New York Journal of Mathematics

New York J. Math. **12** (2006) 123–141.

A motivic Chebotarev density theorem

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Abstract. We define motivic Artin L-functions and show that they specialize to the usual Artin L-functions under the trace of Frobenius. In the last section we use our L-functions to prove a motivic analogue of the Chebotarev density theorem.

Contents

Received March 23, 2006.

Mathematics Subject Classification. 11, 14.

Key words and phrases. Chow motives, L-functions, Chebotarev density theorem.

The first-named author was supported by a University of Western Ontario start up grant during this work. The second-named author was supported by an NSERC discovery grant during this work.

ISSN 1076-9803/06

1. Introduction

Our goal is to prove a motivic analogue of the Chebotarev density theorem. Recall that this theorem classically gives estimates on the growth of the number of points with prescribed Artin symbols; see [7, Section 6.3]. The theorem we obtain, Theorem 6.3, is valid over all fields, however it is only over finite fields that we can use it to construct points with prescribed Artin symbols. Along the way we define non-Abelian motivic L-functions and prove their basic properties. A motivic Chebotarev density theorem without motives can be found in [8] and [7, Chapter 32]. In place of motives, Galois stratification is used in this work. The motivic approach to L-functions is by constructing certain idempotents associated to group actions. It is interesting to note that this use of idempotents was also present in [8] and [7, Chapter 3.1].

This work was first extended to a motivic setting in [6]. In this paper a motivic Igusa zeta function is attached to a Galois formula and used to prove invariance properties of the usual Igusa zeta function. Let us recall that the Igusa zeta function counts solutions in $\mathbb{Z}/p^n\mathbb{Z}$. Denef and Loeser are able to use their motivic function to study the zeta function as p varies.

Our work is in a different direction. We formulate a version of the geometric Chebotarev density theorem. This theorem counts points with prescribed Artin symbol in $\mathbb{F}_{q^n}$.

The Chebotarev density theorem carries key arithmetical information about the splitting of divisors in Galois extensions and is now a basic tool in current arithmetic. For a delightful and informative article about the theorem and its history see [23].

Grothendieck's idea of motives as "a systematic theory of the arithmetic properties of varieties as embodied in their groups of cycles" has proved inspiring and useful in spite of the fact that some of the key conjectures and constructions are not yet established. When one succeeds in lifting some deep arithmetical properties to motives one usually obtains a clear transparent picture and one can try to apply the properties to other situations. The project of transferring arithmetic to algebraic varieties is a long one and can be traced back to Kronecker. For a very good exposition of the basic theory of motives see [1].

The motivic zeta function was first introduced in [11]. The definition was cast in a slightly different light by the elegant constructions of [4]. The rationality of the motivic zeta function is tied to some deep conjectures in the theory of algebraic cycles, [12] and [1]. These are the key facts on which we build our theory of motivic L-functions. Our L-functions clarify some of the properties of usual Artin L-functions. The motivic L-function is just the zeta function of a special motive. The proofs of most of the basic properties are quite elementary. Furthermore, our definition does not need to treat the ramification locus separately because it is built into the definition.

Section 2 is devoted to basic definitions. We explain what a pseudo-Abelian rigid tensor category $\mathbf{C}$ is and, following [4, 12, 16], how to carry out the standard constructions of linear algebra in such a category. Given an object X of such a category with finite group G acting on it and a representation of G we define an L-function. The L-function takes values in the ring $K_o(\mathbf{C})$. The last part of the section is devoted to proving the usual basic properties, direct sums, restriction and induction formulas, of this L-function.

Section 3 specialises to the case where $\mathbf{C}$ is the category $\mathcal{M}_k(E)$ of Chow motives over k with coefficients in E. The L-function behaves just like the L-function of a scheme over a finite field. We prove in Section 5 that it is rational and that when the representation is irreducible and nontrivial it is in fact a polynomial.

When k is a finite field, we prove in Section 5 that our L-function specialises to the usual Artin L-function under the trace of Frobenius.

In Section 6 we define the motive of Artin symbols. Under the trace of Frobenius it just counts points with prescribed Artin symbol. We use the results of the previous sections to derive an expression for it. This expression can be viewed as a Chebotarev density theorem, along the lines of [18].

Notations and conventions. We assume all group actions to be left actions.

k is the ground field, and E a field of characteristic 0 containing all roots of unity.

$(X \otimes V)^G$ is the image of the projection $\frac{1}{|G|}\sum_{g\in G} g$; see Section 2.

$\mathcal{M}_k(E)$ is the category of Chow motives over k with coefficients in E.

$L(M, \rho, t)$ is the L-function of the motive M with respect to the representation ρ; see Section 3.

$\mathrm{Ar}(C, n)$ is the motive of Artin symbols in the conjugacy class C and of degree n; see Section 5.

Acknowledgements. This work would not have been possible without the insights of Professor Michael Fried. We thank him for helpful discussions and correspondence. We would like to thank the referee for numerous suggestions which led to an improvement in the exposition. We are also grateful to Professor Loeser for some comments.

2. The relevant category theory

2.1. Basic definitions. We fix a field E of characteristic 0 that contains all roots of unity. We denote by $\mathbf{C}$ an E-linear additive pseudo-Abelian rigid tensor category. We recall what this means along with the basic properties of $\mathbf{C}$.

