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Symplectic torus bundles and group extensions

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Abstract. Symplectic torus bundles $\xi : T^2 \to E \to B$ are classified by the second cohomology group of B with local coefficients $H_1(T^2)$. For B a compact, orientable surface, the main theorem of this paper gives a necessary and sufficient condition on the cohomology class corresponding to ξ for E to admit a symplectic structure compatible with the symplectic bundle structure of ξ: namely, that it be a torsion class. The proof is based on a group-extension-theoretic construction of J. Huebschmann, 1981. A key ingredient is the notion of fibrewise-localization.

Contents

1. Introduction

A symplectic F-bundle in this paper is a smooth fibre bundle $\xi : F \xrightarrow{i} E \xrightarrow{p} B$ whose structure group is the group of symplectomorphisms $\mathrm{Symp}(F, \sigma)$ for some symplectic form σ on F. For such a bundle, the fibres $F_b = p^{-1}(b)$ admit canonical symplectic forms σ_b, the pullbacks of σ via symplectic trivializations. A natural question to ask about ξ is under what conditions the forms σ_b "piece together" to produce a symplectic form on E. More exactly, when is there a *closed* 2-form β on E such that

$$\beta | F_b = \sigma_b, \qquad \text{for all } b \in B, \tag{1}$$

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with β nondegenerate? When B is connected, an argument of W. Thurston (cf. [8, page 199]) shows that a closed 2-form β satisfying (1) exists if and only if the de Rham cohomology class of σ is contained in image$(i^* : H^2_{\mathrm{DR}}(E) \to H^2_{\mathrm{DR}}(F))$. Thurston further shows that when such a β exists and E is compact and B is symplectic, then β may be modified to be nondegenerate while still satisfying (1). McDuff and Salamon [8, page 202] use Thurston's result to settle the existence question for a large family of surface bundles:

Theorem. *Suppose that F is a closed, oriented, connected surface of genus $\neq 1$, and let $\xi : F \to E \to B$ be a symplectic F-bundle with B a compact, connected symplectic manifold. Then, E admits a symplectic structure inducing the given structures on the fibres.*

Their argument does not apply to the case of torus bundles, however; indeed, they present the following simple counterexample in that case. Consider the composition

$$S^1 \times S^3 \xrightarrow{\mathrm{pr}} S^3 \xrightarrow{\mathfrak{H}} S^2,$$

where $\mathfrak{H}$ is the well-known Hopf map. This composition is the projection of a symplectic torus bundle. No symplectic form can exist on the total space $S^1 \times S^3$, however, because $H^2_{\mathrm{DR}}(S^1 \times S^3) = 0$.

1.1. The results. This paper obtains a necessary and sufficient condition for the existence of β in the case of symplectic torus bundles over surfaces. Before stating our main result, however, we remind the reader of some subsidiary facts. For any fibre bundle $\xi : F \to E \to B$ with group G, the action of G on F produces a $\pi_0(G)$-action on the homology and cohomology of F. When B is a pointed space, there is a well-defined homomorphism $\pi_1(B) \to \pi_0(G)$ that gives each homology or cohomology group of F the structure of a $\mathbb{Z}[\pi_1(B)]$-module. Now suppose that F is the 2-torus T^2 and $G = \mathrm{Symp}(T^2, \sigma)$. It is not hard to show (see Appendices A and B) that $\pi_0(G) \approx \mathrm{SL}(2, \mathbb{Z})$ and that the $\pi_0(G)$-action on $H_1(T^2)$ may be identified with the natural action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{Z}^2$. Given any representation $\rho : \pi_1(B) \to \pi_0(G) = \mathrm{SL}(2, \mathbb{Z})$, we let $\mathbb{Z}^2_\rho$ denote the corresponding $\mathbb{Z}[\pi_1(B)]$-module.

The following proposition and remark follow immediately from known, classical results of algebraic topology, as described in Appendices A and B.

Proposition 1.1. *Assume that B has the homotopy type of a pointed, path-connected CW complex, and choose any representation $\rho : \pi_1(B) \to \mathrm{SL}(2, \mathbb{Z})$. Then there is a natural, bijective correspondence between the based equivalence classes of symplectic torus bundles over B inducing the module structure $\mathbb{Z}^2_\rho$ on $H_1(T^2)$ and the elements of $H^2(B; \mathbb{Z}^2_\rho)$, the second cohomology group of B with local coefficients $\mathbb{Z}^2_\rho$.*

Remark. We call the cohomology class corresponding to the symplectic torus bundle ξ the *characteristic class* of ξ and denote it by $c(\xi)$. The characteristic class $c(\xi)$ vanishes if and only if ξ admits a section. When the representation ρ is trivial, $c(\xi) = 0$ if and only if ξ is trivial.

We can now state the main result of this paper.

Theorem 1.1. *Suppose that ξ is a symplectic torus bundle over a connected surface B. Then the total space of ξ admits a closed form β satisfying* (1) *if and only if the characteristic class $c(\xi)$ is a torsion element of $H^2(B;\mathbb{Z}^2_\rho)$. If, in addition, B is compact and orientable and such a form exists, it can be chosen to be a symplectic form.*

The last statement of the theorem is simply an application of Thurston's argument mentioned above. So our proof of the theorem focuses exclusively on the existence of a closed 2-form β satisfying (1).

The following consequences of the theorem are almost immediate. We give proofs in §5.

Corollary 1.2. *Let B be a connected surface, and let $\rho : \pi_1(B) \to \mathrm{SL}(2,\mathbb{Z})$ be a representation. Among the symplectic torus bundles over B that induce the representation ρ, there are, up to equivalence, only finitely many whose total spaces admit closed forms β satisfying* (1).

Corollary 1.3. *Every principal torus bundle has a canonical structure as a symplectic torus bundle. Let $\xi : T^2 \to E \to B$ be such a bundle, with B a connected surface. Then, E* fails *to admit a closed* 2*-form β satisfying* (1) *if and only if B is closed and orientable* and *ξ is nontrivial.*

A specialization of this corollary perhaps deserves a separate statement.

Corollary 1.4. *Suppose the closed, connected symplectic* 4*-manifold E admits a free T^2-action such that the orbits are symplectic submanifolds. Then, as T^2-manifolds, $E \approx T^2 \times (E/T^2)$.*

Remark. There does not appear to be a reasonable, *nontrivial* sense in which the T^2-equivariant diffeomorphism of this corollary can be taken to be a symplectomorphism. There is simply too much leeway allowed by the hypotheses for symplectic forms on E.

The proof of Theorem 1.1 breaks into three cases according as the base surface B is nonclosed, closed of genus zero, and closed of genus different from zero. The first two cases are substantially easier than the third and are proved at the end of this section and in §4, respectively. In these two cases, the theorem reduces to the following propositions.

Proposition 1.2. *Every symplectic torus bundle over a connected, nonclosed surface admits a section and has a total space that admits a closed* 2*-form β satisfying* (1).

Proposition 1.3. (a) *The total space of a symplectic torus bundle over S^2 admits a closed* 2*-form satisfying* (1) *if and only if the bundle is trivial.* (*See the reference to* [3] *in the remarks at the end of this section.*)

(b) *Let E be the total space of a symplectic torus bundle ξ over $\mathbb{R}P^2$. If the representation ρ corresponding to ξ is trivial, then E admits a closed* 2*-form β satisfying* (1). *If ρ is nontrivial, then E admits such a* 2*-form if and only if $c(\xi) = 0$, that is, if and only if ξ admits a section.*

The case in which B is a closed surface of genus $\neq 0$ forms the heart of the paper and occupies §§2,3. The following two examples suggest the variety of concrete possibilities in this case. In both examples the base space B is itself the torus T^2. Thus, in both, the representation ρ is a homomorphism $\pi_1(T^2) = \mathbb{Z}^2 \to \mathrm{SL}(2,\mathbb{Z})$.

