p, \sigma) = [p]$ . By Lemma 2.16, the corresponding symbol  $\Lambda_p$  will have  $|Z_1(\Lambda_p)| = s$ , implying that the only admissible involution  $\iota$  on  $Z_1$  is trivial. Hence  $c(\Lambda_p, \iota) = [\Lambda_p] = [p] = c(p, \sigma)$ .

Thus we assume  $\mathcal{HC}(p) = \mathcal{HC}^*(p)$  and write  $Z_1 = Z_1(\Lambda_p)$  and  $Z_2 = Z_2(\Lambda_p)$ . We describe a bijection between the cycle structure sets for  $p$  and admissible involutions  $\iota : Z_1 \rightarrow Z_1$ . Images of the orbits of  $\iota$  under the map of Lemma 2.16 form a pairing  $\sigma_\iota$  on the squares of  $\mathcal{HC}(p)$ . Noting that the squares in  $\mathcal{HC}(p)$  alternate between  $\mathcal{H}(p)$  and  $\mathcal{C}(p)$  with increasing row number, Definition 4.3(2) implies that  $\sigma_\iota$  is in fact a pairing between squares of  $\mathcal{H}(p)$  and  $\mathcal{C}(p)$ . Furthermore, it follows directly from the definition that  $\sigma_\iota$  is in fact a cycle structure set for  $p$ . This process is easily reversed, establishing the desired bijection. Write  $\iota_\sigma$  for the admissible involution associated with the cycle structure set  $\sigma$ .

We would like to show that  $c(p, \sigma) = c(\Lambda_p, \iota_\sigma)$ . Lemma 2.16 establishes a bijection between  $S_{\iota_\sigma}$  and  $S_\sigma$ . If  $\tilde{Y}$  represents the image of  $Y \in S_{\iota_\sigma}$ , it is sufficient to show that the symbol  $(\Lambda_p)_Y = \Lambda_{p_{\tilde{Y}}}$  for all  $Y \in S_{\iota_\sigma}$ . It is clear that  $Z_1((\Lambda_p)_Y) = Z_1(\Lambda_{p_{\tilde{Y}}})$  and  $Z_2((\Lambda_p)_Y) = Z_2(\Lambda_{p_{\tilde{Y}}})$ . Consider a square  $s_{ij} \in \tilde{Y}$  and write  $\iota_{ij}$  for the corresponding element of  $Y \subset Z_1$ . It is enough to show that  $\iota_{ij}$  appears in the bottom row of the symbol  $\Lambda_{p_{\tilde{Y}}}$ . With  $k'$  defined as in Section 2.4, note that  $j + k' - i$  is odd. By the definition of the map  $p \rightarrow \Lambda_p$  and Lemma 2.16,  $\iota_{ij}$  must equal  $\frac{j+k'-i-1}{2}$  and hence appears in the bottom row of  $\Lambda_{p_{\tilde{Y}}}$ , as desired.

Finally, since the map of Theorem 2.15 is a bijection and we've established a bijection between cycle structure sets and  $s$ -admissible involutions, every constructible  $W_n$ -module appears as  $c(p, \sigma)$  for some  $p$  and  $\sigma$ , since it appears as  $c(\Lambda, \iota)$  for some  $\Lambda$  and  $\iota$ . □

The above theorem can easily be restated in terms of tableaux. To each tableau  $T \in \text{SDT}_r(n)$  we associate a  $W_n$ -module  $[T]$  in the following manner. For each family of open cycles  $\mathcal{C}$  in  $T$  define  $p_{\mathcal{C}}$  to be the shape of the tableau  $MT(T, \mathcal{C})$  obtained from  $T$  by moving through  $\mathcal{C}$ . Let

$$[T] = \bigoplus_{\mathcal{C} \subset \text{OC}^*(T)} [p_{\mathcal{C}}].$$

The partitions which can be obtained by moving through noncore open cycles in a tableau depend only on the cycle structure of the tableau, hence the modules  $[T]$  and  $[T']$  are equal iff the underlying partitions have the same heart and  $\tilde{cs}^*(T) = \tilde{cs}^*(T')$ . The  $W_n$ -modules obtained in this way are precisely the constructible ones.

**Corollary 4.6.** *Consider a standard domino tableau  $T$ . Then:*

- (1)  $[T]$  is a constructible  $W_n$ -module.
- (2) Every constructible  $W_n$ -module can be obtained in this way.

**Proof.** The module  $[T]$  is precisely  $c(\text{shape}(T), \tilde{c}s^*(T))$ , and hence constructible. Conversely, a constructible module  $c(p, \sigma)$  equals  $[T]$  for some tableau  $T$  by Proposition 2.13.  $\square$

Given a Coxeter system  $(W, S)$  with weight function  $L$ , a family of partitions is an equivalence class defined by the transitive closure of the relation linking  $p$  and  $p'$  iff  $[p]$  and  $[p']$  appear as simple components of some constructible representation of  $W$ . The following relates families and the partitions appearing in combinatorial two-sided cells. It is a version of the result of [6](4.3).