By an E-linear additive category we mean a category with a terminal object and direct sums such that for all objects the set $\mathrm{Hom}_{\mathbf{C}}(A, B)$ has the structure of an E-vector space. The composition law is required to be E-linear. The condition that $\mathbf{C}$ is pseudo-Abelian means that every idempotent endomorphism has a kernel and hence an image. If p is such an endomorphism of the object X we will often denote $\mathrm{Im}(p) = \mathrm{Ker}(1 - p)$ by (X, p). The fact that $\mathbf{C}$ is a tensor category means that there is a bilinear functor

$$\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$$

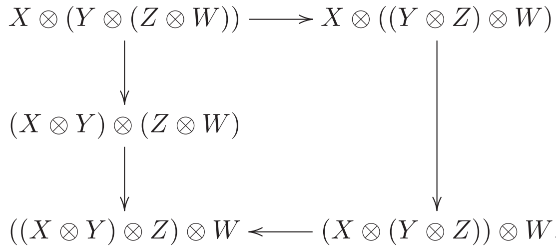
that has an identity and satisfies compatible associativity and commutativity constraints. An *identity* is an object U of $\mathbf{C}$ together with the functorial isomorphism

$$l_X : U \otimes X \xrightarrow{\sim} X \quad \text{and} \quad r_X : X \otimes U \xrightarrow{\sim} X.$$

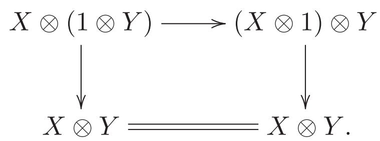
The identity is unique up to isomorphism and we usually denote it by **1**. The associativity constraint is a natural isomorphism

$$a(X, Y, Z) : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z.$$

It is subject to the requirement that the following diagram commutes:



There is a compatibility between the associativity and the identity which is encoded in the following commutative diagram:



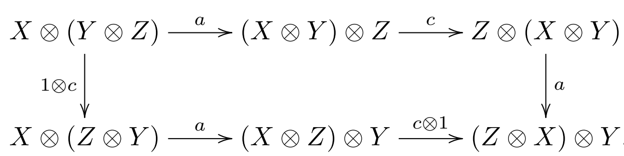
Proposition 2.1. *If F and G are functors $\boldsymbol{C}^n \to \boldsymbol{C}$ obtained from combining $\otimes$ in various orders then it follows that there is a unique isomorphism of functors $F \cong G$ obtained from iterates of a and a^{-1}.*

Proof. See [13] for the proof and precise meaning of iterate. □

The commutativity constraint is a natural isomorphism

$$c(X, Y) : X \otimes Y \to Y \otimes X.$$

We require the following diagram to commute:



Using 2.1, we have a unique, up to canonical isomorphism functor

$$\otimes^n : \mathbf{C}^n \to \mathbf{C}$$

defined by

$$(X_1, X_2, \ldots, X_n) \mapsto X_1 \otimes X_2 \otimes \cdots \otimes X_n.$$

Denote by S_n the symmetric group on n letters. For $\sigma \in S_n$, we define a new functor

$$\otimes^{\sigma, n} : \mathbf{C}^n \to \mathbf{C}$$

by

$$(X_1, X_2, \ldots, X_n) \mapsto X_{\sigma(1)} \otimes X_{\sigma(2)} \otimes \cdots \otimes X_{\sigma(n)}.$$

Proposition 2.2. *There is a unique isomorphism of functors obtained from various iterates of* a, a^{-1} *and* c*:*

$$\otimes^{\sigma,n} \xrightarrow{\sim} \otimes^n.$$

Proof. See [13]. □

Corollary 2.3. *For each object* X *of* $\boldsymbol{C}$ *there is a canonical action of* S_n *on* $X^{\otimes n}$*.*

The fact that $\mathbf{C}$ is rigid means that for every object X of $\mathbf{C}$ there is an object X^* and natural morphisms $\eta_X : \mathbf{1} \to X^* \otimes X$ and $\epsilon_X : X \otimes X^* \to \mathbf{1}$ such that both of the compositions below are the identity

$$X \to X \otimes X^* \otimes X \to X \quad X^* \to X^* \otimes X \otimes X^* \to X^*.$$

Proposition 2.4. *The functor*

$$\otimes X : \boldsymbol{C} \to \boldsymbol{C}$$

has a right adjoint denoted $\mathbb{H}\mathrm{om}(X, -)$*. In other words there are natural isomorphisms*

$$\mathrm{Hom}(Y \otimes X, Z) \xrightarrow{\sim} \mathrm{Hom}(Y, \mathbb{H}\mathrm{om}(X, Z)).$$

Proof. See [5, page 111 to 113]. □

Corollary 2.5. *The functor* $\otimes X$ *preserves direct sums.*

2.2. Idempotents associated to group actions. We recall some facts from [4]. See also [9]. Given a finite-dimensional E vector space V we may form objects $V \otimes X$ and $\mathbb{H}\mathrm{om}(V, X)$. They are characterized by

$$\mathrm{Hom}(V \otimes X, Y) \cong \mathrm{Hom}(V, \mathrm{Hom}(X, Y)) \tag{2.1}$$

$$\mathrm{Hom}(Y, \mathbb{H}\mathrm{om}(V, X)) \cong \mathrm{Hom}(V \otimes Y, X). \tag{2.2}$$

Note that $\mathbb{H}\mathrm{om}(V, X) \cong V^* \otimes X$, canonically. Suppose that the finite group G acts on X. The endomorphism

$$i = \frac{1}{|G|} \sum_{g \in G} g$$

of X is idempotent. We shall denote its image by X^G. If we also have a representation

$$\rho : G \to \mathrm{GL}(V)$$

in a finite-dimensional vector space then there is a G-action on $V \otimes X$ and on $\mathbb{H}\mathrm{om}(V, X)$. We shall denote the images of the respective idempotents by $(V \otimes X)^G$ and $\mathbb{H}\mathrm{om}_G(V, X)$. If G acts on T and S has a trivial action then $\mathrm{Hom}(T^G, S) = \mathrm{Hom}_G(T, S)$. The following formulas then follow:

$$\mathrm{Hom}((V \otimes X)^G, Y) = \mathrm{Hom}_G(V, \mathrm{Hom}(X, Y)) \tag{2.3}$$

$$\mathrm{Hom}(Y, \mathbb{H}\mathrm{om}_G(V, X)) = \mathrm{Hom}_G(V, \mathrm{Hom}(Y, X)). \tag{2.4}$$

Note that if X and the action by G are defined over $\mathbb{Z}$ then so is the motive $(X \otimes V)^G$. This is because the coefficents of our Chow motives are in E.