Example 1. For any $(a,b) \in \mathbb{Z}^2$, define ρ by the equation

$$\rho(a,b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

In this example, one computes that the bundles are classified by $H^2(T^2;\mathbb{Z}^2_\rho) = \mathbb{Z}$. Consequently, up to equivalence, there is only one torus bundle ξ — namely, the one satisfying $c(\xi) = 0$ — for which the total space admits a symplectic form satisfying (1). According to the classification, this is the unique bundle admitting a section. The total space of ξ is the renowned Kodaira–Thurston manifold, the earliest known example of a symplectic manifold that is not Kähler (cf. [8, page 89]).

Example 2. Let m and n be any fixed integers ≥ 0. Then, for $(a,b) \in \mathbb{Z}^2$, define ρ by

$$\rho(a,b) = \begin{pmatrix} -2mn+1 & 2mn^2+n \\ -m & mn+1 \end{pmatrix}^{a+b}.$$

In this example, the bundles are classified by $H^2(T^2;\mathbb{Z}^2_\rho) = \mathbb{Z}_m \oplus \mathbb{Z}_n$. So, when $m,n \neq 0$, there are exactly mn symplectic torus bundles over the torus, and, for every one of them, the total space admits the desired symplectic form.

Both examples proceed by computing H^2 and then applying Theorem 1.1. The computation begins with Poincaré duality for T^2 (with twisted coefficients), which implies that the desired result is just the group of coinvariants of the module $\mathbb{Z}^2_\rho$ (cf. [1, page 57]). We leave this computation to the reader.

1.2. Reformulating Thurston's criterion. We conclude this introduction with a brief reformulation of Thurston's cohomology criterion for the existence of the desired closed 2-forms β in the context of symplectic torus bundles. This will immediately imply Proposition 1.2.

Thurston's criterion is stated in our opening paragraph in terms of de Rham cohomology, but clearly, by de Rham's theorem, it may be equivalently stated in terms of singular cohomology with real coefficients. In fact, a further easy reduction is desirable: namely, we pass to rational coefficients. Indeed, note that since $H^2(T^2;\mathbb{R}) \approx \mathbb{R}$, the existence of a nontrivial class in the image of $i^* : H^2(E;\mathbb{R}) \to H^2(T^2;\mathbb{R})$ is equivalent to the surjectivity of this map, and this in turn is easily checked to be equivalent to the surjectivity of $i^* : H^2(E;\mathbb{Q}) \to H^2(T^2;\mathbb{Q}) \approx \mathbb{Q}$.

Now using rational coefficients, we consider the Serre cohomology spectral sequence for the symplectic torus bundle $\xi : T^2 \xrightarrow{i} E \xrightarrow{p} B$, for which the E_2-term is given by

$$E_2^{p,q} = H^p(B; H^q(T^2;\mathbb{Q})).$$

Therefore, $E_2^{0,2} = H^0(B; H^2(T^2;\mathbb{Q})) = H^2(T^2;\mathbb{Q})^{\pi_1(B)}$, the group of $\pi_1(B)$-invariant classes in $H^2(T^2;\mathbb{Q})$. But $\pi_1(B)$ acts via symplectomorphisms, which are orientation-preserving, so $E_2^{0,2} = H^2(T^2;\mathbb{Q})$. Now when B is a surface, its cohomology vanishes above dimension two, so that $d_2^{0,2}$ is the only possibly nontrivial differential issuing from $E_r^{0,2}, r \geq 2$. Thus

$$\ker(d_2^{0,2} : H^2(T^2;\mathbb{Q}) \to H^2(B; H^1(T^2;\mathbb{Q}))) = E_\infty^{0,2},$$

which equals $i^*(H^2(E;\mathbb{Q}))$. Therefore, in this context, Thurston's cohomology criterion becomes

$$d_2^{0,2} = 0. \tag{2}$$

Proof of Proposition 1.2. Proposition 1.2 now follows easily, using the fact that every connected, nonclosed surface has the homotopy type of a 1-dimensional simplicial complex. Every F-bundle over such a base space admits a section when F is path-connected. Moreover, the target of $d_2^{0,2}$, namely $H^2(B; H^1(T^2;\mathbb{Q}))$, is identically zero, so (2) is satisfied. □

Remarks. To conclude this introduction, I am pleased to to acknowledge my indebtedness to K. Brown for a number of very helpful conversations during the preparation of this paper. I also want to mention two related papers, which were brought to my attention after this work was completed. The first is a paper by Hansjörg Geiges [3], which deals primarily with the case of torus bundles over a torus. However, it also obtains (p. 545) our Proposition 1.3(a) — the case $B = S^2$. The second paper is an e-print by Rafał Walczak [11], who uses Seiberg–Witten theory to answer the question of when the total space of the bundle admits a symplectic structure (whether or not it is compatible with the fibering). This last can be viewed as complementary to the current paper.

2. An interpretation of the main theorem in terms of group extensions

Let B be a connected, closed surface of genus $\neq 0$ and fundamental group π. As is well-known, B is a $K(\pi, 1)$, and so one sees easily that the homotopy exact sequence of the symplectic torus bundle $\xi : T^2 \xrightarrow{i} E \xrightarrow{p} B$ collapses to the short exact sequence

$$\mathbb{E}: \quad \mathbb{Z}^2 \overset{i_*}{\rightarrowtail} G \overset{p_*}{\twoheadrightarrow} \pi, \tag{3}$$

which will be convenient to regard as a group extension of π by $\mathbb{Z}^2$. Thus, the group G equals $\pi_1(E)$, and E is a $K(G, 1)$. Huebschmann [6] uses the cohomology spectral sequence of (3) (which is the same as the Serre spectral sequence of ξ) and obtains group-extension-theoretic interpretations of some of its differentials. We are interested in his interpretation of

$$d_2^{0,2} : H^2(\mathbb{Z}^2;\mathbb{Q}) \to H^2(\pi; H^1(\mathbb{Z}^2;\mathbb{Q})).$$

Here, we follow Huebschmann and use group-cohomology notation for the cohomology groups, but of course these are the same as the cohomology groups of the base and fibre of ξ as before. Since 2-dimensional group cohomology classifies group extensions with abelian kernel, the map $d_2^{0,2}$ may be regarded as mapping extensions of $\mathbb{Z}^2$ by $\mathbb{Q}$ — more precisely, *central* extensions, since $\mathbb{Z}^2$ acts trivially on $\mathbb{Q}$ — to extensions of π by $H^1(\mathbb{Z}^2;\mathbb{Q})$. Huebschmann presents a construction that uses $\mathbb{E}$ to pass from an extension $\mathbb{E}_1$ of the first kind to an extension $\mathbb{E}_2$ of the second.

In what follows, we shall refer to the 2-dimensional cohomology class corresponding to an extension $\mathbb{E}^*$ as $c(\mathbb{E}^*)$.

2.1. Huebschmann's construction. Let $\mathbb{E}_1$ denote an arbitrary *central* extension of $\mathbb{Z}^2$ by $\mathbb{Q}$

$$\mathbb{E}_1 : \quad \mathbb{Q} \rightarrowtail G_1 \overset{r_1}{\twoheadrightarrow} \mathbb{Z}^2. \tag{4}$$

We follow Huebschmann by using $\mathbb{E}$ and $\mathbb{E}_1$ to construct an extension $\mathbb{E}_2$

$$\mathbb{E}_2 : \quad H^1(\mathbb{Z}^2; \mathbb{Q}) \rightarrowtail G_2 \twoheadrightarrow \pi. \tag{5}$$

We do this in several steps.

Step (a): Since, in the extension $\mathbb{E}$, $\mathbb{Z}^2$ is normal in G, inner automorphisms of G determine automorphisms of $\mathbb{Z}^2$. Thus, we have a representation

$$\rho : \pi \to \operatorname{Aut}(\mathbb{Z}^2) = \mathrm{GL}(2, \mathbb{Z}).$$

Recalling that $\mathbb{E}$ comes from a *symplectic* torus bundle, our comment in §1.1 implies that

$$\rho(\pi) \subseteq \mathrm{SL}(2, \mathbb{Z}). \tag{6}$$

We shall also make use of the composition

$$G \overset{p_*}{\twoheadrightarrow} \pi \overset{\rho}{\to} \mathrm{GL}(2, \mathbb{Z}).$$

Now consider an automorphism, h, of G_1. Since $\ker(r_1)$ of (4) may be characterized as the set of all infinitely divisible elements of G_1, we have $h(\ker(r_1)) = \ker(r_1)$, so that h induces an automorphism f of $\mathbb{Z}^2$. The rule $h \mapsto f$, thus, gives a representation

$$\rho_1 : \operatorname{Aut}(G_1) \to \operatorname{Aut}(G_1/\mathbb{Q}) = \mathrm{GL}(2, \mathbb{Z}). \tag{7}$$

Lemma 1. $\mathrm{SL}(2, \mathbb{Z}) \subseteq \rho_1(\operatorname{Aut}(G_1))$.

