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On generators of shy sets on Polish topological vector spaces

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Abstract. We introduce the notion of generators of shy sets in Polish topological vector spaces and give their various interesting applications. In particular, we demonstrate that this class contains specific measures which naturally generate implicitly introduced subclasses of shy sets. Moreover, such measures (unlike σ-finite Borel measures) possess many interesting, sometimes unexpected, geometric properties.

Contents

1. Preliminaries

The main goal of the present work is to construct specific Borel measures in Polish topological vector spaces which naturally generate classes of null sets playing an important role in studying the properties of a function space (see, for example, [1], [4], [6], [8], [12], [16], [20], [21], [23], [25]). In this direction, for a Polish topological vector space V, we introduce the notion of a generator of shy sets which is a Borel measure μ in V such that every

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set of μ-measure zero is a Haar null set (or a shy set). Note that since no nonzero σ-finite Borel measure possesses the above mentioned property, we have to choose measures in the class of nonzero non-σ-finite Borel measures defined on the entire spaces.

The paper is organized as follows.

In Section 2 we introduce the notion of a generator defined in a Polish topological vector space and prove that the class of generators is nonempty. We also show that in the class of generators there does not exist a generator with the property of uniqueness. We prove that every generator defined in the infinite-dimensional topological vector space is non-σ-finite. We demonstrate that a quasifinite translation-quasiinvariant Borel measure, as well as a quasifinite translation-invariant Borel measure, is a generator. Some interesting examples of generators [2], [3], and [19] are considered in this section.

In Section 3 we focus on the so-called quasigenerator μ which is a Borel measure whose every pair of shifts is equivalent or orthogonal. The main result of this section (Theorem 3.1) states that the structure of an arbitrary quasigenerator μ allows one to construct a new quasifinite generator $\overline{G}_\mu$ such that the class of all shy sets $S(\mu)$ defined by any quasigenerator μ coincides with the σ-ideal of null sets $N(\overline{G}_\mu)$ defined by the constructed generator G_μ.

Section 4 presents various examples of Gikhman–Skorokhod measures which were constructed in papers [5], [9], [10], [11], [13], [14], [24], [26]. A simple example of a Gaussian generator is given in an arbitrary Polish topological vector space. A general problem is stated whether a Gaussian generator is quasifinite.

In Section 5 we describe a construction of A.B. Kharazishvili's quasigenerators and, using their structures, we construct an example of a semifinite quasifinite Baker's generator in $\mathbb{R}^{\mathbb{N}}$.

In Section 6, A.B. Kharazishvili's quasigenerator $\nu_{[0,1]^{\mathbb{N}}}$ is used to construct Mankiewicz's generator $G_{\nu_{[0,1]^N}}$ which generates exactly the class of all cube null sets [16] in $\mathbb{R}^{\mathbb{N}}$. Also, we consider the generator defined by a finite-dimensional manifold in Polish topological vector space V. Here we establish that the class of all shy sets defined by any finite-dimensional manifold $L \subseteq V$ coincides with the σ-ideal of $\overline{G}_{\lambda_L}$-null sets, where λ_L denotes a Lebesgue measure concentrated on L.

In Section 7, we construct n-dimensional Preiss–Tišer generators in Banach spaces (in this context, see [20]) and establish their some interesting geometrical properties.

2. Existence and uniqueness of generators

Let V be a Polish topological vector space, by which we mean a vector space with a complete metric for which the addition and the scalar multiplication are continuous. Let $\mathcal{B}(V)$ be the σ-algebra of Borel subsets of V and μ be a nonzero nonnegative measure defined on $\mathcal{B}(V)$. We write $X + a$ for the translation of a set $X \subseteq V$ by a vector $a \in V$.

Definition 2.1 ([12, Definition 1, p. 221])**.** A measure μ is said to be transverse to a Borel set $S \subset V$ if:

(1) There exists a compact set $U \subset V$ for which $0 < \mu(U) < \infty$.
(2) $\mu(S + v) = 0$ for every $v \in V$.

Definition 2.2 ([12, Definition 1, p. 222])**.** A Borel set $S \subset V$ is called shy if there exists a Borel measure which is transverse to S. More generally, a subset of V is called shy if it is contained in a shy Borel set.

The class of all shy sets in V is denoted by $S(V)$.

Definition 2.3 ([12, p. 226])**.** The complement of a shy set is called a prevalent set. We say that almost every element of V satisfies a given property if the subset of V on which the property holds is prevalent.

As Christensen [6, p. 119] notes, there is no σ-finite (equivalently, probability) measure μ such that S being shy is equivalent to $\mu(S) = 0$. Slightly more can be said that any σ-finite measure μ must assign 0 to a prevalent set of points. On these grounds, we introduce the following:

Definition 2.4. A Borel measure μ in V is called a generator (of shy sets) in V, if

$$\overline{\mu}(X) = 0 \quad \Rightarrow \quad X \in S(V),$$

where $\overline{\mu}$ denotes the usual completion of the Borel measure μ.

Definition 2.5. A Borel measure μ in V is called quasifinite if there exists a compact set $U \subset V$ for which $0 < \mu(U) < \infty$.

Definition 2.6. A Borel measure μ in V is called semifinite if for X with $\mu(X) > 0$ there exists a compact subset $F \subset X$ for which $0 < \mu(F) < \infty$.

Definition 2.7. For a Borel measure μ in V and $h \in V$, the shift μ_h is defined by

$$\mu_h(X) = \mu(X + h) \tag{2.1}$$

for $X \in \mathcal{B}(V)$. We say that μ is translation-quasiinvariant if $\mu \sim \mu_h$ (i.e., μ is equivalent to μ_h) for all $h \in V$.

Definition 2.8. A Borel measure μ in V is called translation-invariant if $\mu_h = \mu$ for all $h \in V$.

Definition 2.9. Let K be a class of measures on V. We say that a measure $\mu \in K$ has the property of uniqueness in the class K if μ and λ are equivalent for every $\lambda \in K$.

Now there naturally arises the following:

Question 2.1. *Let V be a Polish topological vector space. Does there exist a generator in V?*

Let V be a Polish vector space. Let L be a proper vector subspace of V. By the Axiom of Choice, we can construct a proper vector subspace $F \subset V$ such that

$$L \cap F = \{\mathbf{0}\} \quad \text{and} \quad L + F = V, \tag{2.2}$$

where $\{\mathbf{0}\}$ denotes the zero of V.

In the sequel we denote by $\mathcal{F}$ such a class of vector subspaces $F \subset V$ satisfying (2.2). We set $L^{\perp} = \tau(\mathcal{F})$, where τ denotes a global operator of choice. A vector subspace $L^{\perp}$ is said to be a linear complement of the vector space L in V. In the sequel we will apply such a notation.

The next assertion gives a positive answer to Question 2.1.

Theorem 2.1. *Let V be a Polish topological vector space. For every $v_0 \in V$, there exists a semifinite inner regular generator λ such that*

$$\lambda(\{\alpha v_0 : \alpha \in [0,1]\}) = 1$$

and λ is non-σ-finite iff $\dim(V) \geq 2$.

Proof. Let L_1 be the one-dimensional vector subspace defined by v_0. Let μ be the classical one-dimensional Borel measure in $L_1 = \{\alpha v_0 : \alpha \in \mathbb{R}\}$ defined by $\mu(Y) = b_1(X)$ for $Y = X v_0, X \in \mathcal{B}(\mathbb{R})$.

We put

$$\lambda(X) = \sum_{v \in L_1^{\perp}} \mu((X + v) \cap L_1)$$

for $X \in \mathcal{B}(V)$. First, we are to show the correctness of the definition of a functional λ. Indeed, let F be any element from the class $\mathcal{F}$ and assume that λ_1 is defined by

$$\lambda_1(X) = \sum_{u \in F} \mu((X + u) \cap L_1)$$

for $X \in \mathcal{B}(V)$

We have $L_1 + F = L_1 + L_1^{\perp}$ which means that

$$\cup_{u \in F}(L_1 + u) = \cup_{v \in L_1^{\perp}}(L_1 + v) = V.$$

The latter equalities imply that for every $u \in F$ there exists $f(u) \in L_1^{\perp}$ such that $L_1 + u = L_1 + f(u)$. It is not hard to show that f is an injective function from F to $L_1^{\perp}$, and so is $f^{-1} : L_1^{\perp} \to F$. Thus, f defines a bijection between F and $L_1^{\perp}$. Since a sum of an arbitrary family of nonnegative real numbers is invariant under their permutations, for $X \in \mathcal{B}(V)$, we have

$$\begin{aligned}\lambda_1(X) &= \sum_{u \in F} \mu((X+u) \cap L_1) = \sum_{f(u) \in f(F)} \mu((X+f(u)) \cap L_1) \\ &= \sum_{v \in L_1^{\perp}} \mu((X+v) \cap L_1) = \lambda(X).\end{aligned}$$

The latter equality shows the definition of the functional λ is correct.

