New York Journal of Mathematics

New York J. Math. **12** (2006) 349–356.

Compatible ideals and radicals of Ore extensions

E. Hashemi

Abstract. For a ring endomorphism α and an α-derivation δ, we introduce α-compatible ideals which are a generalization of α-rigid ideals and study the connections of the prime radical and the upper nil radical of R with the prime radical and the upper nil radical of the Ore extension $R[x;\alpha,\delta]$ and the skew power series $R[[x;\alpha]]$. As a consequence we obtain a generalization of Hong, Kwak and Rizvi, 2005.

Contents

1. Introduction

Throughout the paper R always denotes an associative ring with unity. $R[x;\alpha,\delta]$ will stand for the Ore extension of R, where α is an endomorphism and δ an α-derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for all $a, b \in R$. We use $P(R)$, $N_r(R)$ and $N(R)$ to denote the prime radical, the upper nil radical and the set of all nilpotent elements of R, respectively.

According to Krempa [15], an endomorphism α of a ring R is called *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. R is called an α-rigid ring [9] if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism and α-rigid rings are reduced by Hong et al. [9]. Properties of α-rigid rings have been studied in Krempa [15], Hirano [7] and Hong et al. [11] and [9].

On the other hand, a ring R is called 2-*primal* if $P(R) = N(R)$ (see [2]). Every reduced ring is obviously a 2-primal ring. Moreover, 2-primal rings have been extended to the class of rings which satisfy $N_r(R) = N(R)$, but the converse does not hold ([3], Example 3.3). Observe that R is a 2-primal ring if and only if $P(R) = N_r(R) = N(R)$, if and only if $P(R)$ is a completely semiprime ideal (i.e., $a^2 \in P(R)$ implies $a \in P(R)$ for $a \in R$) of R. We refer to [2, 3, 7, 10, 14, 18, 19]

Received May 29, 2006.

Mathematics Subject Classification. primary 16W20, 16N40, 16S36, secondary 16W60.

Key words and phrases. α-compatible rings, Ore extensions, skew power series, α-rigid rings, prime and upper nil radicals.

ISSN 1076-9803/06

for more detail on 2-primal rings. Recall that a ring R is called *strongly prime* if R is prime with no nonzero nil ideals. An ideal P of R is *strongly prime* if R/P is a strongly prime ring. All (strongly) prime ideals are taken to be proper. We say an ideal P of a ring R is *minimal* (*strongly*) *prime* if P is minimal among (strongly) prime ideals of R. Note that (see [17])

$$N_r(R) = \cap\{P \mid P \text{ is a minimal strongly prime ideal of } R\}.$$

Recall that an ideal P of R is *completely prime* if $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$. Every completely prime ideal of R is strongly prime and every strongly prime ideal is prime.

According to Hong et al. [11], for an endomorphism α of a ring R, an α-ideal I is called an *α-rigid ideal* if $a\alpha(a) \in I$ implies $a \in I$ for $a \in R$. Hong et al. [11] studied connections between the α-rigid ideals of R and the related ideals of some ring extensions. They also studied the relationship of $P(R)$ and $N_r(R)$ of R with the prime radical and the upper nil radical of the Ore extension $R[x; \alpha, \delta]$ of R in the cases when either $P(R)$ or $N_r(R)$ is an α-rigid ideal of R, obtaining the following result: Let $P(R)$ (resp. $N_r(R)$) be an α-rigid δ-ideal of R. Then

$$P(R[x; \alpha, \delta]) \subseteq P(R)[x; \alpha, \delta]$$

(resp. $N_r(R[x; \alpha, \delta]) \subseteq N_r(R)[x; \alpha, \delta]$). Hong et al. [11] provided an example to show that the condition "$P(R)$ is α-rigid" is not superfluous.