Proof. An automorphism $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ can be used to construct a pullback extension

$$f^\sharp \mathbb{E}_1 : \quad \mathbb{Q} \rightarrowtail f^\sharp G_1 \twoheadrightarrow \mathbb{Z}^2 \tag{8}$$

of $\mathbb{E}_1$. By naturality, $c(f^\sharp \mathbb{E}_1) = f^*(c(\mathbb{E}_1))$, where f^* is the automorphism of $H^2(\mathbb{Z}^2; \mathbb{Q}) \approx \mathbb{Q}$ induced by f. One checks easily that this automorphism is multiplication by $\det(f) = \pm 1$. Therefore, if f belongs to $\mathrm{SL}(2, \mathbb{Z})$, we have $c(f^\sharp \mathbb{E}_1) = c(\mathbb{E}_1)$, implying an equivalence of extensions $\mathbb{E}_1 \approx f^\sharp \mathbb{E}_1$. Post-composing this with the canonical map of extensions $f^\sharp \mathbb{E}_1 \to \mathbb{E}_1$, which equals f on $\mathbb{Z}^2$, we obtain an automorphism of the extension $\mathbb{E}_1$, yielding an automorphism $h : G_1 \to G_1$ such that $\rho_1(h) = f$. $\square$

The homomorphisms $\rho \circ p_*$ and ρ_1 allow us to form the fibre product $\Pi = G \times_{\mathrm{GL}(2,\mathbb{Z})} \operatorname{Aut}(G_1)$. Let p_1 and p_2 denote the projections $\Pi \to G$, $\Pi \to \operatorname{Aut}(G_1)$, respectively. Note that the inclusions in (6) and Lemma 1 combine to show that p_1 is surjective.

Step (b): Combining (3) and (4), we have a composite homomorphism

$$\lambda : G_1 \overset{r_1}{\twoheadrightarrow} \mathbb{Z}^2 \overset{i_*}{\rightarrowtail} G. \tag{9}$$

and a homomorphism $\mu : G_1 \to \Pi$ given by

$$\mu(x) = (\lambda(x), \iota_x), \tag{10}$$

where ι_x denotes inner automorphism by x. It is not hard to check that

$$\rho \circ p_*(\lambda(x)) = \rho_1(\iota_x) = I,$$

where I is the 2×2 identity matrix in $\mathrm{GL}(2, \mathbb{Z})$. Therefore, μ does indeed take values in Π. Let G_2 denote the quotient $\Pi/\mathrm{im}(\mu)$ and λ_2 the projection $\Pi \to G_2$.

Step(c): Note that μ vanishes on $\ker(r_1)$ so that it factors as $G_1 \overset{r_1}{\twoheadrightarrow} \mathbb{Z}^2 \rightarrowtail \Pi$, where the second map lifts the injection $i_* : \mathbb{Z}^2 \rightarrowtail G$. It follows that p_1 maps $\mathrm{im}(\mu)$ bijectively onto $\mathrm{im}(i_*)$, which implies that p_1 descends to a surjection $r : G_2 \twoheadrightarrow \pi$, and λ_2 maps $\ker(p_1) = H^1(\mathbb{Z}^2; \mathbb{Q})$ isomorphically onto $\ker(r)$. Therefore, $r : G_2 \twoheadrightarrow \pi$ is an extension of π by $H^1(\mathbb{Z}^2; \mathbb{Q})$, which is the desired extension $\mathbb{E}_2$ (see (5) above). The following diagram of exact sequences summarizes the situation:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathbb{Q} & \xrightarrow{0} & H^1(\mathbb{Z}^2;\mathbb{Q}) & = & H^1(\mathbb{Z}^2;\mathbb{Q}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & G_1 & \xrightarrow{\mu} & \Pi & \xrightarrow{\lambda_2} & G_2 & \longrightarrow & 0 \\
 & & {\scriptstyle r_1}\downarrow & & {\scriptstyle p_1}\downarrow & & {\scriptstyle r}\downarrow & & \\
0 & \longrightarrow & \mathbb{Z}^2 & \xrightarrow{i_*} & G & \xrightarrow{p_*} & \pi & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Theorem (Huebschmann, [6])**.** $d_2^{0,2}(c(\mathbb{E}_1)) = c(\mathbb{E}_2)$.

Huebschmann's result allows us to analyze properties of $d_2^{0,2}$ (e.g., condition (2)) by applying his construction to a certain family of central extensions. Note, however, that the family we are interested in may be described as $H^2(\mathbb{Z}^2; \mathbb{Q}) \approx \mathbb{Q}$, a 1-dimensional vector space over $\mathbb{Q}$. So, to determine the vanishing of $d_2^{0,2}$, it suffices to analyze Huebschmann's construction for any single central extension of $\mathbb{Z}^2$ by $\mathbb{Q}$ that represents a nonzero element of cohomology. We describe such an extension shortly, but first we must make a short preparatory digression.

2.2. Fibrewise-localization. The theory of localization in algebraic topology has been well-known since the work of Quillen, Sullivan, Bousfield, Kan, Dwyer, Hilton, Mislin and others. We summarize only that small fragment of the subject that we need here. A useful reference for the reader is [5]. We shall confine ourselves to *localizing at* 0, i.e., to rationalization, although most of what we describe applies to the general case.

Localization of a nilpotent group N is equivalent to localization of the Eilenberg-MacLane space $K(N, 1)$. We'll use the language of groups here, however, rather than that of topology. For the moment, we restrict entirely to nilpotent groups. A *local group* may be defined here as a nilpotent group that is uniquely p-divisible for all primes p. A localization of the nilpotent group N consists of a localization homomorphism (or localization map) $\ell : N \to N_0$, where N_0 is local, such that ℓ is universal for homomorphisms of N into local groups (i.e., every such homomorphism $h : N \to L$ factors as $h_0\ell$ for a unique homomorphism $h_0 : N_0 \to L$). N_0 and ℓ are uniquely determined up to the obvious equivalence. When N is abelian, N_0 may be taken to be $N \otimes \mathbb{Q}$ and ℓ given by $x \mapsto x \otimes 1$. A key fact about localization is that localization maps induce localization homomorphisms of homology. Localization respects exact sequences. Indeed, it is not hard to show that, given any exact sequence $\mathbb{S}$ of nilpotent groups, we may localize its terms and maps, obtaining an exact sequence $\mathbb{S}_0$ of local groups and a map of exact sequences $\ell_{\mathbb{S}} : \mathbb{S} \to \mathbb{S}_0$ that localizes the individual terms. Thus, we may apply this to group extensions in which all the groups are nilpotent.

Let

$$\mathbb{S} : \quad N' \rightarrowtail N \twoheadrightarrow N''$$

be a short exact sequence of nilpotent groups, and let

$$\mathbb{S}_0 : \quad N'_0 \rightarrowtail N_0 \twoheadrightarrow N''_0$$

denote its localization. Then $\ell_{\mathbb{S}}$ may be thought of as a triple of localization maps $(\ell_{N'}, \ell_N, \ell_{N''})$. We use $\ell_{N''} : N'' \to N''_0$ to pull back the sequence $\mathbb{S}_0$ to an exact sequence

$$\mathbb{S}_{f0} : \quad N'_0 \rightarrowtail N_{f0} \twoheadrightarrow N'',$$

which we call the *fibrewise-localization* of $\mathbb{S}$. The pullback construction produces a natural map of exact sequences $\ell_f : \mathbb{S} \to \mathbb{S}_{f0}$ which on N'' is just the identity and on N' is just the localization map $\ell_{N'} : N' \to N'_0$.