Now let us show that λ is a measure. Let $(X_k)_{k \in \mathbb{N}}$ be a family of pairwise disjoint Borel sets in B. Then we get

$$\begin{aligned}\lambda(\cup_{k \in \mathbb{N}} X_k) &= \sum_{v \in L_1^{\perp}} \mu(((\cup_{k \in \mathbb{N}} X_k) + v) \cap L_1) = \sum_{v \in L_1^{\perp}} \mu((\cup_{k \in \mathbb{N}} (X_k + v)) \cap L_1) \\ &= \sum_{v \in L_1^{\perp}} \sum_{k \in \mathbb{N}} \mu((X_k + v) \cap L_1) = \sum_{k \in \mathbb{N}} \sum_{v \in L_1^{\perp}} \mu((X_k + v) \cap L_1) \\ &= \sum_{k \in \mathbb{N}} \lambda(X_k).\end{aligned}$$

The measure λ is quasifinite. Indeed, for $D = [0,1]v_0$, we have

$$\lambda(D) = \sum_{v \in L_1^{\perp}} \mu((D+v) \cap L_1) = \mu((D+\mathbf{0}) \cap L_1) = 1.$$

The measure λ is translation-invariant. Indeed, for $h \in V$ we have a representation $h = h_1 + h_2$ where $h_1 \in L_1$ and $h_2 \in L_1^{\perp}$. Therefore for $X \in \mathcal{B}(V)$, we have

$$\begin{aligned}\lambda(X+h) &= \sum_{v \in L_1^{\perp}} \mu((X+h+v) \cap L_1) = \sum_{v \in L_1^{\perp}} \mu((X+h_1+h_2+v) \cap L_1) \\ &= \sum_{v \in L_1^{\perp}} \mu((X+h_2+v) \cap L_1) = \sum_{h_2+v \in h_2+L_1^{\perp}} \mu((X+h_2+v) \cap L_1) \\ &= \sum_{h_2+v \in L_1^{\perp}} \mu((X+h_2+v) \cap L_1) = \sum_{s \in L_1^{\perp}} \mu((X+s) \cap L_1) = \lambda(X).\end{aligned}$$

Now let us show that λ is a generator. Let S be a subset of V with $\bar{\lambda}(S) = 0$. Since $\bar{\lambda}$ is a completion of λ, there exists a Borel set $S^{'}$ for which $S \subseteq S^{'}$ and $\lambda(S^{'}) = 0$. Taking into account that D is a compact set in V with $\lambda(D) = 1$ and applying a simple consequence of the translation-invariance of the measure λ, stating that $\lambda(S^{'} + v) = 0$ for $v \in V$, we deduce that λ is transverse to a Borel set $S^{'}$. This means that $S^{'}$ is a Borel shy set. Finally, since S is a subset of a Borel shy set $S^{'}$, we conclude that S is a shy set. One can observe that the generator λ is non-σ-finite if and only if $\dim(V) \geq 2$.

Let us show that the generator λ is inner regular. For this, we are to show that for all X with $0 < \lambda(X) < \infty$ and all $\epsilon > 0$, there exists a compact $F_\epsilon \subseteq X$ such that $\lambda(X \setminus F_\epsilon) < \epsilon$.

Since $0 < \lambda(X) < \infty$, there exists $G_0 \subset L_1^{\perp}$ such that $\mathrm{card}(G_0) \le \aleph_0$ and

$$0 < \lambda(X) = \sum_{g \in G_0} \mu((X+g) \cap L_1) < \infty.$$

We set $G_0 = (g_m)_{m \in N}$. Let n_0 be a natural number such that

$$\sum_{1 \le m \le n_0} \mu((X+g_m) \cap L_1) > \lambda(X) - \frac{\epsilon}{2}.$$

For $\epsilon > 0$ there exists a compact set $F_m \subset V$ such that $F_m \subseteq (X+g_m) \cap L_1$ and

$$\mu(((X+g_m) \setminus F_m) \cap L_1) < \frac{\epsilon}{2^{m+1}}$$

for $1 \le m \le n_0$.

We set $F_\epsilon = \cup_{1 \le k \le n_0}(F_k - g_k)$. It is obvious that F_ϵ is compact in V.

Finally, we get

$$\begin{aligned}
\lambda(X \setminus \cup_{1 \le k \le n_0}(F_k - g_k)) &= \sum_{g \in G_0} \mu(((X \setminus \cup_{1 \le k \le n_0}(F_k - g_k)) + g) \cap L_1) \\
&= \sum_{m \in N} \mu(((X \setminus \cup_{1 \le k \le n_0}(F_k - g_k)) + g_m) \cap L_1) \\
&= \sum_{1 \le m \le n_0} \mu(((X \setminus \cup_{1 \le k \le n_0}(F_k - g_k)) + g_m) \cap L_1) \\
&\quad + \sum_{m > n_0} \mu(((X \setminus \cup_{1 \le k \le n_0}(F_k - g_k)) + g_m) \cap L_1) \\
&\le \sum_{1 \le m \le n_0} \mu(((X \setminus (F_m - g_m)) + g_m) \cap L_1) \\
&\quad + \sum_{m > n_0} \mu((X + g_m) \cap L_1) \\
&\le \sum_{1 \le m \le n_0} \mu(((X + g_m) \setminus F_m) \cap L_1) \\
&\quad + \sum_{m > n_0} \mu((X + g_m) \cap L_1) \\
&\le \sum_{1 \le m \le n_0} \frac{\epsilon}{2^{m+1}} + \frac{\epsilon}{2} \le \sum_{m \in N} \frac{\epsilon}{2^{m+1}} + \frac{\epsilon}{2} = \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Let us show that λ is semifinite. Indeed, if $\lambda(X) > 0$, then there exists $g_0 \in L_1^{\perp}$ such that $0 < \mu((X+g_0) \cap L_1) < \infty$. Using the property of inner regularity of μ we deduce that there exists a compact set $U \subseteq (X+g_0) \cap L_1$ with $0 < \lambda(U) < \infty$. The latter relation means that λ is semifinite. $\square$

Theorem 2.2. *Let V be a Polish topological vector space whose dimension is not equal to one. Then there does not exist a generator with the property of uniqueness in the class of all generators in V.*

Proof. Let us consider two linearly independent elements v_0 and v_1 in V and apply the construction used in the proof of Theorem 2.1 with respect to the above-mentioned elements. Hence we will construct two generators λ_0 and λ_1 such that

$$\lambda_0(\{\alpha v_0 : 0 \leq \alpha \leq 1\}) = 1,$$
$$\lambda_1(\{\alpha v_0 : 0 \leq \alpha \leq 1\}) = 0,$$

which means that λ_0 and λ_1 are not equivalent. From the latter relations it follows that there does not exist a generator in V with the property of uniqueness. $\square$

Theorem 2.3. *Every generator in an infinite-dimensional Polish topological vector space V is non-σ-finite.*

Proof. Assume the contrary and let λ be a σ-finite generator in V. Then there exists a countable family of compact sets $\{K_n : n \in N\}$ such that

$$\lambda(V \setminus \cup_{n\in N} K_n) = 0.$$

Note that the compact set K_n is shy for $n \in N$ (cf. [12, Fact 8, p. 225]). Following Definition 2.4, the set $V \setminus \cup_{n\in N} K_n$ is a shy set because it is of λ-measure zero. Thus, we get that V is shy, which is a contradiction since there does not exists a Borel measure in V which is transverse to the set V. $\square$

Remark 2.1. An n-dimensional classical Borel measure in $\mathbb{R}^n (n \geq 2)$ is an example of a σ-finite generator, but on the same space an example of such a generator which is non-σ-finite can be constructed (cf. Theorem 2.1).