The symmetric group S_n acts on $X^{\otimes n}$. We define the *nth symmetric power* of X by

$$\mathrm{Sym}^n X = (X^{\otimes n})^{S_n}.$$

More generally, given a partition λ of n, there is a corresponding irreducible representation V_λ of S_n. We can define *Schur functors* $S_\lambda : \mathbf{C} \to \mathbf{C}$ by

$$S_\lambda(X) = \mathbb{H}\mathrm{om}_{S_n}(V_\lambda, X^{\otimes n}).$$

2.3. Zeta and L-functions. We will assume from now on that the category $\mathbf{C}$ is small. We denote by $\mathbb{Z}(\mathbf{C})$ the free Abelian group on isomorphism classes of objects of $\mathbf{C}$. The Abelian group $K_0(\mathbf{C})$ is the quotient of $\mathbb{Z}(\mathbf{C})$ by the subgroup generated by

$$[M \oplus N] - [M] - [N].$$

This group becomes a ring under the multiplication induced by the tensor product of $\mathbf{C}$. Let X be an object of $\mathbf{C}$. The *zeta function* of X is the formal power series in $K_0(\mathbf{C})[[t]]$ defined by

$$1 + [X]t + [\mathrm{Sym}^2 X]t^2 + \cdots .$$

We denote it by $Z(X, t)$. Now consider an object X on which there is an action of the finite group G. Consider a representation

$$\rho : G \to \mathrm{GL}(V).$$

We define the corresponding *L-function* to be

$$L(t, X, \rho) = Z((V \otimes X)^G, t).$$

(Recall that $(V \otimes X)^G$ is the image of $(V \otimes X)$ under the idempotent $\frac{1}{|G|}\sum \rho(g) \otimes g$.) We will see later that this definition of L-function specializes to the usual Artin L-function under the trace of Frobenius.

2.4. Direct sums.

Proposition 2.6. *In $K_0(\mathbf{C})$ we have the equality*

$$[\mathrm{Sym}^n(X \oplus Y)] = \sum_{i=0}^{n} [\mathrm{Sym}^i X][\mathrm{Sym}^{n-i} Y].$$

Proof. This follows from the identity [4, 1.8] and the fact that the Littlewood–Richardson coefficients are 1 in this case. □

And hence:

Proposition 2.7. *We have $Z(X \oplus Y, t) = Z(X, t)Z(Y, t)$.*

Proof. This is a restatement of the above proposition. □

Suppose that G acts on X and that the representation $\rho = \rho_1 \oplus \rho_2$ decomposes. There is a corresponding decomposition

$$X \otimes V \cong (X \otimes V_1) \oplus (X \otimes V_2).$$

The G-action respects this decomposition so that

$$(X \otimes V)^G \cong (X \otimes V_1)^G \oplus (X \otimes V_2)^G.$$

So we have:

Proposition 2.8. *In the above situation*

$$L(X,\rho,t) = L(X,\rho_1,t)L(X,\rho_2,t).$$

2.5. Restriction. Let H be a normal subgroup of G and suppose now that G/H acts on X and we are given a representation $\tau : G/H \to \mathrm{GL}(V)$. We have a representation ρ of G obtained by composing with the quotient map. Let $g_1, g_2, \dots, g_k$ be a collection of coset representatives for G/H. We have the following equality of idempotent endomorphisms of X:

$$\frac{1}{|G|}\sum_{g\in G}\rho(g)g = \frac{|H|}{|G|}\sum_{i=1}^{k}\rho(g_i)g_i = \frac{|H|}{|G|}\sum_{h\in G/H}\tau(h)h.$$

It follows that $(X\otimes V)^G = (X\otimes V)^{G/H}$ and therefore we have established the following proposition.

Proposition 2.9. *We have* $L(X,\rho,t) = L(X,\tau,t)$.

2.6. Induction. Let H be a subgroup of G. Suppose that $\rho : H \to \mathrm{GL}(V)$ is a representation. There is an induced representation

$$\mathrm{Ind}_H^G\rho : G \to \mathrm{GL}(W).$$

It follows from formula (2.3) that

$$(W\otimes X)^G \cong (V\otimes X)^H.$$

Proposition 2.10. *We have*

$$L(X,H,\rho,t) = L(X,G,\mathrm{Ind}_H^G\rho,t).$$

3. Chow motives and motivic L-functions

Let $\mathcal{V}_k$ be the category of smooth projective varieties over a ground field k. We denote by $\mathcal{M}_k(E)$ (resp. $\mathcal{M}_k^+(E)$) the category of (resp. effective) cohomological Chow motives with coefficients in E. The fact that they are cohomological amounts to the fact that there is a contravariant functor

$$h : \mathcal{V}_k^{\mathrm{op}} \to \mathcal{M}_k(E).$$

For a precise definition of these categories see [14], [20] or [1].

The category $\mathcal{M}_k(E)$ is a rigid tensor category. Let X be a motive with a group action. Given a representation $\rho : G \to \mathrm{GL}_m(E)$ we obtain an L-function $L(X,\rho,t)$ using the procedure in the previous section.

Given a smooth projective variety X with a group action, then the opposite group G^{op} acts on the motive $h(X)$. A representation $\rho : G \to \mathrm{GL}_m(E)$ produces an opposite representation

$$\rho^{\mathrm{op}}(g^{\mathrm{op}}) = \rho(g^{-1}).$$

We define

$$L(X,\rho,t) \overset{\mathrm{defn}}{=} L(h(X),\rho^{\mathrm{op}},t).$$

4. Rationality of L-functions

We will settle questions regarding the rationality of the L-series using some results of André and Kimura; see [1] and [12]. The symmetric group S_n acts on the motive $X^{\otimes n}$. We consider the signature representation

$$\mathrm{sgn} : S_n \to GL_1(E).$$

If $p = \frac{1}{n!}\sum(\mathrm{sgn}\,\sigma)\sigma$ is the associated idempotent we call the image of p the nth exterior power of X and denote it by $\bigwedge^n X$.