While this construction is perfectly valid, we want to use fibrewise-localization in the case of group extensions with abelian kernel without assuming any nilpotency restrictions. So we present another construction, valid for all such extensions. Consider a group extension with abelian kernel A,

$$\mathbb{S} : \quad A \rightarrowtail B \twoheadrightarrow C, \tag{11}$$

and consider any normalized 2-cocycle ϕ associated with $\mathbb{S}$. This is a function $\phi : C \times C \to A$ subject to normalization and 2-cocyle identities (cf. [1, pp. 91 ff.]). ϕ is defined by choosing a *function* $C \to B$ that splits the surjection $B \twoheadrightarrow C$ in (11) and measuring how far this deviates from being a homomorphism. Now, form the composite $C \times C \xrightarrow{\phi} A \xrightarrow{\ell} A_0$, where ℓ is a localization map. This composite is a new normalized 2-cocycle for an extension of C by A_0. We define this extension to be the fibrewise-localization of $\mathbb{S}$ and denote it by $\mathbb{S}_{f0}$. There is an obvious map of extensions $\mathbb{S} \to \mathbb{S}_{f0}$ with the same properties as before. It is not hard to show, using basic facts about extensions, that, up to equivalence of extensions, this construction is independent of the initial choice of 2-cocycle ϕ corresponding to $\mathbb{S}$ and independent of the choice of localization map ℓ, and it coincides with our earlier description of fibrewise-localization for nilpotent extensions of nilpotent groups with abelian kernels. Note also that this construction shows that if $c(\mathbb{S})$ and $c(\mathbb{S}_{f0})$ are the cohomology classes of the corresponding extensions (i.e., the cohomology classes of the corresponding 2-cocycles), then the homomorphism $H^2(C;A) \to H^2(C;A_0)$ induced by the localization map $\ell : A \to A_0$ sends $c(\mathbb{S})$ to $c(\mathbb{S}_{f0})$.

We now present a useful and well-known extension of $\mathbb{Z}^2$ by $\mathbb{Z}$.

The discrete Heisenberg group $\mathcal{H}$ may be described as the set $\mathbb{Z}^3$ of all integer triples with the following multiplication

$$(r,s,t) \bullet (u,v,w) = (r+u+sw, s+v, t+w). \tag{12}$$

The center $Z[\mathcal{H}]$ and commutator $[\mathcal{H},\mathcal{H}]$ both equal $\mathbb{Z} = \mathbb{Z} \times 0 \times 0$, so that we clearly obtain the central extension

$$\mathbb{H}: \quad \mathbb{Z} \rightarrowtail \mathcal{H} \twoheadrightarrow \mathbb{Z}^2.$$

We call this the *Heisenberg extension.* The following result about $\mathbb{H}$ is well-known. For the convenience of the reader, we present a proof due to K. Brown.

Lemma 2. *The cohomology class $c(\mathbb{H})$ generates $H^2(\mathbb{Z}^2;\mathbb{Z}) \approx \mathbb{Z}$.*

Proof. Let the group H be given by the presentation $\langle x, y : [x,[x,y]], [y,[x,y]]\rangle$. If $a, b \in \mathcal{H}$ are the triples $(0,1,0), (0,0,1)$, respectively, then it is not hard to check that they generate $\mathcal{H}$, that $[a,b] = (1,0,0)$, and that, accordingly, a and b satisfy the relations for x and y in H above. Therefore, the rule $x \mapsto a$, $y \mapsto b$ well-defines a surjective homomorphism $f : H \to \mathcal{H}$. We let the reader check that this is injective as well. Thus, $H \approx \mathcal{H}$, so that, given any group H' and elements $c, d \in H'$ satisfying the stated relations, there is a unique homomorphism $\mathcal{H} \to H'$ sending a to c and b to d.

We apply this last fact to an arbitrary central extension $\mathbb{M} : \mathbb{Z} \rightarrowtail M \twoheadrightarrow \mathbb{Z}^2$, choosing the elements $c, d \in M$ to be arbitrary lifts of $(1,0), (0,1) \in \mathbb{Z}^2$, respectively. Let $h : \mathcal{H} \to M$ be the corresponding homomorphism. h clearly induces a map of extensions $\mathbb{H} \to \mathbb{M}$ which is the identity on $\mathbb{Z}^2$ and is an endomorphism on $\mathbb{Z}$, say multiplication by some integer k. By tracing out the definition of the 2-cocycle corresponding to an extension, it is easy to check that $c(\mathbb{M}) = kc(\mathbb{H})$. Thus, $c(\mathbb{H})$ generates $H^2(\mathbb{Z}^2;\mathbb{Z}) \approx \mathbb{Z}$. □

We now define the extension of $\mathbb{Z}^2$ by $\mathbb{Q}$ that interests us: namely, it is the fibrewise-localization of the Heisenberg extension, $\mathbb{H}_{f0}$.

Corollary 3. *$c(\mathbb{H}_{f0})$ generates the 1-dimensional $\mathbb{Q}$ vector space $H^2(\mathbb{Z}^2;\mathbb{Q})$.*

Proof. Let $\ell_* : H^2(\mathbb{Z}^2;\mathbb{Z}) \to H^2(\mathbb{Z}^2;\mathbb{Q})$ denote the homomorphism induced by the coefficient injection $\mathbb{Z} \to \mathbb{Q}$. As already observed, ℓ_* maps $c(\mathbb{H})$ to $c(\mathbb{H}_{f0})$. At the same time, it is clear that ℓ_* is a localization map, essentially the same as the standard injection $\mathbb{Z} \to \mathbb{Q}$. Therefore, by the foregoing lemma, $c(\mathbb{H}_{f0}) \neq 0$, as desired. □

2.3. Reinterpreting the main theorem. Let us return to the context with which this section opened: namely, to the symplectic torus bundle $\xi : T^2 \xrightarrow{i} E \xrightarrow{p} B$ with B a closed, connected $K(\pi, 1)$ surface. The group π acts via symplectomorphisms on $H_1(T^2) = \mathbb{Z}^2$. Thus, we have a representation ρ and corresponding (left) $\mathbb{Z}[\pi]$-module $\mathbb{Z}^2_\rho$, as explained before. In a similar way, the cohomology group $H^1(T^2; \mathbb{Q}) \approx \mathbb{Q}^2$ receives the structure of a $\mathbb{Z}[\pi]$-module. We want this to be a left $\mathbb{Z}[\pi]$-module also despite the contravariance of cohomology, so we use the standard convention for this, which we may describe here as follows: identify $H^1(T^2; \mathbb{Q})$ with $\mathrm{Hom}(H_1(T^2), \mathbb{Q})$, and for any $\alpha \in \pi$, $h \in \mathrm{Hom}(H_1(T^2), \mathbb{Q})$, and $x \in H_1(T^2)$, let $(\alpha h)(x) = h(\alpha^{-1}x)$.

We now return to our use of group cohomology notation in the following lemma, the proof of which is given in the next section.

Lemma 4. *Let $D : H^1(\mathbb{Z}^2; \mathbb{Q}) \to H_1(\mathbb{Z}^2; \mathbb{Q})$ denote Poincaré duality, and let ψ be the composite*

$$\mathbb{Z}^2 = H_1(\mathbb{Z}^2; \mathbb{Z}) \xrightarrow{\ell} H_1(\mathbb{Z}^2; \mathbb{Q}) \xrightarrow{D^{-1}} H^1(\mathbb{Z}^2; \mathbb{Q}),$$

where, here, ℓ is the localization map induced by the usual injection $\mathbb{Z} \rightarrowtail \mathbb{Q}$. Then, using the module structures described above, ψ is a $\mathbb{Z}[\pi]$-injection and a localization map. Therefore,

$$\psi_\sharp : H^2(\pi; \mathbb{Z}^2) \to H^2(\pi; H^1(\mathbb{Z}^2; \mathbb{Q}))$$

induced by ψ is also a localization map.

We can now state a reinterpretation of Theorem 1.1 in this group-extension context.