Theorem 2.4. *Every quasifinite translation-quasiinvariant Borel measure μ defined in a Polish topological vector space V is a generator.*

Proof. Let $\overline{\mu}(S) = 0$ for $S \subset V$. Since $\overline{\mu}(S) = 0$, there exists a Borel set $S^{'}$ for which $S \subseteq S^{'}$ and $\mu(S^{'}) = 0$. By using the property of translation-quasiinvariance of the Borel measure μ, we have $\mu(S^{'} + h) = 0$ for all $h \in V$. Thus, μ is transverse to the Borel set $S^{'}$ and therefore $S^{'}$ is a Borel shy set. S is a shy set because it is a subset of the Borel shy set $S^{'}$. $\square$

Corollary 2.1. *Since every translation-invariant measure is at the same time translation-quasiinvariant, we deduce that every quasifinite translation-invariant Borel measure in V is a generator.*

Let $\mathbb{R}$ be the real line and $\mathbb{R}^{\mathbb{N}}$ stand for the space of all real-valued sequences equipped with the Tychonoff topology (i.e., the product topology). Denote by $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$ the σ-algebra of all Borel subsets in $\mathbb{R}^{\mathbb{N}}$.

Example 2.1. Let $\mathcal{R}_1$ be the class of all infinite-dimensional rectangles $R \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ of the form

$$R = \prod_{i=1}^{\infty}(a_i, b_i),\ -\infty < a_i \leq b_i < +\infty$$

such that

$$0 \leq \prod_{i=1}^{\infty}(b_i - a_i) := \lim_{n\to\infty}\prod_{i=1}^{n}(b_i - a_i) < \infty.$$

Let τ_1 be a set function on $\mathcal{R}_1$ defined by

$$\tau_1(R) = \prod_{i=1}^{\infty}(b_i - a_i).$$

Following R. Baker [2], the functional λ defined by

$$\lambda(X) = \inf\left\{\sum_{j=1}^{\infty}\tau_1(R_j) : R_j \in \mathcal{R}_1 \ \&\ X \subseteq \cup_{j=1}^{\infty}R_j\right\}$$

for $X \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ is a quasifinite translation-invariant Borel measure in $\mathbb{R}^{\mathbb{N}}$. By Corollary 2.1 we deduce that λ is a generator in $\mathbb{R}^{\mathbb{N}}$.

Example 2.2. Let $\mathcal{R}_2$ be the class of all infinite-dimensional rectangles $R \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ of the form

$$R = \prod_{i=1}^{\infty}R_i,\ R_i \in \mathcal{B}(\mathbb{R})$$

such that

$$0 \leq \prod_{i=1}^{\infty}m(R_i) := \lim_{n\to\infty}\prod_{i=1}^{n}m(R_i) < \infty,$$

where m denotes a one-dimensional classical Borel measure on $\mathbb{R}$.

Let τ_2 be a set function on $\mathcal{R}_2$, defined by

$$\tau_2(R) = \prod_{i=1}^{\infty}m(R_i).$$

R. Baker [3] proved that the functional μ defined by

$$\mu(X) = \inf\left\{\sum_{j=1}^{\infty}\tau_2(R_j) : R_j \in \mathcal{R}_2 \ \&\ X \subseteq \cup_{j=1}^{\infty}R_j\right\}$$

for $X \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ is a quasifinite translation-invariant Borel measure in $\mathbb{R}^{\mathbb{N}}$. By Corollary 2.1 we establish that μ is a generator in $\mathbb{R}^{\mathbb{N}}$.

Remark 2.2. Following [12, Theorem 15.2.1, p. 204] the generator μ is absolutely continuous with respect to the generator λ, and the generators λ and μ are not equivalent. This fact gives an answer to R. Baker's one question posed in [3].

Example 2.3 ([19, Theorem 7.1, p. 119])**.** Let $\mathbb{J}$ be any nonempty subset of the set of all natural numbers $\mathbb{N}$. Then, in the Solovay model [22] there exists a translation-invariant measure $\mu_{\mathbb{J}}$ on the power set of $\mathbb{R}^{\mathbb{J}}$ such that $\mu_{\mathbb{J}}([0,1]^{\mathbb{J}}) = 1$. Following Corollary 2.1, the restriction of the measure $\mu_{\mathbb{J}}$ on $\mathcal{B}(\mathbb{R}^{\mathbb{J}})$ is a generator in $\mathbb{R}^{\mathbb{J}}$.

3. Quasigenerators in Polish topological vector spaces

For a Borel measure μ in V we denote by $N(\mu)$ a class of all subsets which are of $\overline{\mu}$-measure zero, where $\overline{\mu}$ denotes a completion of μ.

Definition 3.1. A set $S \subset V$ is called μ-shy if it is a subset of a Borel set S' with $\mu(S' + v) = 0$ for $v \in V$. The class of all μ-shy sets is denoted by $S(\mu)$.

It is clear that

$$S(V) = \cup_{\mu} S(\mu).$$

If μ has atoms, then $S(\mu) = \{\emptyset\}$. This means that the measures which matter in defining shy sets are the ones without atoms (equivalently, diffused measures) (cf. [23], p. 456).

In the context of the above representation of $S(V)$ there naturally arises the following:

Problem 3.1. *Let μ be a diffused Borel measure on V. Does there exist a quasifinite generator λ in V such that the following equality*

$$S(\mu) = N(\bar{\lambda})$$

holds?

Note that Problem 3.1 is not trivial and its solution depends on the structure of the measure μ. Below we identify a subclass of diffused Borel measures (cf. Definition 3.5) for which Problem 3.1 has a positive solution.

Definition 3.2. A quasifinite Borel measure μ in V is called a Gikhman–Skorokhod measure if for each $h \in V$, the shift μ_h (2.1) is either orthogonal ($\perp$) or equivalent to μ.

Definition 3.3. Let μ be a Borel measure in V. The set of admissible translations of μ in the sense of quasiinvariance is

$$Q_{\mu} = \{h \in V : \mu_h \sim \mu\}.$$

It is clear that $Q_{\mu} = V$ iff μ is translation-quasiinvariant.

Definition 3.4. Let μ be a Borel measure in V. The set of admissible translations of μ in the sense of invariance is

$$I_{\mu} = \{h \in V : \mu_h = \mu\}.$$

It can be shown that $I_\mu = V$ iff μ is translation-invariant.

Remark 3.1. Obviously, $I_\mu \subseteq Q_\mu$ for every measure μ. Note that every quasifinite Borel measure with $Q_\mu = V$ or $I_\mu = V$ is a Gikhman–Skorokhod measure in V.

Now there naturally arises a question whether an analogous result is valid when Q_μ or I_μ are everywhere dense linear manifolds in V.

In this context I.I. Gikhman and A.V. Skorokhod considered the following problem in [10, Chapter 7, Paragraph 2]:

Does there exist a probability Borel measure μ in the Hilbert space ℓ_2 which satisfies the following conditions?

(i) The group Q_μ of all admissible translations (in the sense of quasiinvariance) is an everywhere dense linear manifold in ℓ_2.
(ii) There exists $a \in \ell_2 \setminus Q_\mu$ such that a measure μ is not orthogonal to the measure $\mu^{(a)}$, where
$$\mu^{(a)}(X) = \mu(X - a)$$
for $X \in \mathcal{B}(\ell_2)$.

Gikhman–Skorokhod's positive solution of this problem employs the technique of Gaussian measures in an infinite-dimensional separable Hilbert space. In [18], Gikhman–Skorokhod's result was extended to invariant Borel measures in ℓ_2. In particular, a nonzero σ-finite Borel measure μ is constructed in ℓ_2 which satisfies the following conditions:

(iii) The group I_μ of all admissible (in the sense of invariance) translations for the measure μ is an everywhere dense linear manifold in ℓ_2.
(iv) There exists $a \in \ell_2 \setminus I_\mu$ such that a measure $\mu^{(a)}$ is not orthogonal to the measure μ.

Definition 3.5. A quasifinite Borel measure μ in V is called a quasigenerator if the following two conditions are satisfied:

(i) Q_μ is a linear manifold in V.
(ii) There exists a σ-compact F such that for all $u \in Q_\mu$ and $v \in Q_\mu^\perp \setminus \{\mathbf{0}\}$, $\mu(V \setminus (F \cap (F + u))) = 0$ and $\mu(F \cap (F + v)) = 0$.

Remark 3.2. Note that every quasigenerator is a Gikhman–Skorokhod measure, but the converse statement is not always valid. Indeed, consider an arbitrary generator λ in an infinite-dimensional Polish topological vector space V. Obviously, $Q_\lambda = V$, which means that λ is a Gikhman–Skorokhod measure. If we assume that λ is a quasigenerator, then there will be a σ-compact set F such that $\lambda(V \setminus (F \cap (F + \mathbf{0}))) = 0$ because $\mathbf{0} \in Q_\lambda$. Therefore, $\lambda(V \setminus F) = 0$, from which by Definition 2.4 it follows that $V \setminus F \in S(V)$. So $F \in S(V)$ (cf. [12, Fact 8, p. 225]), we get $V = (V \setminus F) \cup F \in S(V)$, which is a contradiction.