**Proposition 4.7.** *Consider  $W$  is of type  $B_n$  with a weight function  $L$  and parameter  $s$  and let  $\mathfrak{D}$  be a combinatorial two-sided cell. Then the family of  $p \in \pi(\mathfrak{D})$  is precisely  $\pi(\mathfrak{D})$ .*

**Proof.** If  $p, p'$  lie in the same family, then they must have the same heart, implying  $p, p' \in \pi(\mathfrak{D})$ . We show the converse. If  $k_{\mathfrak{D}} = 0$ , then  $|\pi(\mathfrak{D})| = 1$ , and we are done. Otherwise, note that  $\mathcal{HC}(p) = \mathcal{HC}^*(p)$  and we can let  $p^\uparrow$  be the partition with the same heart as  $p$  but with the top-most  $k_{\mathfrak{D}}$  squares of  $\mathcal{HC}(p)$  filled. We will show that  $p$  and  $p^\uparrow = p'^{\uparrow}$  lie in the same family, implying the result.

If  $p \neq p^\uparrow$ , order elements of  $\mathcal{HC}(p)$  by their row number, and let  $s$  be the greatest empty square of  $\mathcal{HC}_{\mathfrak{D}}$  preceding the greatest filled square in  $\mathcal{HC}_{\mathfrak{D}}$ . Let  $t$  be the least filled square following  $s$  in  $\mathcal{HC}_{\mathfrak{D}}$ . The pair  $\{s, t\}$  can be extended to a cycle structure set for  $p$  via Proposition 2.14. Let  $p^\uparrow$  be the partition obtained from  $p$  by filling  $s$  and emptying  $t$ . Then by Corollary 4.6,  $p$  and  $p^\uparrow$  lie in the same family. This process can be repeated successively producing a sequence  $p, p^\uparrow, (p^\uparrow)^\uparrow, \dots$  of partitions in the same family which terminates in  $p^{\uparrow\uparrow}$ .  $\square$

**Example 4.8.** Consider the symbol

$$\Lambda = \begin{pmatrix} 0 & 1 & 3 & 4 \\ & & 2 & \end{pmatrix}$$

of defect  $s = 3$ . Its set of singles has four 3-admissible involutions  $(0, 1)$ ,  $(1, 2)$ ,  $(2, 3)$ , and  $(3, 4)$  which, according to the above proposition, produce the constructible representations

$$\begin{aligned} \mathcal{S}_{(0,1)} &: \left[ \begin{pmatrix} 1 & 2 & 3 & 4 \\ & 0 & & \end{pmatrix} \right] \oplus \left[ \begin{pmatrix} 0 & 2 & 3 & 4 \\ & 1 & & \end{pmatrix} \right] & \mathcal{S}_{(1,2)} &: \left[ \begin{pmatrix} 0 & 2 & 3 & 4 \\ & 1 & & \end{pmatrix} \right] \oplus \left[ \begin{pmatrix} 0 & 1 & 3 & 4 \\ & 2 & & \end{pmatrix} \right] \\ \mathcal{S}_{(2,3)} &: \left[ \begin{pmatrix} 0 & 1 & 3 & 4 \\ & 2 & & \end{pmatrix} \right] \oplus \left[ \begin{pmatrix} 0 & 1 & 2 & 4 \\ & 3 & & \end{pmatrix} \right] & \mathcal{S}_{(3,4)} &: \left[ \begin{pmatrix} 0 & 1 & 2 & 4 \\ & 3 & & \end{pmatrix} \right] \oplus \left[ \begin{pmatrix} 0 & 1 & 2 & 3 \\ & 4 & & \end{pmatrix} \right]. \end{aligned}$$

By using the identification from Section 2.4, we can rephrase this list in terms of partitions of rank  $r = s - 1 = 2$ . The symbol  $\Lambda$  corresponds to the

partition  $(4, 3^2, 1)$  and the constructible representations can be rewritten in terms of partitions as

$$\begin{aligned} \mathcal{S}_{(0,1)} &: [(4, 3, 2, 1^2)] \oplus [(4, 3, 2^2)] & \mathcal{S}_{(1,2)} &: [(4, 3, 2^2)] \oplus [(4, 3^2, 1)] \\ \mathcal{S}_{(2,3)} &: [(4, 3^2, 1)] \oplus [(4^2, 2, 1)] & \mathcal{S}_{(3,4)} &: [(4^2, 2, 1)] \oplus [(5, 3, 2, 1)]. \end{aligned}$$

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