Theorem 2.1. *Let $\mathbb{H}_{f0}$ be the fibrewise-localization of the Heisenberg extension, and let $\mathbb{E}$ be the group extension* (3) *described at the start of* §2. *Apply Huebschmann's construction to these, obtaining an extension $\mathbb{E}_2$ as in* (5). *Then,*

$$\psi_\sharp(c(\mathbb{E})) = -c(\mathbb{E}_2).$$

We prove Theorem 2.1 in the next section. We close this section by using it to prove Theorem 1.1 in case B is closed, connected of genus $\neq 0$:

Proof. Let $\xi : T^2 \xrightarrow{i} E \xrightarrow{p} B$ be a symplectic torus bundle with corresponding group extension $\mathbb{E}$. As discussed in Appendix C, the classes $c(\xi)$and $c(\mathbb{E})$ are the same, so we may deal exclusively with the latter. Suppose it has finite order. Then, by Huebschmann's theorem and Theorem 2.1,

$$d_2^{0,2}(c(\mathbb{H}_{f0})) = c(\mathbb{E}_2) = -\psi_\sharp(c(\mathbb{E})) = 0.$$

By Corollary 3 of §2.2, this implies that $d_2^{0,2} = 0$, which is condition (2). Therefore, as already argued, the desired form β exists. The converse follows by reversing the steps. □

3. Proof of Theorem 2.1

The basic idea of the proof of Theorem 2.1 is to produce suitable 2-cocycles f and F for the extensions $\mathbb{E}$ and $\mathbb{E}_2$, respectively, and then to show that, if $\psi_\flat$ is the chain map induced by ψ, then $\psi_\flat(f) = -F$. To carry this out, we need to be more explicit about ψ and about the groups and maps occurring in Huebschmann's construction.

3.1. The map ψ. We begin with a proof of Lemma 4 of §2.

Proof. That $\psi = D^{-1}\ell$ is a localization map and injective is obvious. Choose any $\alpha \in \pi$, and let a be a symplectomorphism of T^2 representing α. This is a degree-one map. Therefore, the standard cap product identity yields $a_* D a^* = D$, or $a_* D = D(a^*)^{-1}$, that is $\alpha\ D = D\alpha$. So, D is $\mathbb{Z}[\pi]$-equivariant. That ℓ is also equivariant is immediate from definitions. Hence ψ is a map of $\mathbb{Z}[\pi]$-modules.

It remains to show that $\psi_\sharp : H^2(\pi;\mathbb{Z}^2) \to\ H^2(\pi; H^1(\mathbb{Z}^2;\mathbb{Q}))$ is a localization map. By definition, $\psi_\sharp$ factors as

$$H^2(\pi;\mathbb{Z}^2) \xrightarrow{\ell_\sharp} H^2(\pi;\mathbb{Q}^2) \underset{\approx}{\xrightarrow{(D^{-1})_\sharp}} H^2(\pi; H^1(\mathbb{Z}^2;\mathbb{Q})).$$

So, $\psi_\sharp$ is equivalent to $\ell_\sharp$. But π is finitely-presented, hence of type FP_2 ([1, page 197]). It follows without difficulty that $\ell_\sharp$ is equivalent to the standard localization map $H^2(\pi;\mathbb{Z}^2) \to H^2(\pi;\mathbb{Z}^2) \otimes \mathbb{Q}$. □

For computations which follow below, it will be useful to obtain an alternative description of ψ. Accordingly, we let e_1 and e_2 be the standard generators of $H_1(\mathbb{Z}^2;\mathbb{Z}) = \mathbb{Z}^2$; we may write $a_1e_1 + a_2e_2$ as (a_1, a_2). Let e_1^*, e_2^* denote the basis of $H^1(\mathbb{Z}^2;\mathbb{Q})$ dual to $\ell(e_1), \ell(e_2)$, using this to write elements of $H^1(\mathbb{Z}^2;\mathbb{Q})$ as pairs. Then, one easily computes, $\psi(e_1) = e_2^*$ and $\psi(e_2) = -e_1^*$, so that, in pair notation,

$$\psi(a_1, a_2) = (-a_2, a_1). \tag{13}$$

3.2. $\mathbb{E}$ and the 2-cocycle f. Recall that $\mathbb{E}$ is the extension

$$\mathbb{Z}^2 \overset{i_*}{\rightarrowtail} G \overset{p_*}{\twoheadrightarrow} \pi.$$

Choose an arbitrary *function* $s : \pi \to G$ splitting p_* and define the normalized 2-cocycle f by the usual rule

$$i_*(f(x, y)) = s(x)s(y)s(xy)^{-1}. \tag{14}$$

Now f, together with the representation $\rho : \pi \to \mathrm{GL}(2,\mathbb{Z})$ induced by $\mathbb{E}$, can be used to form another extension $\mathbb{E}'$ of π as follows: In the cartesian product $\mathbb{Z}^2 \times \pi$ define a group multiplication $\bullet$ by the rule

$$\mathbb{E}' : \qquad (u, x) \bullet (v, y) = (u + \rho(x)(v) + f(x, y), xy). \tag{15}$$

Define homomorphisms $\mathbb{Z}^2 \rightarrowtail \mathbb{Z}^2 \times \pi$ and $\mathbb{Z}^2 \times \pi \twoheadrightarrow \pi$ by the rules $u \mapsto (u, \epsilon)$ and $(u, x) \mapsto\ x$, respectively, where ϵ denotes the identity of π. These piece together to give the extension $\mathbb{E}'$. It is a classical fact that $\mathbb{E}$ and $\mathbb{E}'$ are equivalent extensions, and so $c(\mathbb{E}) = c(\mathbb{E}')$. Therefore, without losing generality, we may assume that $\mathbb{E} = \mathbb{E}'$.

With this assumption, the map $\lambda : \mathcal{H}_{f0} = G_1 \to\ G$ defined in (9) can now be expressed as follows:

$$\lambda(a, b, c) = (b, c, \epsilon),$$

where we omit extra parentheses when harmless. We want to get a similar explicit representation of the map μ used above to define G_2, and for this, we need some computational information about $\mathcal{H}_{f0}$ and $\mathrm{Aut}(\mathcal{H}_{f0})$.

3.3. Computational information about $\mathcal{H}_{f0}$ and Aut($\mathcal{H}_{f0}$). We shall always regard $\mathcal{H}$ as embedded in $\mathcal{H}_{f0}$ via the inclusion $\mathbb{Z}^3 \subseteq \mathbb{Q} \times \mathbb{Z}^2$.

Given elements x and y in some group, we let ${}^x y$ denote the conjugate xyx^{-1}. The following lemma may be easily derived by the reader from the definition of the operation (12).

Lemma 5. *In* $\mathcal{H}_{f0}$,

$$ {}^{(a,b,c)}(x, y, z) = (x + bz - cy, y, z), $$
$$ [(a, b, c), (x, y, z)] = (bz - yc, 0, 0). $$

Corollary 6. *The center* $Z[\mathcal{H}_{f0}]$ *equals* $\mathbb{Q} \times 0 \times 0$*, setwise and as abelian groups.*

Thus, the surjection $\mathcal{H}_{f0} \to \mathbb{Z}^2$ in $\mathbb{H}_{f0}$ is just the projection $\mathcal{H}_{f0} \to \mathcal{H}_{f0}/Z[\mathcal{H}_{f0}]$. Recall that we have denoted this r_1 in our description of Huebschmann's construction (cf. (4)).