Let μ be an arbitrary Borel measure in V. Let us define a functional G_μ by

$$G_\mu(X) = \sum_{v \in Q_\mu^\perp} \mu(X+v)$$

for $X \in \mathcal{B}(V)$. The main result of the present section is formulated as follows.

Theorem 3.1. *Let μ be an arbitrary Borel measure in V. Then G_μ is a generator in V such that*

$$S(\mu) = N(\overline{G}_\mu),$$

where $\overline{G}_\mu$ denotes a usual completion of G_μ. If μ is a quasigenerator, then the generator G_μ is quasifinite.

Proof. *Step* (i) We show that if $G_\mu(X) = 0$ then $G_\mu(X+h) = 0$ for every $h \in V$. For $h \in V$ we have the representation $h = h_1 + h_2$, where $h_1 \in Q_\mu$ and $h_2 \in Q_\mu^\perp$. Thus

$$\begin{aligned} G_\mu(X) = 0 &\Leftrightarrow \sum_{v \in Q_\mu^\perp} \mu(X+v) = 0 \Leftrightarrow \sum_{v \in Q_\mu^\perp} \mu(X+h_1+v) = 0 \\ &\Leftrightarrow \sum_{v \in Q_\mu^\perp} \mu(X+h_1+h_2+v) = 0 \Leftrightarrow \sum_{v \in Q_\mu^\perp} \mu((X+h)+v) = 0 \\ &\Leftrightarrow G_\mu(X+h) = 0. \end{aligned}$$

Step (ii) We show the measure G_μ is quasifinite if μ is a quasigenerator. Indeed, in such a situation there exists a σ-compact set F such that for all $u \in Q_\mu$ and $v \in Q_\mu^\perp \setminus \{\mathbf{0}\}$, $\mu(V \setminus (F \cap (F+u))) = 0$ and $\mu(F \cap (F+v)) = 0$.

Since $\mu(V \setminus F) = 0$, there exists a compact set $U \subset F$ with $0 < \mu(U) < \infty$. Thus, $\mu(U+v) = 0$ for all $v \in Q_\mu^\perp \setminus \{\mathbf{0}$, as

$$\mu(U+v) = \mu((U+v) \cap F) \leq \mu((F+v) \cap F) = 0$$

for $v \in Q_\mu^\perp \setminus \{\mathbf{0}\}$.

Hence

$$G_\mu(U) = \sum_{v \in Q_\mu^\perp} \mu(U+v) = \mu(U+\mathbf{0}) = \mu(U).$$

Thus G_μ is a quasifinite generator.

Step (iii) We show that $S(\mu) = N(\overline{G}_\mu)$. Let $X \in S(\mu)$. This means that there exists a Borel set $X^{'}$ such that $X \subseteq X^{'}$ and $\mu(X^{'} + v) = 0$ for $v \in V$. The latter relation means that

$$\sum_{v \in Q_\mu^\perp} \mu(X^{'} + v) = 0$$

which implies that $X^{'} \in N(G_\mu)$ and $X \in N(\overline{G}_\mu)$.

Now let $X \in N(\overline{G}_\mu)$. This means that there exists a Borel set X', where $X \subseteq X'$ and

$$\sum_{v \in Q_\mu^\perp} \mu(X' + v) = 0.$$

The latter relation implies that $\mu(X' + v) = 0$ for $v \in Q_\mu^\perp$.

Now let $h = h_1 + h_2$ be an arbitrary element of V, where $h_1 \in Q_\mu$ and $h_2 \in Q_\mu^\perp$. Since $\mu(X' + h_2) = 0$ and $h_1 \in Q_\mu$, we conclude that $\mu(X' + h) = 0$, which means that $X' \in S(\mu)$. Therefore $X \in S(\mu)$ because $X \subseteq X'$. $\square$

4. On the quasifiniteness problem for Gaussian generators

The problem of equivalence and orthogonality relations between two measures in infinite-dimensional topological vector spaces, which underlies the notion of a quasigenerator, has been investigated by many authors. In this direction, a special mention should be made of the result of S. Kakutani [13] stating that if one has equivalent probability measures μ_i and ν_i on the σ-algebra L_i of subsets of a set $\Omega_i, i = 1, 2, \dots,$ and if μ and ν denote respectively the infinite product measures $\prod_{i \in \mathbb{N}} \mu_i$ and $\prod_{i \in \mathbb{N}} \nu_i$ on the infinite product σ-algebra generated on the infinite product set Ω, then μ and ν are either equivalent or orthogonal. Similar dichotomies have revealed themselves in the study of Gaussian stochastic processes. C. Cameron and W.E. Martin showed in [5] that if one considers the measures induced on a path space by a Wiener process on the unit interval, then the measures are orthogonal provided that the variances of the processes are different. Results of this kind were generalized by many authors (cf. [9], [11], [26] and others).

Let μ be a Gaussian measure in V. The question whether G_μ is quasifinite depends on the structure of μ. Below we give a simple example of such a generator in an infinite-dimensional Polish topological vector space V.

Example 4.1. Let $e_1, \dots, e_k$ be any family of linearly independent vectors in V. We set $L(e_1, \dots, e_k) = \mathrm{span}(e_1, \dots, e_k)$. We denote by $\lambda_{L(e_1,\dots,e_k)}$ a Gaussian Borel measure on $L(e_1, \dots, e_k)$ defined by

$$\lambda_{L(e_1,\dots,e_k)}(\{t_1 e_1 + \dots + t_k e_k : (t_1, \dots, t_k) \in X\}) = \gamma_k(X),$$

where $X \in \mathcal{B}(\mathbb{R}^k)$ and γ_k is a standard k-dimensional Gaussian Borel measure in $\mathbb{R}^k$.

Clearly, $\lambda_{L(e_1,\dots,e_k)}$ is a quasigenerator in V and $G_{\lambda_{L(e_1,\dots,e_k)}}$ is a quasifinite Gaussian generator in V such that

$$S(\lambda_{L(e_1,\dots,e_k)}) = N(\overline{G}_{\lambda_{L(e_1,\dots,e_k)}}).$$

Remark 4.1. Let μ_k be the standard Gaussian probability Borel measure in $\mathbb{R}$ for $k \in \mathbb{N}$. Let $X = \mathbb{R}^\mathbb{N}$ and $\mu = \prod_{k \in \mathbb{N}} \mu_k$ be the canonical Gaussian Borel probability measure in $\mathbb{R}^\mathbb{N}$. Following Kakutani [14], $Q_\mu = \ell_2$, which implies that μ is a Gikhman–Skorokhod measure. It follows from Theorem 3.1 that G_μ is a generator and

$$N(\overline{G}_\mu) = S(\mu),$$

but we do not know whether G_μ is a quasifinite.

Remark 4.2. We remind the reader that a Borel measure μ in an infinite-dimensional separable Hilbert space is called Gaussian if an arbitrary continuous linear functional $\ell_z(x) = (z, x)(z, x \in H)$ is a normally distributed random variable (cf. [10, Chapter v, Paragraph 6]).

Following A.M. Vershik [26], a group Q_μ of arbitrary Gaussian measure μ in an infinite-dimensional separable Hilbert space is linear manifold. In particular, if μ is a Gaussian measure in H with zero mean and correlation operator B, then $Q_\mu = B^{\frac{1}{2}}(H)$ (cf. [10]).

Following Phelps [21], a Borel set is called Gaussian null if it is null for every Gaussian measure in H. A set is called Gaussian null if it is contained in a Borel Gaussian null set.

The class of all Gaussian null sets in H is denoted by $\mathrm{GN}(H)$. Let Γ be a class of all Gaussian measures in H. *Then the representation*

$$\mathrm{GN}(H) = \cap_{\gamma \in \Gamma} N(\overline{G}_\gamma)$$

is valid. The proof of this fact employs the result of Theorem 3.1.

In the context of Example 4.1, there naturally arises a question *whether any Gaussian measure γ in H is a quasigenerator or whether G_γ is quasifinite.*

5. On generators defined by Kharazishvili's quasigenerators

The problem of the existence of a partial analog of a Lebesgue measure in an infinite-dimensional topological vector space is interesting and important in itself and has been studied for over 50 years by many authors. Among their results the result of V. Sudakov [24] should be mentioned specially. This result asserts that an arbitrary σ-finite quasiinvariant Borel measure in an infinite-dimensional locally convex topological vector space is identically zero. According to this result, the properties of the σ-finiteness and the translation-invariance are not consistent for nonzero Borel measures in infinite-dimensional topological vector spaces. A.B. Kharazishvili [15] constructed an example of a nonzero non-translation-invariant σ-finite Borel measure in the Hilbert space ℓ_2 which is invariant with respect to an everywhere dense (in ℓ_2) linear manifold. R. Baker gave constructions of quasifinite non-σ-finite translation-invariant Borel measures in the infinite-dimensional topological vector space $\mathbb{R}^{\mathbb{N}}$ (cf. Examples 2.1–2.2). A similar construction (cf. [17]) is given in the well-known Solovay's model [22].