In [6], the authors defined α-compatible rings, which are a generalization of α-rigid rings. A ring R is called *α-compatible* if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$. Moreover, R is said to be *δ-compatible* if for each $a, b \in R$, $ab = 0 \Rightarrow a\delta(b) = 0$. If R is both α-compatible and δ-compatible, we say that R is *(α, δ)-compatible.* In this case, clearly the endomorphism α is injective. In ([6], Lemma 2.2), the authors showed that R is α-rigid if and only if R is α-compatible and reduced. Thus the α-compatible ring is a generalization of an α-rigid ring to the more general case where R is not assumed to be reduced.

Motivated by the above facts, for an endomorphism α of a ring R, we define α-compatible ideals in R which are a generalization of α-rigid ideals. For an ideal I, we say that I is an *α-compatible ideal* of R if for each $a, b \in R$, $ab \in I \Leftrightarrow a\alpha(b) \in I$. Moreover, I is said to be a *δ-compatible ideal* if for each $a, b \in R$, $ab \in I \Rightarrow a\delta(b) \in I$. If I is both α-compatible and δ-compatible, we say that I is an *(α, δ)-compatible ideal.* The definition is quite natural, in the light of its similarity with the notion of α-rigid ideals. Here, in Proposition 2.4, we will show that I is an α-rigid ideal if and only if I is an α-compatible ideal and completely semiprime.

In this paper we first give some examples of (α, δ)-compatible ideals which are not α-rigid. Then, we study connections between (α, δ)-compatible ideals of R and related ideals of some ring extensions. Also we investigate the relationship of $P(R)$ and $N_r(R)$ of R with the prime radical and the upper nil radical of the Ore extension $R[x; \alpha, \delta]$ and the skew power series $R[[x; \alpha]]$.

Recall that an ideal I of R is called an *α-ideal* if $\alpha(I) \subseteq I$; I is called *α-invariant* if $\alpha^{-1}(I) = I$; I is called a *δ-ideal* if $\delta(I) \subseteq I$; I is called an *(α, δ)-ideal* if it is both an α- and a δ-ideal. If I is an (α, δ)-ideal, then $\bar{\alpha} : R/I \to R/I$ defined by $\bar{\alpha}(a + I) = \alpha(a) + I$ is an endomorphism and $\bar{\delta} : R/I \to R/I$ defined by $\bar{\delta}(a + I) = \delta(a) + I$ is an $\bar{\alpha}$-derivation.

2. The prime and upper nil radicals of Ore extensions

In this section, our focus of study will be on the prime and the upper nil radicals of a ring R and those of the Ore extension $R[x;\alpha,\delta]$ and the skew power series ring $R[[x;\alpha]]$.

Proposition 2.1. *Let I be an ideal of a ring R. Then the following statements are equivalent:*

(1) *I is an (α,δ)-compatible ideal.*
(2) *R/I is $(\bar{\alpha},\bar{\delta})$-compatible.*

Proof. (1)$\Rightarrow$(2) Clearly I is an (α,δ)-ideal. Let $(a+I)(b+I)=0$ in R/I. Then $ab\in I$. Hence $a\alpha(b)\in I$. Thus $(a+I)\bar{\alpha}(b+I)=0$. Similarly, $(a+I)\bar{\alpha}(b+I)=0$ implies $(a+I)(b+I)=0$. If $(a+I)(b+I)=0$, then $ab\in I$, so that $a\delta(b)\in I$. Hence $(a+I)\bar{\delta}(b+I)=0$.

(2)$\Rightarrow$(1) This is similar to the proof of (1)$\Rightarrow$(2). □

Lemma 2.2. *Let I be an α-compatible ideal of a ring R. Then I is α-invariant.*

Proof. Let $\alpha(a)\in I$. Then $1\alpha(a)\in I$ and so $a\in I$, since I is α-compatible. Thus I is an α-invariant ideal. □

Recall from [16] that a one-sided ideal I of a ring R has the *insertion of factors property* (or simply, IFP) if $ab\in I$ implies $aRb\subseteq I$ for $a,b\in R$ (H.E. Bell in 1970 introduced this notion for $I=0$).

The following proposition extends ([6], Lemma 2.1).