Following Kimura we say that a motive X is *oddly finite-dimensional* if there is an integer n so that $\mathrm{Sym}^n X = 0$. It follows that $\mathrm{Sym}^m X = 0$ for all $m > n$, [12, 5.9].

A motive X is said to be *evenly finite-dimensional* if there is an integer n so that $\bigwedge^n X = 0$. Similarly by Kimura, we have $\bigwedge^m X = 0$ for all bigger m.

A motive is said to be *finite-dimensional* if there is a decomposition

$$X = X^+ \oplus X^-$$

with X^+ evenly finite-dimensional and X^- oddly finite-dimensional.

Theorem 4.1. *Let X be a smooth projective curve over k. The motives $h^0(X)$ and $h^2(X)$ are evenly finite-dimensional. The motive $h^1(X)$ is oddly finite-dimensional.*

Proof. See [12]. □

Let us record the following:

Lemma 4.2. *We have the following identity in $K_0(\mathcal{M}_k(E))[[t]]$:*

$$\left(\sum_{k=0}^{\infty}[\wedge^k X](-t)^k\right)\left(\sum_{k=0}^{\infty}[\mathrm{Sym}^k X]t^k\right) = 1.$$

Proof. One may deduce this from [4, Section 1.] or see [1, Section 13.3]. □

Corollary 4.3 (André)**.** (1) *If M^+ is an evenly finite-dimensional motive then $Z(M^+, t)^{-1}$ is a polynomial.*
(2) *If M^- is an oddly finite-dimensional motive then $Z(M^-, t)$ is a polynomial.*
(3) *If M is finite-dimensional then $Z(M, t)$ is rational.*

Proof. The proof is by the above lemma definitions and 2.7. □

Corollary 4.4 (Kapranov)**.** *The Zeta function of a curve is rational.*

Proposition 4.5. *Let X be a smooth projective curve with an action of the finite group G. Let*

$$\rho : G \to GL(V)$$

be an irreducible nontrivial representation. Then the power series $L(X, \rho, t)$ is a polynomial.

Proof. There is an induced action of G on each of the pieces $h^i(X)$. If a motive is evenly (resp. oddly) finite-dimensional then every direct summand of it is evenly (resp. oddly) finite-dimensional. So it suffices to show that

$$Z((h^0(X) \otimes V)^G, t) = Z((h^2(X) \otimes V)^G, t) = 1.$$

In other words both the motives $(h^0(X) \otimes V)^G$ and $(h^2(X) \otimes V)^G$ vanish. We will prove this for h^0 and leave the other case to the reader.

We first need to observe that the action of G on $h^0(X)$ is trivial. To see this, first assume that X has a rational point $x \in X(k)$. Then the inclusion

$$h(\mathrm{spec}(k)) = h^0(X) \hookrightarrow h(X)$$

is given by the cycle $[X] \in CH^0(X)$. The inclusion is split by the cycle

$$[x] \in CH^1(X).$$

As the G-action is defined over k the composition

$$h(\mathrm{spec}(k)) \to h(X) \xrightarrow{g^*} h(X) \to h(\mathrm{spec}(k))$$

is the identity. When X has no rational points we may find a Galois extension k'/k with Galois group Γ such that $X' = X \otimes k'$ has a k' rational point. The result follows from the observation that $h(X')^\Gamma = h(X)$ and the projection is compatible with the decomposition $h(X) = h^0(X) \oplus h^1(X) \oplus h^2(X)$.

For an arbitrary smooth projective variety Y there is a canonical isomorphism

$$CH^*(V \otimes h^0(X) \otimes Y) \cong CH^*(h^0(X) \otimes Y) \otimes V$$

compatible with G-actions. The G-action on the last term is entirely on V. As V is irreducible as a G-module, we have $V^G = 0$ and hence the fixed part of the above module is trivial for every smooth projective variety Y. The Manin identity principle; see [20], shows that our motive vanishes. □

5. Relationship with the usual Artin L-function

We assume in this section that the ground field k is in fact a finite field. Then there is a ring homomorphism, given by taking the trace of the Frobenius:

$$\mathrm{Tr} : K_0(\mathcal{M}_k(E)) \to \mathbb{Z}.$$

Here we mean the alternating sum of the traces on the graded pieces of the cohomology groups. In this section we want to prove:

Theorem 5.1. *Suppose that X is a smooth projective curve with an action of the finite group G. Let $\rho : G \to GL(V)$ be a representation of G. Then*

$$\mathrm{Tr}(L(X, \rho, t)) = L^{\mathrm{Ar}}(X, \rho, t).$$

The function on the right-hand side is the usual Artin L-function.

5.1. Extensions of fields. Let us do a warm up exercise to illustrate the proof. We will also use this exercise in the next section. Given an extension of finite fields $\mathbb{F}_{q^n}/\mathbb{F}_q$ with Galois group G and a representation of the Galois group $\rho : G \to \mathrm{GL}(V)$ we define the Artin L-function by

$$L^{\mathrm{Ar}}(\mathbb{F}_{q^n}, \rho, t) = \det(1 - t\rho(f))^{-1}.$$

Here f is the Frobenius element in G. We have an associated motivic L-function

$$L(h(\mathbb{F}_{q^n}), \rho, t) = 1 + [(h(\mathbb{F}_{q^n}) \otimes V)^G]t + [\mathrm{Sym}^2(h(\mathbb{F}_{q^n}) \otimes V)^G]t^2 + \cdots ,$$

so let us see if the two coincide under the trace of Frobenius. We start by assuming $\dim V = 1$ and the general case will reduce to this below. We have

$$L^{\mathrm{Ar}}(\mathbb{F}_{q^n}, \rho, t) = (1 - t\rho(f))^{-1} = 1 + \rho(f)t + \rho(Ff)t^2 + \cdots .$$

The main tool for showing that the two formulas are the same is the Lefschetz trace formula:

Theorem 5.2. *Let Y be a smooth projective variety over an algebraically closed field and let ϕ be an endomorphism of Y. Then*

$$(\Gamma_\phi.\Delta) = \sum(-1)^i \mathrm{Tr}(\phi|H^i(\overline{Y}, \mathbb{Q}_l)),$$

where Γ_ϕ is the graph of ϕ and Δ is the diagonal in $Y \times Y$.