Below we present the construction of A.B. Kharazishvili's quasigenerators in $\mathbb{R}^{\mathbb{N}}$.

Let $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}}$ be sequences of real numbers such that $a_i < b_i$ for all $i \in \mathbb{N}$.

We set

$$A_n = \mathbb{R}_0 \times \cdots \times \mathbb{R}_n \times \prod_{i>n} \Delta_i,$$

for $n \in \mathbb{N}$, where $\mathbb{R}_i = \mathbb{R}$ for $i = 1, \dots, n$ and $\Delta_i = [a_i, b_i[$ for $i > n$. We also set

$$\Delta = \prod_{i\in\mathbb{N}} \Delta_i \text{ and } B_\Delta = \cup_{n\in\mathbb{N}} A_n.$$

For an arbitrary natural number $i \in \mathbb{N}$, consider the Lebesgue measure μ_i in the space $\mathbb{R}_i$ and satisfying the condition $\mu_i(\Delta_i) = 1$. Let us denote by λ_i the normalized Lebesgue measure defined on the interval Δ_i.

For an arbitrary $n \in \mathbb{N}$, let us denote by ν_n the measure defined by

$$\nu_n = \prod_{1\le i\le n} \mu_i \times \prod_{i>n} \lambda_i,$$

and by $\overline{\nu}_n$ the Borel measure in the space $\mathbb{R}^{\mathbb{N}}$ defined by

$$\overline{\nu}_n(X) = \nu_n(X \cap A_n)$$

for $X \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}$.

Lemma 5.1 ([19, Lemmas 15.3.2–15.3.3, p. 207])**.** *The functional ν_Δ defined by*

$$\nu_\Delta(X) = \lim_{n\to\infty} \overline{\nu}_n(X),$$

is a quasifinite Borel measure for which

$$I_{\nu_\Delta} = Q_{\nu_\Delta} = \left\{ h = (h_1, h_2, \dots) \in \mathbb{R}^{\mathbb{N}} : \sum_{i\in\mathbb{N}} \frac{|h_i|}{b_i - a_i} \quad \textit{is convergent} \right\}$$

and for all $u \in Q_{\nu_\Delta}$ and $v \in Q^{\perp}_{\nu_\Delta} \setminus \{\mathbf{0}\}$,

$$\mu(\mathbb{R}^{\mathbb{N}} \setminus (\cup_{n\in N} A_n \cap (\cup_{n\in N} A_n + u))) = 0,$$
$$\mu((\cup_{n\in N} A_n) \cap (\cup_{n\in N} A_n + v)) = 0.$$

As a consequence of Lemma 5.1 we get:

Corollary 5.1. *The functional ν_Δ is a quasigenerator in $\mathbb{R}^{\mathbb{N}}$.*

Remark 5.1. The construction of ν_Δ belongs to A.B. Kharazishvili [15]. For this reason, a quasigenerator ν_Δ is called A.B. Kharazishvili's quasigenerator. Note that a generator G_{ν_Δ} is a semifinite inner regular translation-invariant Borel measure in $\mathbb{R}^{\mathbb{N}}$, which assigns value one to Δ (cf. [19]).

Now let K be the class of all positive sequences $(a_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that

$$0 < \prod_{k\in\mathbb{N}} a_k < \infty.$$

Let $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \in K$. We say that $(a_k)_{k\in\mathbb{N}} \simeq (b_k)_{k\in\mathbb{N}}$ if

$$\nu_{\prod_{k\in\mathbb{N}}[0,a_k]} = \nu_{\prod_{k\in\mathbb{N}}[0,b_k]}.$$

Lemma 5.2 ([19, Lemma 15.3.3, p. 207])**.** *The relation $\simeq$ is an equivalence relation on K.*

Lemma 5.3 ([19, Lemma 15.3.4, p. 207])**.** *Suppose $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \in K$ are not equivalent. Then $\nu_{\prod_{k\in\mathbb{N}}[0,a_k]} \perp \nu_{\prod_{k\in\mathbb{N}}[0,b_k]}$.*

Definition 5.1. Suppose $\mathcal{R}_1$ is the class of all infinite-dimensional rectangles $R \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ of a form

$$R = \prod_{k=1}^{\infty}(a_i, b_i),\ -\infty < a_i \le b_i < \infty$$

such that $0 \le \prod_{k=1}^{\infty}(b_k - a_k) < \infty$. Following [2], a translation-invariant Borel measure λ in $\mathbb{R}^{\mathbb{N}}$ is called Baker's generator if

$$\lambda(R) = \prod_{k=1}^{\infty}(b_k - a_k)$$

for $R \in \mathcal{R}_1$.

Theorem 5.1 ([19, Theorem 15.3.1, p. 208])**.** *There exists a semifinite inner regular Baker's generator λ in $\mathbb{R}^{\mathbb{N}}$.*

Proof. Let us consider the equivalence classes $(K_i)_{i\in I}$ of K generated by the equivalence relation $\simeq$ (cf. [19, Lemma 15.3.3]). Let $\left(a_k^{(i)}\right)_{k\in\mathbb{N}} \in K_i$ for $i \in I$. We set $\Delta_i = \prod_{k\in\mathbb{N}}\left[0, a_k^{(i)}\right]$, $B_i = B_{\Delta_i}$ and $\mu_i = \nu_{\Delta_i}$. We put

$$\lambda(X) = \sum_{i\in I}\sum_{g\in\ell_1^{\perp}} \mu_i((X-g)\cap B_i),$$

for $X \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$, where $\ell_1^{\perp}$ denotes a linear complement of the vector subspace ℓ_1 in $\mathbb{R}^{\mathbb{N}}$.

We shall show that λ is a translation-invariant Borel measure in $\mathbb{R}^{\mathbb{N}}$ such that

$$\lambda\left(\prod_{k=1}^{\infty}(a_k, b_k)\right) = \prod_{k=1}^{\infty}(b_k - a_k).$$

We do this in steps.

Step (i) We show λ is σ-additive. Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of pairwise disjoint Borel subsets in $\mathbb{R}^{\mathbb{N}}$. Then

$$\begin{aligned}\lambda(\cup_{k\in\mathbb{N}}X_k) &= \sum_{i\in I}\sum_{g\in\ell_1^{\perp}} \mu_i((\cup_{k\in\mathbb{N}}X_k - g)\cap B_i)\\ &= \sum_{i\in I}\sum_{g\in\ell_1^{\perp}} \mu_i((\cup_{k\in\mathbb{N}}(X_k - g))\cap B_i)\\ &= \sum_{i\in I}\sum_{g\in\ell_1^{\perp}}\sum_{k\in\mathbb{N}} \mu_i((X_k - g)\cap B_i)\\ &= \sum_{k\in\mathbb{N}}\sum_{i\in I}\sum_{g\in\ell_1^{\perp}} \mu_i((X_k - g)\cap B_i) = \sum_{k\in\mathbb{N}}\lambda(X_k).\end{aligned}$$

Step (ii)We show that λ is translation-invariant. Taking into account the equality $G_\Delta = \ell_1$ (cf. [19, Lemma 15.3.2, p. 214]) and the fact that an arbitrary element $h \in \mathbb{R}^{\mathbb{N}}$ can be written in the form $h = h_1 + h_2$, where $h_1 \in \ell_1$ and $h_2 \in \ell_1^{\perp}$, we get

$$\begin{aligned}\lambda(X+h) &= \sum_{i\in I}\sum_{g\in\ell_1^{\perp}} \mu_i(((X+h_1+h_2)-g)\cap B_i)\\ &= \sum_{i\in I}\sum_{g\in\ell_1^{\perp}} \mu_i(((X+h_2)-g+h_1)\cap B_i)\\ &= \sum_{i\in I}\sum_{g\in\ell_1^{\perp}} \mu_i(((X+h_2)-g)\cap B_i)\\ &= \sum_{i\in I}\sum_{g\in\ell_1^{\perp}} \mu_i((X-(g-h_2))\cap B_i)\\ &= \sum_{i\in I}\sum_{g-h_2\in\ell_1^{\perp}-h_2} \mu_i((X-(g-h_2))\cap B_i)\\ &= \sum_{i\in I}\sum_{g-h_2\in\ell_1^{\perp}} \mu_i((X-(g-h_2))\cap B_i)\\ &= \sum_{i\in I}\sum_{\widetilde{g}\in\ell_1^{\perp}} \mu_i((X-\widetilde{g})\cap B_i) = \lambda(X).\end{aligned}$$