Proposition 2.3. *Let I be an (α,δ)-compatible ideal of R and $a,b\in R$.*

(1) *If $ab\in I$, then $a\alpha^n(b)\in I$ and $\alpha^n(a)b\in I$ for every positive integer n. Conversely, if $a\alpha^k(b)$ or $\alpha^k(a)b\in I$ for some positive integer k, then $ab\in I$.*
(2) *If $ab\in I$, then $\alpha^m(a)\delta^n(b),\delta^n(a)\alpha^m(b)\in I$ for any nonnegative integers m,n.*

Proof. (1) If $ab\in I$, then $\alpha^n(a)\alpha^n(b)\in I$, since I is an α-ideal. Hence $\alpha^n(a)b\in I$, since I is α-compatible. If $\alpha^k(a)b\in I$, then $\alpha^k(a)\alpha^k(b)\in I$, since I is α-compatible. Hence $\alpha^k(ab)\in I$ and $ab\in I$, since I is α-invariant, by Lemma 2.2.

(2) It is enough to show that $\delta(a)\alpha(b)\in I$. If $ab\in I$, then by (1) and δ-compatibility of I, $\alpha(a)\delta(b)\in I$. Hence $\delta(a)b=\delta(ab)-\alpha(a)\delta(b)\in I$. Thus $\delta(a)b\in I$ and $\delta(a)\alpha(b)\in I$, since I is α-compatible. □

Proposition 2.4. *Let R be a ring, I be an ideal of R and $\alpha:R\to R$ be an endomorphism of R. Then the following conditions are equivalent:*

(1) *I is an α-rigid ideal of R.*
(2) *I is α-compatible, semiprime and satisfies the IFP property.*
(3) *I is α-compatible and completely semiprime.*

If δ is an α-derivation of R, then the following are equivalent:

(4) *I is an α-rigid δ-ideal of R.*
(5) *I is (α,δ)-compatible, semiprime and satisfies the IFP property.*
(6) *I is (α,δ)-compatible and completely semiprime.*

Proof. (1)⇒(2) This follows from ([11], Propositions 2.2 and 2.4).

(2)⇒(1) Let $a\alpha(a) \in I$. Then $a^2 \in I$, since I is α-compatible. Hence $aRa \subseteq I$, since I satisfies the IFP property. Thus $a \in I$, since I is semiprime.

(4)⇒(6) By (1)⇒(3), I is α-compatible and completely semiprime. We show that $a\delta(b) \in I$, when $ab \in I$. If $ab \in I$, then $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b \in \delta(I) \subseteq I$. Thus $(\alpha(a)\delta(b))^2 = \delta(ab)\alpha(a)\delta(b) - \delta(a)b\alpha(a)\delta(b) \in I$, because $\delta(ab), b\alpha(a) \in I$. Since I is completely semiprime, we have $\alpha(a)\delta(b) \in I$ and so $a\delta(b) \in I$, by Proposition 2.3. □

In [6], the authors give some examples of (α, δ)-compatible rings which are not α-rigid. Note that there exists a ring R for which every nonzero proper ideal is α-compatible but R is not α-compatible. For example, consider the ring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field, and the endomorphism α of R is defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$ for $a, b, c \in F$.

The following examples show that there exist α-compatible ideals that are not α-rigid.