Lemma 7. *Every endomorphism* h *of* $\mathcal{H}$ *(resp.,* $\mathcal{H}_{f0}$*) is uniquely determined by the values* $h(0, 1, 0)$ *and* $h(0, 0, 1)$*.*

Proof. The result is obvious for $\mathcal{H}$, since $(0, 1, 0)$ and $(0, 0, 1)$ generate it. So, suppose h is an endomorphism of $\mathcal{H}_{f0}$. Similarly to our discussion above Equation (7), we observe here that the center $Z[\mathcal{H}_{f0}] = \mathbb{Q} \times 0 \times 0$ may be characterized as the set of all infinitely-divisible elements of $\mathcal{H}_{f0}$, which implies that $h(Z[\mathcal{H}_{f0}]) \subseteq Z[\mathcal{H}_{f0}]$. Thus, $h|Z[\mathcal{H}_{f0}]$ may be identified with an endomorphism of $\mathbb{Q}$. But every such endomorphism is uniquely determined by its value at any single nonzero element. Therefore, $h|Z[\mathcal{H}_{f0}]$ is determined by $h(1, 0, 0) = [h(0, 1, 0), h(0, 0, 1)]$. Since $\mathcal{H}_{f0}$ is generated by $\mathcal{H} \cup Z[\mathcal{H}_{f0}]$, the result holds for $\mathcal{H}_{f0}$. □

Lemma 8. *For any triples* $(a, b, c), (d, e, f) \in \mathcal{H}_{f0}$*, there exists an endomorphism* h *of* $\mathcal{H}_{f0}$ *satisfying* $h(0, 1, 0) = (a, b, c)$ *and* $h(0, 0, 1) = (d, e, f)$*.* h *is an automorphism if and only if the determinant*

$$ \begin{vmatrix} b & c \\ e & f \end{vmatrix} = \pm 1 $$

Proof. By Lemma 5 and Corollary 6, the commutator $[(a, b, c), (d, e, f)]$ belongs to $Z[\mathcal{H}_{f0}]$, so by the argument in the proof of Lemma 2 of §2.2, there is a unique homomorphism $k : \mathcal{H} \to \mathcal{H}_{f0}$ satisfying $k(0, 1, 0) = (a, b, c)$ and $k(0, 0, 1) = (d, e, f)$. By Lemma 5, $k(1, 0, 0) = (bf - ec, 0, 0)$, so it belongs to $Z[\mathcal{H}_{f0}]$, and there is a unique extension of $k|Z[\mathcal{H}]$ to an endomorphism of $Z[\mathcal{H}_{f0}]$. Every element y of $\mathcal{H}_{f0}$ can be written as a product zx, with $z \in Z[\mathcal{H}_{f0}]$ and $x \in \mathcal{H}$, so we attempt to define h by the rule, $h(y) = k(z)k(x)$. It is an easy exercise to verify that this gives a well-defined endomorphism. Now suppose that h is an automorphism. Then it induces an automorphism of $\mathbb{Z}^2$ given by the matrix

$$ \begin{pmatrix} b & c \\ e & f \end{pmatrix}, $$

which immediately shows that the stated determinant must equal ± 1. Conversely, if the determinant is ± 1, then by what was just said, the endomorphism of $\mathbb{Z}^2$ induced by h is an automorphism, and, by the equation $h(1, 0, 0) = (bf - ec, 0, 0)$, so is the endomorphism of $Z[\mathcal{H}_{f0}]$. The Five-Lemma then implies that h is an automorphism. □

We now introduce some convenient 'matrix' notation for automorphisms $h \in \mathrm{Aut}(\mathcal{H}_{f0})$. If $h(0,1,0) = (a,b,c)$ and $h(0,0,1) = (d,e,f)$, as above, we associate with h the matrix

$$\begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}.$$

We may occasionally wish to abbreviate this by letting, say, u denote the top row and, say, M the remaining 2×2 submatrix and writing the above matrix as

$$\begin{pmatrix} u \\ M \end{pmatrix}.$$

Of course, the identity automorphism has the obvious matrix representation

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Slightly less obvious, but useful, is the matrix representation of the inner automorphism ι_x, where $x = (a,b,c)$. An easy application of Lemma 5 and Equation (13) above shows that this is

$$\begin{pmatrix} -c & b \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \psi(b,c) \\ I \end{pmatrix},$$

where I is the 2×2 identity matrix. It is possible to work out the multiplication, i.e., composition, in $\mathrm{Aut}(\mathcal{H}_{f0})$ in terms of this notation, but the formula is complicated and not particularly useful here — in addition to the usual quadratic terms of linear algebra, there are also third and fourth order terms. We do record one special case, however: namely, the case of elements of the kernel of the natural projection $\rho_1 : \mathrm{Aut}(\mathcal{H}_{f0}) \to \mathrm{GL}(2,\mathbb{Z})$ in (7). In matrix notation, these elements consist of all matrices of the form,

$$\begin{pmatrix} u \\ I \end{pmatrix}.$$

In this case, one computes easily that

$$\begin{pmatrix} u \\ I \end{pmatrix} \circ \begin{pmatrix} v \\ I \end{pmatrix} = \begin{pmatrix} u+v \\ I \end{pmatrix}.$$

Thus, the kernel is isomorphic, as an abelian group, to $\mathbb{Q}^2$. Now, in fact, we know this for other reasons: the kernel is known to be isomorphic to $\mathrm{Hom}(\mathbb{Z}^2,\mathbb{Q}) \approx H^1(\mathbb{Z}^2;\mathbb{Q}) \approx \mathbb{Q}^2$. However, it is convenient for our computations to have an explicit realization as $\mathbb{Q}^2$.

The following lemma provides a critical ingredient in the proof of Theorem 2.1:

Lemma 9. $\rho_1 : \mathrm{Aut}(\mathcal{H}_{f0}) \to \mathrm{Aut}(\mathbb{Z}^2) = \mathrm{GL}(2,\mathbb{Z})$ *is a split surjection.*

Proof. That ρ_1 is surjective is an immediate corollary of Lemma 8. To show that it splits, we consider the extension $H^1(\mathbb{Z}^2;\mathbb{Q}) \rightarrowtail \mathrm{Aut}(\mathcal{H}_{f0}) \overset{\rho_1}{\twoheadrightarrow} \mathrm{GL}(2,\mathbb{Z})$, which represents an element of $H^2(\mathrm{GL}(2,\mathbb{Z}); H^1(\mathbb{Z}^2;\mathbb{Q}))$. Now, the virtual cohomological dimension of $\mathrm{GL}(2,\mathbb{Z})$ is 1 ([1, page 229]). It follows easily that $H^i(\mathrm{GL}(2,\mathbb{Z}); V) = 0$

for all $i \geq 2$ and all $\mathbb{Q}[\mathrm{GL}(2,\mathbb{Z})]$-modules V. Thus, $H^2(\mathrm{GL}(2,\mathbb{Z}); H^1(\mathbb{Z}^2;\mathbb{Q})) = 0$, implying that ρ_1 splits. □

Remark. A stronger result holds than what is given by this lemma. Specifically, it is possible to define an explicit splitting of the the surjection $\mathrm{Aut}(\mathcal{H}) \to \mathrm{Aut}(\mathbb{Z}^2)$, which then yields a splitting of ρ_1. A description of this is somewhat lengthy, so we have opted for the more abstract, shorter proof above.

Choose and fix an arbitrary (homomorphic!) splitting $\tau : \mathrm{GL}(2,\mathbb{Z}) \to \mathrm{Aut}(\mathcal{H}_{f0})$.