Step (iii) We now show

$$\lambda\left(\prod_{k=1}^{\infty}(a_k,b_k)\right) = \prod_{k=1}^{\infty}(b_k-a_k).$$

Assume that K_{i_0} is a class of equivalence of K such that $(b_k-a_k)_{k\in\mathbb{N}} \in K_{i_0}$. By using the translation-invariance of λ we have

$$\begin{aligned}\lambda\left(\prod_{k=1}^{\infty}(a_k,b_k)\right) &= \lambda\left(\prod_{k=1}^{\infty}(a_k,b_k)-(a_k)_{k\in\mathbb{N}}\right) = \lambda\left(\prod_{k=1}^{\infty}(0,b_k-a_k)\right)\\ &= \sum_{i\in I}\sum_{g\in\ell_1^{\perp}}\mu_i\left(\left(\prod_{k=1}^{\infty}(0,b_k-a_k)-g\right)\cap B_i\right)\\ &= \mu_{i_0}\left(\left(\prod_{i=1}^{\infty}(0,b_k-a_k)-\mathbf{0}\right)\cap B_{i_0}\right) = \prod_{k=1}^{\infty}(b_k-a_k).\end{aligned}$$

Finally, one can prove that if $\prod_{k=1}^{\infty}(b_k-a_k)=0$, then $\lambda(\prod_{k=1}^{\infty}(a_k,b_k))=0$.

Step (iv) We now show the generator λ is inner regular. To this end, we must to show that for all X with $0<\lambda(X)<\infty$ and all $\epsilon>0$, there exists a compact $F_\epsilon \subseteq X$ with $\lambda(X\setminus F_\epsilon)<\epsilon$.

Since $0<\lambda(X)<\infty$, there exist $I_0\subseteq I$ and $G_0\subseteq \ell_1^{\perp}$ with $\operatorname{card}(I_0\times G_0)\le \aleph_0$ and

$$0<\lambda(X)=\sum_{i\in I_0}\sum_{g\in G_0}\mu_i((X-g)\cap B_i)<\infty.$$

We set $I_0\times G_0=(i_k,g_m)_{k,m\in\mathbb{N}}$. Let n_0 be a natural number such that

$$\sum_{1\le k\le n_0}\sum_{1\le m\le n_0}\mu_{i_k}((X-g_m)\cap B_{i_k})>\lambda(X)-\frac{\epsilon}{2}.$$

For $\epsilon>0$ there exists a compact set $F_{k,m}$ such that $F_{k,m}\subseteq (X-g_m)\cap B_{i_k}$ and

$$\mu_{i_k}(((X-g_m)\cap B_{i_k})\setminus F_{k,m})<\frac{\epsilon}{2^{k+m+1}}$$

for $1\le k,m\le n_0$.

We set $F_\epsilon=\cup_{1\le p,q\le n_0}(F_{p,q}+g_q)$. It is clear that $F_\epsilon\subseteq X$. We get

$$\begin{aligned}&\lambda(X\setminus F_\epsilon)\\ &=\lambda(X\setminus \cup_{1\le p,q\le n_0}(F_{p,q}+g_q))\\ &=\sum_{i\in I}\sum_{g\in\ell_1^{\perp}}\mu_i(((X\setminus \cup_{1\le p,q\le n_0}(F_{p,q}+g_q))-g)\cap B_i)\\ &=\sum_{i\in I_0}\sum_{g\in G_0}\mu_i(((X\setminus \cup_{1\le p,q\le n_0}(F_{p,q}+g_q))-g)\cap B_i)\\ &\quad+\sum_{(i,g)\in I\times\ell_1^{\perp}\setminus I_0\times G_0}\mu_i(((X\setminus \cup_{1\le p,q\le n_0}(F_{p,q}+g_q))-g)\cap B_i)\\ &\le\sum_{i\in I_0}\sum_{g\in G_0}\mu_i(((X\setminus \cup_{1\le p,q\le n_0}(F_{p,q}+g_q))-g)\cap B_i)\end{aligned}$$

$$+ \sum_{(i,g)\in I\times \ell_1^{\perp}\setminus I_0\times G_0} \mu_i((X-g)\cap B_i)$$
$$= \sum_{i\in I_0}\sum_{g\in G_0} \mu_i(((X\setminus \cup_{1\le p,q\le n_0}(F_{p,q}+g_q))-g)\cap B_i)$$
$$= \sum_{k\in \mathbb{N}}\sum_{m\in \mathbb{N}} \mu_{i_k}(((X\setminus \cup_{1\le p,q\le n_0}(F_{p,q}+g_q))-g_m)\cap B_{i_k})$$
$$= \sum_{1\le k\le n_0}\sum_{1\le m\le n_0} \mu_{i_k}(((X\setminus \cup_{1\le p,q\le n_0}(F_{p,q}+g_q))-g_m)\cap B_{i_k})$$
$$+ \sum_{(k,m)\in \mathbb{N}\times\mathbb{N}\setminus\{1,\dots,n_0\}\times\{1,\dots,n_0\}} \mu_{i_k}(((X\setminus \cup_{1\le p,q\le n_0}(F_{p,q}+g_q))-g_m)\cap B_{i_k})$$
$$\le \sum_{1\le k\le n_0}\sum_{1\le m\le n_0} \mu_{i_k}((X\setminus (F_{k,m}+g_m)-g_m)\cap B_{i_k})$$
$$+ \sum_{(k,m)\in \mathbb{N}\times\mathbb{N}\setminus\{1,\dots,n_0\}\times\{1,\dots,n_0\}} \mu_{i_k}((X-g_m)\cap B_{i_k})$$
$$= \sum_{1\le k\le n_0}\sum_{1\le m\le n_0} \mu_{i_k}(((X-g_m)\setminus F_{k,m})\cap B_{i_k})$$
$$+ \sum_{(k,m)\in \mathbb{N}\times\mathbb{N}\setminus\{1,\dots,n_0\}\times\{1,\dots,n_0\}} \mu_{i_k}((X-g_m)\cap B_{i_k})$$
$$= \sum_{1\le k\le n_0}\sum_{1\le m\le n_0} \mu_{i_k}(((X-g_m)\cap B_{i_k})\setminus (F_{k,m}\cap B_{i_k}))$$
$$+ \sum_{(k,m)\in \mathbb{N}\times\mathbb{N}\setminus\{1,\dots,n_0\}\times\{1,\dots,n_0\}} \mu_{i_k}((X-g_m)\cap B_{i_k})$$

since $F_{k,m}\subseteq (X-g_m)\cap B_{i_k}$ we have $F_{k,m}\cap B_{i_k}=F_{k,m}$

$$= \sum_{1\le k\le n_0}\sum_{1\le m\le n_0} \mu_{i_k}(((X-g_m)\cap B_{i_k})\setminus F_{k,m})$$
$$+ \sum_{(k,m)\in (\mathbb{N}\times\mathbb{N})\setminus(\{1,\dots,n_0\}\times\{1,\dots,n_0\})} \mu_{i_k}((X-g_m)\cap B_{i_k})$$
$$\le \sum_{1\le k,m\le n_0} \frac{\epsilon}{2^{k+m+1}}+\frac{\epsilon}{2}$$
$$\le \sum_{k,m\in\mathbb{N}} \frac{\epsilon}{2^{k+m+1}}+\frac{\epsilon}{2}=\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

Step (v) We show that λ is semifinite. Indeed, if $\lambda(X)>0$, then there exist $i_0\in I$ and $g_0\in \ell_1^{\perp}$ such that $0<\mu_{i_0}((X-g_0)\cap B_{i_0})$. Using the property of inner regularity of μ_{i_0} we deduce that there exists a compact set $F\subseteq X$ with $0<\lambda(F)<\infty$. This means that λ is a semifinite measure. □

Remark 5.2. If we consider R. Baker's measure constructed in [2], we observe that it is a nonsemifinite quasifinite Baker's generator. Indeed, it takes the value $+\infty$ on the set $(3\mathbb{Z})^{\mathbb{N}}$, while every compact subset of $(3\mathbb{Z})^{\mathbb{N}}$ is null with respect to the same measure. Note that the set $(3\mathbb{Z})^{\mathbb{N}}$ is null with respect to A.B. Kharazishvili's generator λ, because $\mu_i(((3\mathbb{Z})^{\mathbb{N}}-g)\cap B_i))=0$ for all $i\in I$ and $g\in \ell_1^{\perp}$.