Example 2.5. Let F be a field. Let $R = \left\{\begin{pmatrix} f & f_1 \\ 0 & f \end{pmatrix} \;\middle|\; f, f_1 \in F[x]\right\}$, where $F[x]$ is the ring of polynomials over F. Then R is a subring of the 2×2 matrix ring over the ring $F[x]$. Let $\alpha : R \to R$ be an automorphism defined by $\alpha\left(\begin{pmatrix} f & f_1 \\ 0 & f \end{pmatrix}\right) = \begin{pmatrix} f & uf_1 \\ 0 & f \end{pmatrix}$, where u is a fixed nonzero element of F. Let $p(x)$ be an irreducible polynomial in $F[x]$. Let $I = \left\{\begin{pmatrix} 0 & f_1 \\ 0 & 0 \end{pmatrix} \;\middle|\; f_1 \in \langle p(x)\rangle\right\}$, where $\langle p(x)\rangle$ is the principal ideal of $F[x]$ generated by $p(x)$. Then I is an α-compatible ideal of R but it is not α-rigid. Indeed, since $\begin{pmatrix} 0 & g(x) \\ 0 & 0 \end{pmatrix} \alpha\left(\begin{pmatrix} 0 & g(x) \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in I$, but $\begin{pmatrix} 0 & g(x) \\ 0 & 0 \end{pmatrix} \notin I$ for $g(x) \notin \langle p(x)\rangle$. Thus I is not α-rigid.

Example 2.6 ([12], Example 2)**.** Let $\mathbb{Z}_2$ be the field of integers modulo 2 and $A = \mathbb{Z}_2[a_0, a_1, a_2, b_0, b_1, b_2, c]$ be the free algebra of polynomials with zero constant term in noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$. Note that A is a ring without identity. Consider an ideal of $\mathbb{Z}_2 + A$, say I, generated by a_0b_0, $a_1b_2 + a_2b_1$, $a_0b_1 + a_1b_0$, $a_0b_2 + a_1b_1 + a_2b_0$, a_2b_2, a_0rb_0, a_2rb_2, $(a_0 + a_1 + a_2)r(b_0 + b_1 + b_2)$ with $r \in A$ and $r_1r_2r_3r_4$ with $r_1, r_2, r_3, r_4 \in A$. Then I satisfies the IFP property. Let $\alpha : R \to R$ be an inner automorphism (i.e., there exists an invertible element $u \in R$ such that $\alpha(r) = u^{-1}ru$ for each $r \in R$). Then I is α-compatible, since I satisfies the IFP property. But I is not α-rigid, since I is not completely semiprime.

Definition 2.7. Given α and δ as above and integers $j \geq i \geq 0$, let us write f_i^j for the sum of all "words" in α and δ in which there are i factors of α and $j - i$ factors of δ. For instance, $f_j^j = \alpha^j$, $f_0^j = \delta^j$ and $f_{j-1}^j = \alpha^{j-1}\delta + \alpha^{j-2}\delta\alpha + \cdots + \delta\alpha^{j-1}$.

Note that if I is an (α, δ)-ideal of R, then $I[x; \alpha, \delta]$ is an ideal of the Ore extension $R[x; \alpha, \delta]$.

Theorem 2.8. *Let I be an (α,δ)-compatible semiprime ideal of R. Assume $f(x)=\sum_{i=0}^{n} a_i x^i$ and $g(x)=\sum_{j=0}^{m} b_j x^j \in R[x;\alpha,\delta]$. Then the following statements are equivalent:*

(1) $f(x)R[x;\alpha,\delta]g(x)\subseteq I[x;\alpha,\delta]$.
(2) $a_iRb_j\subseteq I$ *for each* i,j.

Proof. (1)⇒(2) Assume $f(x)R[x;\alpha,\delta]g(x)\subseteq I[x;\alpha,\delta]$. Then

$$(\dagger)\qquad (a_0+\cdots+a_nx^n)c(b_0+\cdots+b_mx^m)\in I[x;\alpha,\delta] \text{ for each } c\in R.$$