Proof. See [17, 12.3]. □

Next observe the following trivial fact:

Lemma 5.3. *Let V be a vector space and p an idempotent endomorphism of V. If f is another endomorphism then*

$$\mathrm{Tr}(fp) = \mathrm{Tr}(f \mid pV).$$

In our context this means that we have to study the trace of the endomorphism

$$fp = pf$$

where f is the Frobenius and $p = \frac{1}{|G|}\sum_{g\in G}\rho(g^{-1})g$ is an idempotent correspondence. By the Lefschetz fixed point formula we are left to count fixed points of $\mathbb{F}_{q^n} \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ under the endomorphisms gf where f is the Frobenius element of $\overline{\mathbb{F}}_q/\mathbb{F}_q$. The scheme $\mathrm{Spec}(\mathbb{F}_{q^n} \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q)$ is a disjoint union of n points permuted by f. It follows that gf has a fixed point if and only if $g = f^{-1}$, in which case it has $n = |G|$ fixed points. This shows that the first terms agree under the trace of Frobenius.

Let us look at the second terms. Unwinding the definitions we wish to understand the trace of Frobenius on the image of the projection

$$\frac{1}{2|G|^2}\sum_{(g_1,g_2)}\rho(g_1^{-1}g_2^{-1})(1+\sigma)(g_1,g_2) : H^*(\mathbb{F}_{q^n}\otimes\mathbb{F}_{q^n},\mathbb{Q}_\ell) \\ \to H^*(\mathbb{F}_{q^n}\otimes\mathbb{F}_{q^n},\mathbb{Q}_\ell).$$

In the above formula σ is the transposition in S_2. Arguing as before we are reduced to counting fixed points. The scheme $\mathrm{Spec}(\mathbb{F}_{q^n} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n})$ has n^2 geometric points. The endomorphism $(g_1, g_2)f$ has fixed points if and only if $g_1 = f^{-1} = g_2$. There are $n^2 = |G|^2$ of them. If the point $(p_1,p_2)$is fixed by $(g_1, g_2)\sigma f$ then a calculation shows

$$p_1 = g_1 f p_2 \qquad p_2 = g_2 f p_1.$$

One sees that $g_1^{-1}g_2^{-1} = f^2$ if there is a fixed point. Note that the G-action commutes with f as it is defined over the base field. The point $(p, g^{-1}f^{-1}p)$ is then a fixed point, fixed by $(g, g^{-1}f^{-2})$. There are again $|G|^2$ possibilities.

The proof for the higher-order terms is similar. We do not provide it here, but we will spell things out carefully in the next section for covers of curves, which is more general.

Proposition 5.4. $\mathrm{Tr}(L(h(\mathbb{F}_{q^n}) \otimes V, \rho, t)) = L^{\mathrm{Ar}}(\mathbb{F}_{q^n}, \rho, t).$

Proof. We have proved the result for degree 1 representations. One can prove restriction, induction and direct sum formulas for Artin L-functions, see [15]. By [21, 10.7], every representation is a direct sum of representations that are induced from degree 1 representations of subgroups. The corresponding direct sum and induction formulas prove the result in general. □

5.2. Extensions of curves. The curve X, the action of the group G, and the representation $\rho : G \to GL_m(E)$ will remain fixed throughout. We will break the proof into parts. We begin by assuming that $m = 1$, the general case will be reduced to this case. Under this assumption, let us unwind definitions a bit. The nth term of the zeta function is

$$((X^G)^{\otimes n})^{S_n}.$$

The G-action is twisted by the representation in the above. There is a representation

$$\rho^n : G^n \to \mathrm{GL}_1(E)$$

given by taking products. On $h(X)^{\otimes n}$ we have two commuting idempotents

$$p_2 = \frac{1}{|G|^n} \sum_{\mathbf{g} \in G^n} \rho^n(\mathbf{g}^{-1})\mathbf{g} \quad \text{and} \quad p_1 = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma.$$

Recall from the previous section that the functor from varieties to motives is contravariant, hence the appearance of the inverse in the definition of p_2. As the idempotents commute we may think of $((X^G)^{\otimes n})^{S_n}$ as the image of $p_1 p_2 = p$.

We use again our Lemma 5.3. In our context this means that we have to study the trace of the endomorphism

$$fp = pf$$

where f is the Frobenius and p is an idempotent correspondence. As the trace is additive with respect to addition of correspondences this implies that we will end up studying the trace of the endomorphisms fg where f is the Frobenius and g is an element of a group, or more generally an endomorphism of the n-fold fibered product X^n. Both S_n and the group G^n act on X^n.

Let $Y = X/G$ be the quotient. We will write Y_k for the set of degree k points of Y that are unramified in X. We will write Y_k^- for the set of degree k points that are ramified and the restriction of ρ to the inertia subgroup is nontrivial. Finally we write Y_k^+ for the degree k points that are ramified but ρ gives a trivial representation of the inertia. The key lemma for the comparison theorem is:

Lemma 5.5. *Let $\sigma = (123\ldots n)$ and write $\boldsymbol{g} = (g_1, g_2, \ldots, g_n) \in G^n$. Let f be the Frobenius endomorphism acting on $\overline{X}^n = X^n \otimes \bar{k}$. If $\#(\boldsymbol{g}\sigma f)$ denotes the number of fixed points of this endomorphism then*

$$\frac{1}{|G|^n} \sum_{\boldsymbol{g} \in G^n} \rho^n(\boldsymbol{g}^{-1}) \#(\boldsymbol{g}\sigma f) = \sum_{\alpha | n} \alpha \left(\sum_{y \in Y_\alpha \cup Y_\alpha^+} \rho(f_y^{n/\alpha}) \right).$$

In this formula f_y is the Artin symbol at y. By the Lefschetz fixed point theorem, this is the same as the trace of the induced endomorphism on the cohomology of X^n.