6. Representation of cube null sets by P. Mankiewicz's generator

The following assertion plays key role in our further discussion.

Theorem 6.1 ([19, Theorem 15.3.2, p. 209])**.** *There exists a semifinite inner regular translation-invariant Borel measure λ_1 in $\mathbb{R}^{\mathbb{N}}$ which satisfies the following conditions:*

(1) $\lambda_1([0,1]^{\mathbb{N}})=1$.

(2) *There exists a rectangle $\prod_{k=1}^{\infty}(a_k,b_k)$ with $0<\prod_{k=1}^{\infty}(b_k-a_k)<\infty$ such that*

$$\lambda_1\left(\prod_{k=1}^{\infty}(a_k,b_k)\right)=0.$$

Remark 6.1. The inner regularity and semifiniteness can be proved by the scheme of the proof of Theorem 5.1.

Remark 6.2. Note that the generator λ_1 is absolutely continuous with respect to R. Baker's generator λ [2] but the converse relation is not valid (cf. Theorem 6.1, condition (2)). This means that the generators λ and λ_1 are not equivalent.

Let $(\mu_k)_{k\in\mathbb{N}}$ be a family of linear Lebesgue measures on $[0,1]$ such that $\mu_k([0,1])=1$ for all k. We set $\mu=\prod_{k\in\mathbb{N}}\mu_k$.

Definition 6.1. A Borel subset $X\subset\mathbb{R}^{\mathbb{N}}$ is said to be a standard cube null set if $\mu((X+a)\cap[0,1]^{\mathbb{N}})=0.$ for all $a\in\mathbb{R}^{\mathbb{N}}$. More generally, a set is called a standard cube null set if it is contained in a Borel standard cube null set.

The class of all standard cube null sets in $\mathbb{R}^{\mathbb{N}}$ is denoted by $\mathcal{SCN}(\mathbb{R}^{\mathbb{N}})$. It is clear that $\mathcal{SCN}(\mathbb{R}^{\mathbb{N}})\subset\mathcal{S}(\mathbb{R}^{\mathbb{N}})$.

Now naturally there arises the following:

Problem 6.1. *Does there exist a semifinite inner-regular generator ν in $\mathbb{R}^{\mathbb{N}}$ such that $\mathcal{N}(\overline{\nu})=\mathcal{SCN}(\mathbb{R}^{\mathbb{N}})$?*

The answer to this problem is contained in the following assertion.

Theorem 6.2 ([19, Theorem 15.3.3, p. 211])**.** *The generator λ_1 is a solution of Problem* 6.1, i.e.,

$$\mathcal{N}(\overline{\lambda}_1)=\mathcal{SCN}(\mathbb{R}^{\mathbb{N}}).$$

Remark 6.3. Following Theorem 6.2, the measure λ_1 is a standard cube generator which can be called P. Mankiewicz's generator in $\mathbb{R}^{\mathbb{N}}$. It is reasonable to note that P. Mankiewicz's generator λ_1 is defined by A.B. Kharazishvili's standard quasigenerator $\nu_{[0,1]^{\mathbb{N}}}$, i.e.,

$$\lambda_1 = G_{\nu_{[0,1]^{\mathbb{N}}}},$$

and the following equality holds:

$$\mathcal{SCN}(\mathbb{R}^{\mathbb{N}}) = \mathcal{N}(\overline{G}_{\nu_{[0,1]^{\mathbb{N}}}}).$$

Since the quasigenerator $\nu_{[0,1]^{\mathbb{N}}}$ can be defined uniquely by the Haar measure μ in $[0,1]^{\mathbb{N}}$, we can use the notation $G(\mu)$ for the generator $G_{\nu_{[0,1]^{\mathbb{N}}}}$. Such notation will be applied below.

Remark 6.4. Let λ and μ be R. Baker's generators in $\mathbb{R}^{\mathbb{N}}$ considered in Examples 2.1 and 2.2, respectively. Since[1]

$$G_{\nu_{[0,1]^{\mathbb{N}}}} \ll \mu \ll \lambda,$$

we deduce that the following strict inclusions

$$\mathcal{N}(\lambda) \subset \mathcal{N}(\mu) \subset \mathcal{N}(G_{\nu_{[0,1]^{\mathbb{N}}}})(= \mathcal{B}(\mathbb{R}^{\mathbb{N}}) \cap \mathcal{SCN}(\mathbb{R}^{\mathbb{N}}))$$

are valid.

Now we consider one interesting application of Theorem 6.2 in a separable Banach space with basis.

Let B be a separable Banach space with basis $e_1, e_2, \dots$ such that

$$\sum_{k \in \mathbb{N}} ||e_k|| < \infty.$$

Definition 6.2. Following P. Mankiewicz [16], a Borel set $X \subset B$ is called a cube null set, if it is null for every nondegenerate cube measure. (Nondegenerate cube measures in B can be defined as distributions of the random variables of the form $a + \sum_{k \in \mathbb{N}} X_k e_k$, where $a \in B$ and $(X_k)_{k \in N}$ are uniformly distributed independent random variables with values in $[0,1]$.)

The class of all cube null sets in B is denoted by $\mathcal{CNS}(B)$.

Let us denote by $\mathcal{C}$ the class of all cube measures defined in B. For $\mu \in \mathcal{C}$ we denote by $G(\mu)$ Mankiewicz' generator defined by μ. Then it is clear that

$$\mathcal{CNS}(B) = \cap_{\mu \in \mathcal{C}} \mathcal{N}(\overline{G}(\mu)).$$

Again let V be a Polish topological vector space.

Definition 6.3 ([25, Definition 6, p. 225])**.** We call a finite-dimensional subspace $L \subset V$ a probe for a set $X \subset V$ if an n-dimensional classical Borel measure λ_L supported on L is transverse to a Borel set containing the complement of X.

---

[1] Let μ and λ be two measures such that $\mathrm{dom}(\mu) = \mathrm{dom}(\lambda)$. We say that μ is absolutely continuous with respect to λ and write $\mu \ll \lambda$, if $\lambda(X) = 0$ implies $\mu(X) = 0$.

Let λ_L be an n-dimensional classical Borel measure in an n-dimensional vector subspace $L \subset V$, where $n \in \mathbb{N}$. It is obvious that G_{λ_L} is a quasifinite translation-invariant generator in V.

It can be easily established that

$$\mathcal{N}(\overline{G}_{\lambda_L}) \subset \mathcal{S}(V)$$

for all finite-dimensional vector subspace $L \subset V$.

One can show that every subset $X \subset V$ is of $\overline{G}_{\lambda_L}$-measure zero if and only if the linear manifold L is a probe for the sets $V \setminus X$. The class of all subsets of V whose complement have n-dimensional probes is denoted by $n(V)$. It is clear that $\cup_{n\in\mathbb{N}} n(V)$ is a subclass of the σ-ideal $S(V)$ and we have the equality

$$\cup_{n\in\mathbb{N}} n(V) = \cup_{L\subset V} \mathcal{N}(\overline{G}_{\lambda_L}),$$

where the union on the right side is considered over all finite-dimensional manifolds.

7. On the geometric aspects of Preiss–Tišer generators

Definition 7.1. A Borel set S in a separable Banach space B is said to be an n-dimensional Preiss–Tišer null set if every Lebesgue measure μ concentrated on any n-dimensional vector space Γ is transverse to S. Any subset A of such a Borel set B is said to be an n-dimensional Preiss–Tišer null set.

We denote the class of all n-dimensional Preiss–Tišer null sets in B by $\mathcal{PTNS}(B, n)$.

Let $(\Gamma_i)_{i\in I}$ be a family of all n-dimensional vector spaces and let μ_i be an n-dimensional Lebesgue measure concentrated on Γ_i for $i \in I$.

Let $\Gamma_i^{\perp}$ be a linear complement of the vector space Γ_i in B for $i \in I$. We put

$$\lambda(X) = \sum_{i\in I} \sum_{g\in\Gamma_i^{\perp}} \mu_i((X - g) \cap \Gamma_i)$$

for $X \in \mathcal{B}(B)$.

Theorem 7.1. *A functional λ is a quasifinite generator in B such that*

$$\mathcal{PTNS}(B, n) = \mathcal{N}(\bar{\lambda}).$$

Proof. By using the construction of the proof of Theorem 5.1 one can easily show that λ is a Borel measure in B.