Hence $a_n\alpha^n(cb_m)\in I$. Thus $a_ncb_m\in I$, since I is α-compatible. Therefore $a_nf_i^j(cb_m)\in I$, by Proposition 2.3. Next, replace c by $cb_{m-1}da_ne$, where $c,d,e\in R$. Then $(a_0+\cdots+a_nx^n)cb_{m-1}da_ne(b_0+\cdots+b_{m-1}x^{m-1})\in I[x;\alpha,\delta]$. Hence $a_n\alpha^n(cb_{m-1}da_neb_{m-1})\in I$ and $a_ncb_{m-1}da_neb_{m-1}\in I$, since I is α-compatible. Thus $(Ra_nRb_{m-1})^2\subseteq I$. Hence $Ra_nRb_{m-1}\subseteq I$, since I is semiprime. Continuing this process, we obtain $a_nRb_k\subseteq I$, for $k=0,1,\cdots,m$. Hence by (α,δ)-compatibility of I, we get $(a_0+\cdots+a_nx^n)R[x;\alpha,\delta](b_0+\cdots+b_{m-1}x^{m-1})\subseteq I[x;\alpha,\delta]$. Using induction on $n+m$, we obtain $a_iRb_j\subseteq I$ for each i,j.

(2)⇒(1) This follows from Proposition 2.3. □

Corollary 2.9. *If I is a (semi)prime (α,δ)-compatible ideal of R, then $I[x;\alpha,\delta]$ is a (semi)prime ideal of $R[x;\alpha,\delta]$.*

Proof. Assume that I is a prime (α,δ)-compatible ideal of R. Let $f(x)=\sum_{i=0}^{n}a_ix^i$ and $g(x)=\sum_{j=0}^{m}b_jx^j\in R[x;\alpha,\delta]$ such that $f(x)R[x;\alpha,\delta]g(x)\subseteq I[x;\alpha,\delta]$. Then $a_iRb_j\subseteq I$ for each i,j, by Theorem 2.8. Assume $g(x)\notin I[x;\alpha,\delta]$. Hence $b_j\notin I$ for some j. Thus $a_i\in I$ for each $i=0,1,\cdots,n$, since I is prime. Therefore $f(x)\in I[x;\alpha,\delta]$. Consequently, $I[x;\alpha,\delta]$ is a prime ideal of $R[x;\alpha,\delta]$. □

Theorem 2.10. *If each minimal prime ideal of R is (α,δ)-compatible, then $P(R[x;\alpha,\delta])\subseteq P(R)[x;\alpha,\delta]$.*

Proof. This follows from Corollary 2.9. □

The following example shows that there exists a ring R such that all minimal prime ideals are α-compatible, but are not α-rigid.

Example 2.11. Let $R=Mat_2(\mathbb{Z}_4)$ be the 2×2 matrix ring over the ring $\mathbb{Z}_4$. Then $P(R)=\left\{\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} \,\middle|\, a_{ij}\in \overline{2}\mathbb{Z}\right\}$ is the only prime ideal of R. Let $\alpha: R\to R$ be the endomorphism defined by $\alpha\left(\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\right)=\begin{pmatrix} a_{11} & -a_{12}\\ -a_{21} & a_{22}\end{pmatrix}$. Then α is an automorphism of R and $P(R)$ is α-compatible. However, $P(R)$ is not α-rigid, since $\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}\alpha\left(\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}\right)\in P(R)$, but $\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}\notin P(R)$.

Theorem 2.12. *If P is a completely (semi)prime (α,δ)-compatible ideal of R, then $P[x;\alpha,\delta]$ is a completely (semi)prime ideal of $R[x;\alpha,\delta]$.*

Proof. Let P be a completely prime ideal of R. R/P is a domain, hence it is a reduced ring. R/P is a $(\overline{\alpha},\overline{\delta})$-compatible ring, hence R/P is $\overline{\alpha}$-rigid, by ([6], Lemma 2.2). Let $\overline{f(x)},\overline{g(x)}\in R/P[x;\overline{\alpha},\overline{\delta}]$ such that $\overline{f(x)}\;\overline{g(x)}=0$. Then $\overline{f(x)}=0$ or $\overline{g(x)}=0$, by ([9], Proposition 6). Thus $R[x;\alpha,\delta]/P[x;\alpha,\delta]\cong R/P[x;\overline{\alpha},\overline{\delta}]$ is a domain and $P[x;\alpha,\delta]$ is a completely prime ideal of $R[x;\alpha,\delta]$. □