Proof. Consider the projection $\pi : \overline{X}^n \to \overline{Y}^n$. If $y^* = (y_1, y_2, \ldots, y_n) \in \overline{Y}^n$ is fixed by σf then it is of the form

$$(y_1, fy_1, \ldots, f^{n-1}y_1)$$

and furthermore we need $\deg y = \alpha | n$ where y is the image of y_1 in Y. Note that there are α different points of $\overline{Y}$ projecting to y.

The projection $\pi : \overline{X}^n \to \overline{Y}^n$ is a G^n quotient. If y is unramified then for each

$$x^* = (x_1, x_2, \ldots, x_n) \in \pi^{-1}(y^*)$$

there is a unique $\mathbf{g} \in G^n$ so that $\mathbf{g}\sigma f$ fixes x^*. An easy calculation shows that

$$\rho^n(\mathbf{g}^{-1}) = \rho(f^n) = \rho(f_y^{n/\alpha}).$$

If the point y is ramified then either the restriction of ρ to the inertia group I_y is trivial or the following sum vanishes:

$$\sum_{g \in I_y} \rho(g).$$

From this observation the result follows. □

The number appearing in this lemma is important so we will give it a name. Define

$$A(n) = \sum_{\alpha | n} \alpha \left(\sum_{y \in Y_\alpha \cup Y_\alpha^+} \rho(f_y^{n/\alpha}) \right).$$

Proposition 5.6. *Let $\boldsymbol{n} = (n_1, n_2, \ldots, n_k)$ be a partition of n. Let $\sigma \in S_n$ be a cycle of type $\boldsymbol{n}$. The the trace of*

$$\frac{1}{|G|^n} \sum_{\boldsymbol{g} \in G^n} \rho^n(\boldsymbol{g}^{-1})(\boldsymbol{g}\sigma f)$$

on the cohomology of X is equal to

$$A(n_1)A(n_2)\ldots A(n_k).$$

Proof. By the Künneth formula we have that

$$\mathrm{Tr}(M \otimes N) = \mathrm{Tr}(M)\mathrm{Tr}(N)$$

for motives M and N. The G^n action preserves the product X^n. Furthermore the action of σ preserves the product

$$X^{n_1} \times X^{n_2} \times \cdots \times X^{n_k} \cong X^n.$$

The result follows from the previous lemma and the above observation. □

Theorem 5.7. *In the above situation of a degree one representation the L-function specializes to the Artin L-function under the trace of the Frobenius.*

Proof. We begin by recalling the definition of the local factor in the Artin L-function corresponding to $y \in Y$. The local factor is given by the formula:

$$\det(I - \rho(f_y)|V^I)$$

where I is the inertia at y. It follows that in our 1-dimensional case that the elements of Y_α^- give no contribution to the L-function.

Set $\hat{Y}_\alpha = Y_\alpha \cup Y_\alpha^+$. We define

$$H_\alpha = \prod_{y \in \hat{Y}_\alpha} (1 + \rho(f_y)t + \rho(f_y^2)t^2 + \cdots),$$

so that the Artin L-function is the product of the H's. We also write

$$\sum_{y \in \hat{Y}_\alpha} \rho(f_y^l) = C_\alpha(l).$$

So that

$$A(k) = \sum_{\alpha | k} \alpha C_\alpha(k/\alpha).$$

A calculation shows that

$$H_\alpha = \exp\left(\sum_{\alpha | s} \alpha C_\alpha(s/\alpha) \frac{t^s}{s}\right).$$

Hence

$$\begin{aligned}
L^{\mathrm{Ar}}(X, \rho, t) &= \prod_\alpha H_\alpha \\
&= \exp\left(\sum_s \left(\sum_{\alpha | s} \alpha C_\alpha(s/\alpha) \frac{t^s}{s}\right)\right) \\
&= \prod_s \exp\left(\sum_{\alpha | s} \alpha C_\alpha(s/\alpha) \frac{t^s}{s}\right) \\
&= \prod_s \sum_{m_s=0}^{\infty} \frac{(\sum_{\alpha | s} \alpha C_\alpha(s/\alpha) t^s)^{m_s}}{s^{m_s} m_s!} \\
&= \sum_{n \geq 0} t^n \left(\prod_{\sum s m_s = n} s^{m_s} m_s!\right)^{-1} \left(\sum_{\sum s_i = n} A(s_1)A(s_2)\ldots A(s_k)\right).
\end{aligned}$$

This last term is the required trace using the above proposition. □

Finally we can prove the main result:

Proof of 5.1. By [21, 10.5] every character of G is a linear combination of characters induced from degree 1 characters of subgroups. One implies the induction and direct sum formulas to deduce the result. □

6. The motivic Chebotarev density theorem

We preserve the following setup throughout this section. We fix an inclusion

$$\mathbb{Q}_\ell \hookrightarrow \mathbb{C},$$

where ℓ is any prime. Let G be a finite group acting on the smooth projective curve X. We denote $Y = X/G$. The set of conjugacy classes of G is written $\mathrm{conj}(G)$. Let

$C \in \text{conj}(G)$. The class function that is 1 on C and 0 otherwise will be denoted $\mathbb{I}_C$. Denote by $\chi_0, \chi_1, \ldots, \chi_k$ the irreducible characters of G, with χ_0 being the character of the trivial representation. There are rational numbers $m_{C,0}, \ldots, m_{C,k}$ so that

$$\mathbb{I}_C = m_{C,0}\chi_0 + m_{C,1}\chi_1 + \cdots + m_{C,k}\chi_k.$$

Note that $m_{C,0} = \frac{|C|}{|G|}$.

6.1. The power set. In this section we make use of the basic properties of Möbius functions associated to a poset; see [22, 3.7].