Now we are to show that λ is quasifinite and translation-invariant.

Let $i_0 \in I$ and let $(e_k)_{1\le k\le n}$ be any basis of Γ_{i_0}. The quasifiniteness is obvious because

$$0 < \mu_{i_0}\left(\left\{ \sum_{i=1}^{n} \alpha_k e_k \ : \ (\alpha_k)_{1\le k\le n} \in [0,1]^n \right\}\right) < \infty$$

and

$$\lambda\left(\left\{\sum_{i=1}^{n}\alpha_k e_k \ : \ (\alpha_k)_{1\le k\le n}\in[0,1]^n\right\}\right)$$
$$= \mu_{i_0}\left(\left\{\sum_{i=1}^{n}\alpha_k e_k \ : \ (\alpha_k)_{1\le k\le n}\in[0,1]^n\right\}\right).$$

Let us show that λ is translation-invariant. Indeed, let $X\in\mathcal{B}(B)$ and $h\in B$. For every $i\in I$ we have $h=h_i^{(0)}+h_i^{(1)}$, where $h_i^{(0)}\in\Gamma_i$ and $h_i^{(1)}\in\Gamma_i^{\perp}$. Then we obtain

$$\begin{aligned}
\lambda(X+h) &= \sum_{i\in I}\sum_{g\in\Gamma_i^{\perp}}\mu_i(((X+h)-g)\cap\Gamma_i)\\
&= \sum_{i\in I}\sum_{g\in\Gamma_i^{\perp}}\mu_i((X+h_i^{(0)}+h_i^{(1)}-g)\cap\Gamma_i)\\
&= \sum_{i\in I}\sum_{g\in\Gamma_i^{\perp}}\mu_i((X+h_i^{(2)}-g)\cap\Gamma_i)\\
&= \sum_{i\in I}\sum_{g\in\Gamma_i^{\perp}}\mu_i((X-(g-h_i^{(2)}))\cap\Gamma_i)\\
&= \sum_{i\in I}\sum_{g-h_i^{(2)}\in\Gamma_i^{\perp}-h_i^{(2)}}\mu_i((X-(g-h_i^{(2)}))\cap\Gamma_i)\\
&= \sum_{i\in I}\sum_{g-h_i^{(2)}\in\Gamma_i^{\perp}}\mu_i((X-(g-h_i^{(2)}))\cap\Gamma_i)\\
&= \sum_{i\in I}\sum_{s\in\Gamma_i^{\perp}}\mu_i((X-s)\cap\Gamma_i)=\lambda(X).
\end{aligned}$$

Now let us assume that $Y\in\mathcal{PTNS}(B,n)$. Then there exists a Borel set X such that $Y\subseteq X$ and $\mu_i((X-g)\cap\Gamma_i))=0$ for $g\in\Gamma_i^{\perp}$. Therefore,

$$\sum_{i\in I}\sum_{g\in\Gamma_i^{\perp}}\mu_i((X-g)\cap\Gamma_i))=0.$$

This means that $\lambda(X)=0$ and we obtain that $Y\in\mathcal{N}(\bar{\lambda})$.

Now let $\bar{\lambda}(Y)=0$. This means that there exists a Borel set X such that $Y\subseteq X$ and $\mu_i(X-g\cap\Gamma_i)=0$ for $g\in\Gamma_i^{\perp}$ and $i\in I$. Let Γ be an arbitrary n-dimensional vector subspace of B, and λ_Γ be a Lebesgue measure concentrated on Γ. We are to show that λ is transverse to X. Indeed, assume the contrary. Then for $h\in B$ we have $\lambda_\Gamma(X-h\cap\Gamma)>0$. Let i_0 be an element of I such that $\Gamma_{i_0}=\Gamma$. Then we have

$$\lambda_\Gamma((X-h)\cap\Gamma)=\mu_{i_0}((X-h)\cap\Gamma_{i_0})>0.$$

Since $h = h_{i_0}^{(0)} + h_{i_0}^{(1)}$, where $h_{i_0}^{(0)} \in \Gamma_{i_0}$ and $h_{i_0}^{(1)} \in \Gamma_{i_0}^{\perp}$, we get

$$\begin{aligned} 0 &< \mu_{i_0}((X-h)\cap\Gamma_{i_0}) = \mu_{i_0}((X-h_{i_0}^{(1)})\cap\Gamma_{i_0}) \\ &\le \sum_{i\in I}\sum_{g\in\Gamma_i^{\perp}} \mu_i((X-g)\cap\Gamma_i) = \lambda(X) = 0. \end{aligned}$$

giving a contradiction. □

Remark 7.1. Following Theorem 7.1, the generator λ can be called an n-dimensional Preiss–Tišer generator.

Below we present one interesting property of the one-dimensional Preiss–Tišer generator in a Banach space whose dimension is larger than 1.

Theorem 7.2. *Let $(B, ||\cdot||)$ be a Banach space whose dimension is larger than* 1*. Then there exists a one-dimensional Preiss–Tišer generator λ (not σ-finite) such that for an arbitrary broken line $A_0A_1\cdots$ defined by*

$$A_0A_1\cdots \equiv [A_0, A_1[\, \cup \cdots$$

we have

$$\lambda(A_0A_1\cdots) = \sum_{k=0}^{\infty} ||A_{k+1} - A_k||.$$

Proof. Let B_0 be the unit sphere in B, i.e.,

$$B_0 = \{h \in B \ : \ ||h|| = 1\}.$$

For $h_1, h_2 \in B_0$ we say that $h_1 \approx h_2$ if h_1 and h_2 are antipodal points of the unit sphere. It is clear that $\approx$ is an equivalence relation in B_0. Let $(K_i)_{i\in I}$ be the set of equivalent classes. Let $(a_i)_{i\in I}$ be a family of elements of B_0 such that $a_i \in K_i$ for $i \in I$. Let Γ_i be the one-dimensional vector subspace in B generated by the vector a_i for $i \in I$. Let $\Gamma_i^{\perp}$ be a co-space of the vector space Γ_i for $i \in I$. Let μ_i be the one-dimensional classical Borel measure on Γ_i with

$$\mu_i(\{ta_i \ : \ 0 \le t \le 1\}) = 1.$$

We put

$$\lambda(X) = \sum_{i\in I}\sum_{g\in\Gamma_i^{\perp}} \mu_i((X-g)\cap\Gamma_i).$$

for $X \in \mathcal{B}(B)$. Now one can easily check that all conditions formulated for the measure λ in Theorem 7.2, are satisfied. □

In context of Theorem 7.2 the following assertion is of a certain interest.

Theorem 7.3. *Let $\mathbb{R}^3$ be a three-dimensional Euclidean vector space. Then there exists a two-dimensional Preiss–Tišer generator μ (not σ-finite) in $\mathbb{R}^3$ such that for an arbitrary polyhedron $P \subset \mathbb{R}^3$ we have*

$$\mu(B(P)) = S(B(P)),$$

where $B(P)$ denotes the surface of the polyhedron P and $S(B(P))$ the surface area of P.

Proof. Let B_0 be the unit sphere in $\mathbb{R}^3$, i.e.,

$$B_0 = \{h \in \mathbb{R}^3 \ : \ ||h||_3 = 1\},$$

where $||\cdot||_3$ denotes a usual norm in $\mathbb{R}^3$. For $h_1, h_2 \in B_0$ we say that $h_1 \approx h_2$ if h_1 and h_2 are antipodal points. It is clear that $\approx$ is an equivalence relation in B_0. Let $(K_i)_{i\in I}$ be the set of equivalence classes. Let $(a_i)_{i\in I}$ be a family of elements of B_0 such that $a_i \in K_i$ for $i \in I$. Let Γ_i be a two-dimensional plane which has a normal a_i for $i \in I$ and which contains the zero of $\mathbb{R}^3$. Let $\Gamma_i^{\perp}$ be a linear complement of the vector subspace Γ_i for $i \in I$. Let μ_i be a standard two-dimensional classical Borel measure in Γ_i which takes the value π on the set $\Gamma_i \cap B_0$.

We put

$$\mu(X) = \sum_{i\in I} \sum_{g\in\Gamma_i^{\perp}} \mu_i((X - g) \cap \Gamma_i).$$

for $X \in \mathcal{B}(\mathbb{R}^3)$. Now one can easily check that all conditions formulated for the measure μ in Theorem 7.3 are satisfied. □

Remark 7.2. Other interesting geometric applications of generators of shy sets in infinite-dimensional separable Banach spaces with basis can be found in [19, Chapters 13 and 15].

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