Corollary 2.13 ([11], Proposition 3.8). *If $P(R)$ is an α-rigid δ-ideal of R, then $P(R[x;\alpha,\delta]) \subseteq P(R)[x;\alpha,\delta]$.*

Proof. $P(R)$ is an α-rigid δ-ideal, hence $P(R)$ is a completely semiprime (α,δ)-compatible ideal of R, by Proposition 2.4. Therefore $P(R[x;\alpha,\delta]) \subseteq P(R)[x;\alpha,\delta]$, by Theorem 2.12. □

Theorem 2.14. *If P is a strongly (semi)prime (α,δ)-compatible ideal of R, then $P[x;\alpha,\delta]$ is a strongly (semi)prime ideal of $R[x;\alpha,\delta]$.*

Proof. By Corollary 2.9, $P[x;\alpha,\delta]$ is a prime ideal of $R[x;\alpha,\delta]$. Hence

$$R[x;\alpha,\delta]/P[x;\alpha,\delta] \cong R/P[x;\bar{\alpha},\bar{\delta}]$$

is a prime ring. We claim that zero is the only nil ideal of $R/P[x;\bar{\alpha},\bar{\delta}]$. Let J be a nil ideal of $R/P[x;\bar{\alpha},\bar{\delta}]$. Assume I be the set of all leading coefficients of elements of J. First we show that I is an ideal of R/P. Clearly I is a left ideal of R/P. Let $\bar{a} \in I$ and $\bar{r} \in R/P$. Then there exists $\overline{f(x)} = \bar{a}_0 + \cdots + \bar{a}_{n-1}x^{n-1} + \bar{a}x^n \in J$. Hence $(\overline{f(x)}\bar{r})^m = 0$ for some nonnegative integer m. Thus $\bar{a}\ \bar{\alpha}^n(\bar{r}\bar{a})\cdots\bar{\alpha}^{(m-1)n}(\overline{ra})\bar{\alpha}^{mn}(\bar{r}) = 0$, since it is the leading coefficient of $(\overline{f(x)}\bar{r})^m$. Therefore $(\overline{ar})^m = 0$, since R/P is $\bar{\alpha}$-compatible. Consequently, I is an ideal of R/P. Clearly I is a nil ideal of R/P. Hence $I = 0$ and so $J = 0$. Therefore $P[x;\alpha,\delta]$ is a strongly prime ideal of $R[x;\alpha,\delta]$. □

Theorem 2.15. *If each minimal strongly prime ideal of R is (α,δ)-compatible, then $N_r(R[x;\alpha,\delta]) \subseteq N_r(R)[x;\alpha,\delta]$.*

Corollary 2.16 ([11], Proposition 3.8). *If $N_r(R)$ is an α-rigid δ-ideal of R, then $N_r(R[x;\alpha,\delta]) \subseteq N_r(R)[x;\alpha,\delta]$.*

Proof. $N_r(R)$ is an α-rigid δ-ideal, hence $N_r(R)$ is a completely semiprime (α,δ)-compatible ideal of R, by Proposition 2.4, and $N_r(R)$ is a strongly semiprime ideal of R. Therefore $N_r(R[x;\alpha,\delta]) \subseteq N_r(R)[x;\alpha,\delta]$, by Theorem 2.15. □

As a parallel result to Theorems 2.8, 2.10, 2.12, 2.14 and 2.15, we have the following results for the skew power series ring $R[[x;\alpha]]$.