If C is a conjugacy class then define

$$P_n(C) = \{C' \in \text{conj}(G) | \text{if } x \in C' \text{ then } x^n \in C\}.$$

We define a relation $\leq$ on the set $\mathbb{N} \times \text{conj}(G)$ by

$$(d, C') \leq (n, C) \quad \text{if and only if} \quad d|n \text{ and } C' \in P_{n/d}(C).$$

This gives $\mathbb{N} \times \text{conj}(G)$ the structure of a poset. We wish to bound the associated Möbius function μ. Recall that

$$\mu((d, C'), (n, C)) = c_0 - c_1 + c_2 - \cdots$$

where c_i is the number of complete chains of length i in $[(d, C'), (n, C)]$; see [22, 3.8.5].

Lemma 6.1. $\mu((1, C'), (n, C)) \leq |\text{conj}(G)| n^2$.

Proof. This bound is classical. If we let $H(n)$ be the number of ordered factorizations of the number n then E. Hille was the first to find a precise bound for $H(n)$ up to a constant; see [10]. Later the constant was found to be one; see [3] and [2]. From these works we have

$$c_0 + c_1 + c_2 + \cdots \leq |\text{conj}(G)| n^\rho \leq |\text{conj}(G)| n^2 \quad \text{where} \quad \rho = \zeta(2)^{-1}. \qquad \square$$

6.2. Local factors. Let $\rho : G \to \text{GL}(V)$ be a representation. Let $p \in Y$ and denote by D (resp. I) the decomposition (resp. inertia) group at p. The fiber $X \times_Y p$ is a disjoint union of points. Let q be one of them. The extension $k(q)/k(p)$ is Galois with Galois group D/I. There is a restricted representation, also denoted ρ,

$$\rho : D/I \to \text{GL}(V^I).$$

We define the local factor at p to be

$$L_p(X, \rho, t) = L(\text{spec}(k(q)), \rho, t).$$

By Subsection 5.1, it specializes to the usual local factor under the trace of Frobenius. We define the unramified L-function by

$$L^*(X, \rho, t) = L(X, \rho, t) \prod L_p(X, \rho, t)^{-1}$$

where the product is over the ramified points.

6.3. The motive of Artin symbols. We wish to define the *motive of Artin symbols of degree n*

$$\operatorname{Ar}(X,G,C,n)=\operatorname{Ar}(C,n)\in K_0(\mathcal{M}_k(E))\otimes\mathbb{Q}.$$

The elements of this last ring will be referred to as *virtual motives.* In order to define the motives of Artin symbols we form the generating functions

$$L(X,C,t)=\exp\left(\sum_{n>0}\frac{t^n}{n}\left(\sum_{d|n,\ C'\in P_{n/d}(C)}\operatorname{Ar}(C',d)\right)\right).$$

We define the motives of Artin symbols by the formula

$$L(X,C,t)=L^*(X,\chi_0,t)^{m_0}L^*(X,\chi_1,t)^{m_1}\dots L^*(X,\chi_k,t)^{m_k}. \tag{6.1}$$

Some remarks are in order. Note that by the results of the first section an L-function is completely determined by its character. So $L(X,\chi_i,t)$ is the L-function coming from the irreducible representation corresponding to χ_i. Raising to a fractional power is only a formal operation here, as the purpose of the above formula is to define $\operatorname{Ar}(C,n)$ only and one needs to take logarithms in the above to write down a formula for $\operatorname{Ar}(C,n)$. Note that the formula is recursive. As $P_1(C)=\{C\}$ the coefficient of t^n involves $\operatorname{Ar}(C,n)$ and $\operatorname{Ar}(C',d)$. We may assume by induction that the $\operatorname{Ar}(C',d)$ have already been defined. (This is essentially Mobius inversion.)

Aside. Let us calculate the first few terms in the case when $X\to Y=X/G$ is unramified. The ramified case is similar but more complicated as one needs to take care of the local factors coming from the ramification. Let $V_0,\dots,V_k$ be the irreducible G-modules corresponding to the characters χ_i. Note that $(h(X)\otimes V_0)^G=h(Y)$. Taking logarithms we find that the first two terms of $\log L(X,C,t)$ are

$$t(\operatorname{Ar}(C,1))+\frac{t^2}{2}\left(\operatorname{Ar}(C,2)+\sum_{C'\in P_2(C)}\operatorname{Ar}(C',1)\right)+\cdots.$$

Equating with the other side and noting that $m_{C,0}=|C|/|G|$ we find that

$$\operatorname{Ar}(C,1)=\frac{|C|}{|G|}[h(Y)]+m_{C,1}[(h(X)\otimes V_1)^G]+\cdots+m_{C,k}[(h(X)\otimes V_k)^G]$$

and

$$\begin{aligned}&\frac{1}{2}\left(\operatorname{Ar}(C,2)+\sum_{C'\in P_2(C)}\operatorname{Ar}(C',1)\right)\\&\quad=\frac{|C|}{|G|}([\operatorname{Sym}^2(h(Y))]-1/2[h(Y)]^2)\\&\qquad+\cdots+m_{C,k}([\operatorname{Sym}^2((h(X)\otimes V_k)^G)]-1/2[(h(X)\otimes V_k)^G]^2).\end{aligned}$$

Now assume k is a global field. Let G_k be its absolute Galois group. For every prime p in k we let f_p be the Frobenius element at p. It is determined up to conjugacy. We say that a motive M is *pure of weight i* if for all but finitely many p the eigenvalues of f_p on the ℓ-adic realisation of M have absolute value $q^{i/2}$. We will denote X_p the base change of our curve to the residue field of p. Note that we have an ℓ-adic realisation homomorphism

$$K_0(\mathcal{M}_k(E)) \otimes \mathbb{Q} \to K_0(G_k\text{-modules}) \otimes \mathbb{Q}.$$

For every prime p we have ring homomorphisms

$$\mathrm{Tr}_p : K_0(\mathcal{M}_k(E)) \otimes \mathbb{Q} \to \mathbb{Q}$$

obtained by taking the alternating sum of the traces of f_p on cohomology. Here ℓ is a prime different from the characteristic of the residue field.