3.4. The proof of Theorem 2.1. We begin by rewriting the definition of the map $\mu : \mathcal{H}_{f0} \to \Pi = G \times_{\mathrm{GL}(2,\mathbb{Z})} \mathrm{Aut}(\mathcal{H}_{f0})$ in terms of the notation just introduced. Recall that, for $z \in \mathcal{H}_{f0}$, $\mu(z) = (\lambda(z), \iota_z)$, as above in (10) and ff. Setting $z = (a,b,c)$ and using results in §§3.2, 3.3, we have

$$\mu(a,b,c) = \left((b,c,\epsilon), \begin{pmatrix} -c & b \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\right). \tag{16}$$

We now proceed to define a 2-cocycle F for the extension $\mathbb{E}_2$ by first defining a function $t : \pi \to G_2$ that splits the surjection $r : G_2 \twoheadrightarrow \pi$. Recall that the standard projection $\Pi \to G_2 = \Pi/\mathrm{im}(\mu)$ is denoted λ_2. For any $w \in \Pi$, let us write $\lambda_2(w) = [w]$. Then, for any $x \in \pi$, we define $t(x)$ by

$$t(x) = [(0,0,x), \tau(\rho(x))]. \tag{17}$$

Now we define F by the usual formula:

$$j(F(x,y)) = t(x)t(y)t(xy)^{-1}, \tag{18}$$

where $j : H^1(\mathbb{Z}^2;\mathbb{Q}) \to G_2$ is the inclusion onto $\ker(r)$. Let us make j more explicit. Choose any $\phi \in H^1(\mathbb{Z}^2;\mathbb{Q}) = \mathrm{Hom}(\mathbb{Z}^2,\mathbb{Q})$. Then $j(\phi)$ is precisely the image under λ_2 of the following pair in $G \times_{\mathrm{GL}(2,\mathbb{Z})} \mathrm{Aut}(\mathcal{H}_{f0}) = \Pi$:

$$\left((0,0,\epsilon), \begin{pmatrix} \phi(e_1) & \phi(e_2) \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\right), \tag{19}$$

where, as before, e_1, e_2 are the standard generators of $\mathbb{Z}^2$. Now using Equation (15), which gives the multiplication in G, we can compute $t(x)t(y)$:

$$\begin{aligned} t(x)t(y) &= [(0,0,x)(0,0,y), \tau(\rho(x))\tau(\rho(y))] \\ &= [(f(x,y),xy), \tau(\rho(xy))] \\ &= \left[(f(x,y),\epsilon), \begin{pmatrix} 0 \\ I \end{pmatrix}\right][(0,0,xy), \tau(\rho(xy))]. \end{aligned}$$

Note that the second and third equalities follow from the definition of the multiplication in G, as given in Equation (15), as well as the fact that τ and ρ are homomorphisms! Now, using Equation (17), we get

$$t(x)t(y) = \left[(f(x,y),\epsilon), \begin{pmatrix} 0 \\ I \end{pmatrix}\right] t(xy),$$

which, when combined with (18), yields

$$j(F(x,y)) = \left[(f(x,y),\epsilon), \begin{pmatrix} 0 \\ I \end{pmatrix}\right].$$

Setting $f(x,y) = (f_1, f_2) = f_1 e_1 + f_2 e_2 \in \mathbb{Z}^2$ and applying Equations (16) and (19), this becomes

$$\begin{aligned} j(F(x,y)) &= \left[(0,0,\epsilon), \begin{pmatrix} f_2 & -f_1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \\ &= j(-\psi(f(x,y))). \end{aligned}$$

Since j is injective, $\psi(f(x,y)) = -F(x,y)$, or $\psi_\flat(f) = -F$. This completes our proof of Theorem 2.1.

4. The main theorem when $B = S^2$ or $\mathbb{R}P^2$

Let $\xi : T^2 \xrightarrow{i} E \xrightarrow{p} B$ be a symplectic torus bundle with B a closed genus zero surface. In this case, Theorem 1.1 reduces to Proposition 1.3, which we prove in this section by methods essentially unrelated to our earlier arguments.

First we deal with the case $B = S^2$.

Proof of Proposition 1.3(a)**.** As we explain in Appendix B, the classification of symplectic torus bundles over a simply-connected space is the same as the classification of principal torus bundles over that space. It is well-known that when $B = S^2$ these are classified by $\pi_1(T^2)$. Indeed, the homotopy class corresponding to ξ may be described as follows (cf. [10, page 98]). Consider the following portion of the exact homotopy sequence of ξ:

$$\pi_2(S^2) \xrightarrow{\partial} \pi_1(T^2) \to \pi_1(E) \to 0.$$

Then the required homotopy class is $\pm\partial(\iota) \in \pi_1(T^2)$, where ι is the class of the identity map of S^2. Since $\pi_1(E)$ is a homomorphic image of $\pi_1(T^2)$, it is abelian and thus equals $H_1(E)$. It follows that this last has rank one or two according as ξ is nontrivial or trivial, respectively. By Poincaré duality, which applies because E is closed and orientable, the same is true of the rank of $H^3(E)$.

We now turn to the following portion of the Wang sequence for ξ:

$$H^2(E;\mathbb{Q}) \xrightarrow{i^*} H^2(T^2;\mathbb{Q}) \xrightarrow{\theta} H^1(T^2;\mathbb{Q}) \to H^3(E;\mathbb{Q}) \to 0.$$

Clearly, i^* in this sequence is onto when $H^3(E)$ has rank two and 0 when $H^3(E)$ has rank one. Since the surjectivity of i^* with rational coefficients is equivalent to the existence of the desired form β, this concludes the proof of Proposition 1.3(a). □

We now deal with the case $B = \mathbb{R}P^2$. Let $\pi : S^2 \to \mathbb{R}P^2$ be the double cover, and let $\widetilde{E}$ be the total space of the pullback $\pi^*\xi$, a symplectic torus bundle over S^2. Then we have the following lemma:

Lemma 10. *The total space E of ξ admits a closed 2-form β satisfying* (1) *if and only if $\widetilde{E}$ does.*

Proof. ($\Rightarrow$): Let $\bar{\pi} : \widetilde{E} \to E$ be the bundle map over π given by the pullback construction. If β is a closed 2-form on E satisfying (1), then $\bar{\pi}^*(\beta)$ is a closed 2-form on $\widetilde{E}$ satisfying (1).

($\Leftarrow$): Let $b : \widetilde{E} \to \widetilde{E}$ be the nontrivial deck transformation. It is not hard to check, using the definition of the pullback construction, that b maps fibres of $\widetilde{E}$ to fibres so as to preserve the pullback symplectic structures. Now let γ be a closed 2-form on $\widetilde{E}$ satisfying (1), and define

$$\widetilde{\beta} = \frac{1}{2}(\gamma + b^*\gamma).$$

Since $\widetilde{\beta}$ is invariant under deck transformations it descends to a closed 2-form β on E. It clearly also satisfies (1), which implies the same for β. □

This lemma immediately implies the first statement of Proposition 1.3(b).

Corollary 11. *Suppose that the representation $\rho : \pi_1(\mathbb{R}P^2) \to \mathrm{GL}(2, \mathbb{Z})$ is trivial. Then E admits a closed 2-form β satisfying (1).*

Proof. If the module structure on $\mathbb{Z}^2$ is trivial, then $H^2(\mathbb{R}P^2; \mathbb{Z}^2_\rho) \approx (\mathbb{Z}_2)^2$. Clearly then the map $\pi^* : H^2(\mathbb{R}P^2; \mathbb{Z}^2_\rho) \to H^2(S^2; \mathbb{Z}^2) \approx \mathbb{Z}^2$ is trivial. By the classification theorem, it follows that the pullback $\pi^*(\xi)$ is trivial. But Proposition 1.3(a) then implies that the total space of this pullback admits the desired 2-form. Therefore, by the lemma, so does E. □

It remains to deal with the case $B = \mathbb{R}P^2$, ρ nontrivial. Since we are dealing with a symplectic torus bundle, ρ must take values in $\mathrm{SL}(2, \mathbb{Z})$, which easily implies that $\mathrm{im}(\rho) = \{\pm I\}$. We now consider the cohomology Serre spectral sequence of the covering $\pi : S^2 \to \mathbb{R}P^2$, which has

$$E_2^{p,q} = H^p(\mathbb{Z}_2; H^q(S^2; \mathbb{Z}^2_\rho))$$

and converges to $H^*(\mathbb{R}P^2; \mathbb{Z}^2_\rho)$. Here, the group $H^q(S^2; \mathbb{Z}^2_\rho)$ is the ordinary cohomology of S^2 with $\mathbb{Z}^2$ coefficients, but the action of $\mathbb{Z}_2$ is a joint action, simultaneous on (the chains of) S^2 (via the antipodal map) and on $\mathbb{Z}^2$ via ρ. It is easy to see that $H^0(S^2; \mathbb{Z}^2_\rho) \approx \mathbb{Z}^2_\rho$ as $\mathbb{Z}[\mathbb{Z}_2]$-modules, and $H^2(S^2; \mathbb{Z}^2_\rho) \approx \mathbb{Z}^2$, i.e., $\mathbb{Z}^2$ with the trivial $\mathbb{Z}_2$-action.