Proposition 2.17. *Let I be an α-compatible semiprime ideal of R. Assume $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x;\alpha]]$. Then the following statements are equivalent:*

(1) *$f(x)R[[x;\alpha]]g(x) \subseteq I[[x;\alpha]]$.*
(2) *$a_i R b_j \subseteq I$ for each i,j.*

Proof. (1)⇒(2) Assume $f(x)R[[x;\alpha]]g(x) \subseteq I[[x;\alpha]]$. Let c be an arbitrary element of R. Then we have the following:

$$(\S) \qquad \sum_{k=0}^{\infty}\left(\sum_{i+j=k} a_i x^i c b_j x^j\right) = \sum_{k=0}^{\infty}\left(\sum_{i+j=k} a_i \alpha^i(c b_j)\right) x^k \in I[[x;\alpha]].$$

By induction on $i+j$, we show that $a_i R b_j \subseteq I$ for all i,j. From Equation (§), we obtain $a_0 R b_0 \subseteq I$. This prove for $i+j=0$. Now suppose that $a_i R b_j \subseteq I$ for

$i + j \le n-1$. From Equation (§), we have

$$(§§)\qquad \sum_{i+j=k} a_i\alpha^i(cb_j) \in I \text{ for all } k \ge 0.$$

Let c, d, e be arbitrary elements of R. For $k = n$ replace c by $cb_n da_0 e$. Then we obtain $a_0 cb_n da_0 eb_n \in I$, since $a_0 Rb_j \subseteq I$ for each $j \le n-1$. Hence $(Ra_0Rb_n)^2 \subseteq I$ and so $Ra_0Rb_n \subseteq I$, since I is semiprime. Continuing this process (replacing c by $cb_j da_{n-j}e$ in Equation (§§) for $j = 1, 2, \cdots, n-1$ and using α-compatibility of I), we obtain $a_iRb_j \subseteq I$ for $i + j = n$.

(2)⇒(1) This follows from Proposition 2.3. □

Corollary 2.18. *If I is a (semi)prime α-compatible ideal of R, then $I[[x;\alpha]]$ is a (semi)prime ideal of $R[[x;\alpha]]$.*

Theorem 2.19. *If each minimal prime ideal of R is α-compatible, then $P(R[[x;\alpha]]) \subseteq P(R)[[x;\alpha]]$.*

Proof. This follows from Corollary 2.18. □

Theorem 2.20. *If P is a completely (semi)prime α-compatible ideal of R, then $P[[x;\alpha]]$ is a completely (semi)prime ideal of $R[[x;\alpha]]$.*

Proof. Let P be a completely prime ideal of R. R/P is a domain, hence it is a reduced ring. $\overline{R/P}$ is an $\bar{\alpha}$-compatible ring, hence $\overline{R/P}$ is $\bar{\alpha}$-rigid, by ([6], Lemma 2.2). Let $\overline{f(x)}, \overline{g(x)} \in R/P[[x;\bar{\alpha}]]$ such that $\overline{f(x)}\ \overline{g(x)} = 0$. Then $\overline{f(x)} = 0$ or $\overline{g(x)} = 0$, by ([9], Proposition 17). Thus $R[[x;\alpha]]/P[[x;\alpha]] \cong R/P[[x;\bar{\alpha}]]$ is a domain and $P[[x;\alpha]]$ is a completely prime ideal of $R[[x;\alpha]]$. □

Corollary 2.21 ([11], Proposition 3.12)**.** *If $P(R)$ is an α-rigid ideal of R, then $P(R[[x;\alpha]]) \subseteq P(R)[[x;\alpha]]$.*

Theorem 2.22. *If P is a strongly prime α-compatible ideal of R, then $P[[x;\alpha]]$ is a strongly prime ideal of $R[[x;\alpha]]$.*

Theorem 2.23. *If each minimal strongly prime ideal of R is α-compatible, then $N_r(R[[x;\alpha]]) \subseteq N_r(R)[[x;\alpha]]$.*

Corollary 2.24 ([11], Proposition 3.12)**.** *If $N_r(R)$ is an α-rigid ideal of R, then $N_r(R[[x;\alpha]]) \subseteq N_r(R)[[x;\alpha]]$.*

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DEPARTMENT OF MATHEMATICS, SHAHROOD UNIVERSITY OF TECHNOLOGY, SHAHROOD, IRAN, P.O.BOX: 316-3619995161
eb_hashemi@yahoo.com or eb_hashemi@shahrood.ac.ir

This paper is available via http://nyjm.albany.edu/j/2006/12-20.html.