Proposition 6.2. *For all but finitely many p we have*

$$\mathrm{Tr}_p(\mathrm{Ar}(C,n)) = n.\#\left\{p \in Y^* \mid \deg p = n,\ (p|X/Y) \in C\right\}.$$

In the above $(p|X/Y)$is the Artin symbol for the cover $X \to Y$ with $Y = X/G$. Here Y^* denotes the set of unramified points of the cover. Note that the multiplication by n amounts to counting geometric points over p.

Proof. The set of primes mentioned in the statement of the proposition is the set where X has good reduction and such that the G-action is defined over them. On the one hand, using Theorem 5.1 and standard facts about Artin L-functions (see [19, Lemma 9.14]) we have

$$\begin{aligned} t\frac{d}{dt}\mathrm{Tr}(f_P|\log L(X,C,t)) &= \sum_{y \in Y^*} \sum_{l=1}^{\infty} \sum_{i=1}^{k} m_i \chi_i((y|X/Y)^l) t^{l \deg y} \\ &= \sum_{y \in Y^*} \sum_{l=1}^{\infty} \mathbb{I}_C((y|X/Y)^l) t^{l \deg y} \\ &= \sum_{l=1}^{\infty} \sum_{y \in Y^*, (y|X/Y)^l \in C} t^{l \deg y} \\ &= \sum_{n=1}^{\infty} t^n \left(\sum_{y \in Y^*, l \deg y = n, (y|X/Y) \in P_l(C)} 1 \right). \end{aligned}$$

Note that the other side of this equation is just

$$\sum_{n>0} t^n \left(\sum_{d|n, C' \in P_{n/d}(C)} Tr_p(\mathrm{Ar}(C',d)) \right).$$

We compare coefficents and use an induction, to obtain the result. □

Let M be a motive. We define virtual motives $W_n(M)$ by the formula

$$\sum_{n=1}^{\infty} W_n(M) t^n = \frac{tZ'(M,t)}{Z(M,t)}.$$

The prime denotes the formal derivative in the above formula.

For each of our characters χ_i there is an irreducible representation

$$\rho_i : G \to \mathrm{GL}(V_i).$$

We define motives by

$$M_i = (h^1(X) \otimes V_i)^G.$$

Using the results of Section 4, in particular 4.5 and its proof, we have

$$L_i(X, \chi_i, t) = Z(M_i, t) \quad i > 0$$

are polynomials and

$$L_0(X, \chi_0, t) = \frac{Z(M_0, t)}{(1-t)(1-\mathbb{L}t)} = Z(Y, t).$$

Here $\mathbb{L}$ is the Lefschetz motive. It is isomorphic to $h^2(X)$; see [20]. Let $p_1, p_2, \ldots, p_l$ be the ramification points of the cover $X \to Y$. We choose preimages $q_1, q_2, \ldots, q_l$. Let I_i (resp. D_i) denote the corresponding inertia (resp. decomposition) groups. We let

$$N_{ij} = (h^0(\mathrm{spec}(k(q_j))) \otimes V_i^{I_j})^{D_j/I_j, \rho_i},$$

where ρ_i indicates that D_j/I_j acts via ρ_i on $V_i^{I_j}$. So that the local factors are given by

$$L_{p_j}(X, \rho_i, t) = Z(N_{ij}, t).$$

Theorem 6.3. *In the above situation we have*

$$\mathrm{Ar}(C, n) = \sum_{d|n,\ C' \in P_{n/d}(C)} \mu((d, C'), (n, C)) \Bigg(\frac{|C'|}{|G|} \mathbb{L}^d + \sum_{i=0}^{k} m_{C',i} W_d(M_i) - \sum_{i=0}^{k} \sum_{j=1}^{l} m_{C',i} W_d(N_{ij}) \Bigg).$$

Proof. We take logarithmic derivatives of (6.1) and equate coefficients to obtain:

$$\sum_{d|n,\ C' \in P_{n/d}(C)} \mathrm{Ar}(C', d) = m_{C,0}\mathbb{L}^n + m_{C,0} + \sum_{i=0}^{k} m_{C,i} W_n(M_i) - \sum_{i=0}^{k} \sum_{j=1}^{l} m_{C,i} W_n(N_{ij}).$$

Observe that $m_{C,0} = \frac{|C|}{|G|}$. Applying Möbius inversion we obtain the desired result. □

We may deduce the usual geometric Chebotarev density theorem from this theorem.

Corollary 6.4. *For all but finitely many p we have*

$$\mathrm{Tr}_p(\mathrm{Ar}(C, n)) = \frac{|C|}{|G|} q^n + O(n^2 q^{n/2})$$

where q is the cardinality of the residue field at p.

Proof. It follows from 6.1 that $\mu((d, C'), (n, C)) \le n^2 |\mathrm{conj}(G)|$. Next observe that $\mathbb{L}$ specializes to q under the trace of Frobenius.

Next we study the terms $W_n(M_i)$ under the trace of Frobenius. It is a theorem of A. Weil that

$$L^{\mathrm{Ar}}(X, \chi_i, t) = \prod_{j=1}^{e_i} (1 - \alpha_{ij} t)$$

with $|\alpha_{ij}| = q^{1/2}$ when χ_i is nontrivial. This follows from the fact that $h^1(X)$ is pure of weight one; see [17]. A calculation shows that

$$t\frac{d}{dt}\log L^{\mathrm{Ar}}(X,\chi_i,t) = \sum_{n>0} t^n \left(\sum \alpha_{ij}^n\right).$$

It follows that

$$|\mathrm{Tr}_p(W_n(M_i))| = O(q^{n/2}).$$

A similar result is true for $i = 0$ as the higher degree terms come from the Lefschetz motive.

Finally it remains to study the terms coming from the ramification. But using the argument above, they are easily bounded in terms of the degree of the representation. This completes the proof. □

Note that the error term $O(n^2 q^{n/2})$ is not as sharp as the error term $O(q^{n/2})$ in [18]. It should be possible to improve this estimate by bounding the Möbius function more carefully.

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This paper is available via http://nyjm.albany.edu/j/2006/12-7.html.