A direct computation (e.g., see [1, pages 58-9]) yields the following values for $E_2^{p,q}$:

$$E_2^{p,q} = \begin{cases} \mathbb{Z}^2 & \text{if } (p,q) = (0,2); \\ (\mathbb{Z}_2)^2 & \text{if } q = 0 \text{ and } p \text{ odd, or if } q = 2 \text{ and } p > 0 \text{ and even;} \\ 0, & \text{otherwise.} \end{cases}$$

It follows easily from this that we have an exact sequence

$$0 \to H^2(\mathbb{R}P^2; \mathbb{Z}^2_\rho) \xrightarrow{\pi^*} H^2(S^2; \mathbb{Z}^2_\rho) = \mathbb{Z}^2 \to (\mathbb{Z}_2)^2 \to 0.$$

Thus, $H^2(\mathbb{R}P^2; \mathbb{Z}^2_\rho) \approx \mathbb{Z}^2$, and π^* is injective. Therefore, in this case Theorem 1.1 reduces to the following, which is an elaboration of the second statement of Proposition 1.3(b):

Proposition 4.1. *The symplectic bundles $\xi : T^2 \to E \to \mathbb{R}P^2$ inducing a nontrivial $\mathbb{Z}_2$-module structure $\mathbb{Z}^2_\rho$ on $H_1(T^2) = \mathbb{Z}^2$ are classified by $H^2(\mathbb{R}P^2; \mathbb{Z}^2_\rho) \approx \mathbb{Z}^2$. For such a ξ, E admits a closed 2-form satisfying (1) if and only if $c(\xi) = 0$.*

Proof. The foregoing calculation implies the first statement of the proposition. The second follows from the injectivity of π^*, Lemma 10, and Proposition 1.3(a). $\square$

This concludes our proof of Theorem 1.1.

5. Proofs of the main corollaries

Proof of Corollary 1.2. For any connected surface B, $H^2(B;\mathbb{Z}^2_\rho)$ is a finitely-generated abelian group, hence, its torsion subgroup is finite. The result now follows from Proposition 1.1 and Theorem 1.1. $\square$

Proof of Corollary 1.3. The group of a principal torus bundle is T^2 acting on itself by translations. If σ denotes the standard symplectic form on T^2, then the translations clearly preserve σ, i.e., $T^2 \subseteq \operatorname{Symp}(T^2,\sigma)$, so the bundle has a canonical symplectic structure. The corresponding representation

$$\rho : \pi_1(B) \to \pi_0(\operatorname{Symp}(T^2,\sigma))$$

factors through $\pi_0(T^2) = 0$, so it is trivial. Hence, when B is a connected surface, the only cases in which the characteristic classes $c(\xi) \in H^2(B;\mathbb{Z}^2)$ do not have finite order are when B is closed and orientable and ξ is nontrivial. $\square$

Proof of Corollary 1.4. Let $i : T^2 \to E$ be the inclusion onto a fixed orbit, and let $p : E \to E/T^2$ be the usual projection. Then, it is a standard fact that $T^2 \to E \to E/T^2$ is a principal torus bundle, say ξ. By Corollary 1.3, ξ has a canonical structure as a symplectic torus bundle. Let σ be the standard symplectic form on T^2, and let σ_b be the corresponding symplectic forms on the fibres (equiv., orbits). By hypothesis, E admits a symplectic form with respect to which all the fibres are symplectic submanifolds. Thus, the restriction map $i^* : H^2_{\mathrm{DR}}(E) \to H^2_{\mathrm{DR}}(T^2)$ is surjective, and, by Thurston's result, there is a closed 2-form β on E satisfying condition (1), that is, $\beta|T^2_b = \sigma_b$, for all $b \in E/T^2$. Assuming that the closed, connected surface E/T^2 is orientable, we can then apply the preceding corollary to conclude that ξ is trivial, *as a symplectic torus bundle*. Thus, it admits a section. But the existence of a section is independent of the group of the bundle. Therefore, ξ has a section as a principal T^2 bundle, and, and therefore it is trivial as a principal T^2 bundle, which implies the stated result. It remains to verify that E/T^2 is orientable. But this follows from a standard fact about smooth fibre bundles that are orientable, that is, for which the fibres can be given orientations that are locally coherent over the base. For such a bundle — for example ξ — the orientability of the base is equivalent to the orientability of the total space. $\square$

APPENDICES

The main arguments of the paper make use of certain known classification results in order to pass from statements about smooth fibre bundles to statements about group extensions. The following three appendices briefly explain these results, starting with facts about torus bundles, then passing to the classification of $K(A,1)$-fibrations, and ending with a comparison between that classification and the classification of corresponding group extensions.

Appendix A. T^2-bundles and T^2-fibrations

Let $\mathcal{E}(T^2)$ (resp., $\mathcal{E}_+(T^2)$) denote the monoid of self homotopy equivalences (resp., orientation-preserving self homotopy equivalences) of T^2. These receive the compact-open topology. Let $\mathrm{Diff}_+(T^2)$ (resp., $\mathrm{Diff}_0(T^2)$) denote the subgroup of orientation-preserving diffeomorphisms of T^2 (resp., the identity component of $\mathrm{Diff}(T^2)$). Finally, let ω be any symplectic form on T^2, and let $\mathrm{Symp}(T^2, \omega)$ be the group of symplectomorphisms of (T^2, ω). These groups of diffeomorphisms are usually given the C^k topology, for $1 \leq k \leq \infty$. The choice of k does not make a difference for our discussion. Regarding T^2 as acting on itself by translation, we have $T^2 \subseteq \mathrm{Diff}_0(T^2)$.

Proposition A.1. *The following inclusions are homotopy equivalences:*
(a) $T^2 \to \mathrm{Diff}_0(T^2)$.
(b) $\mathrm{Diff}(T^2) \to \mathcal{E}(T^2)$.
(c) $\mathrm{Diff}_+(T^2) \to \mathcal{E}_+(T^2)$.
(d) $\mathrm{Symp}(T^2, \omega) \to \mathrm{Diff}_+(T^2)$.

Proof. (a),(b): These are well-known results, due originally to Earle and Eells (cf. Gramain [4]). (c) follows immediately from (b). (d): Given any orientation-preserving diffeomorphism h, $h^*(\omega)$ is homologous to ω, since h has degree one. Thus, Moser's method ([8, pages 93-97]) may be applied to the family of symplectic forms $\omega_t = (1-t)\omega + th^*\omega$, producing an isotopy ψ_t between the identity and a diffeomorphism ψ_1 that satisfies $\psi_1^* h^* \omega = \omega$. Therefore, $h_t = h\psi_t$ is an isotopy between h and a symplectomorphism h_1. The isotopy can be constructed so as to be continuous in h and remain in Symp if h is a symplectomorphism. It follows that the map given by $h \mapsto h_1$ is a homotopy inverse for the inclusion map. □

Since, as is well-known, $B\mathcal{E}_+(T^2)$ classifies oriented T^2-fibrations, statements (c) and (d) immediately give the following result:

Corollary A.2. *Let B be a smooth, connected manifold. Equivalence classes of symplectic torus bundles over B correspond bijectively to fibre-homotopy equivalence classes of oriented T^2-fibrations over B.*

Appendix B. K(A,1)-fibrations

Let A be an abelian group. Following C.A. Robinson [9], for any $n \geq 1$, we let $K(A, n)$ be a CW complex which is a topological abelian group on which $\mathrm{Aut}(A)$ acts by cellular automorphisms. Let Q be a CW complex of type $K(\mathrm{Aut}(A), 1)$ and $\widetilde{Q}$ its universal cover. Thus, there is a free, diagonal left-action of $\mathrm{Aut}(A)$ on the cartesian product $K(A, 2) \times \widetilde{Q}$, with respect to which the projection $K(A, 2) \times \widetilde{Q} \to \widetilde{Q}$ is equivariant. Therefore, it descends to a fibration $p : \hat{K}(A, 2) \to Q$ with fibre $K(A, 2)$.

Robinson shows that $\hat{K}(A, 2)$ classifies Hurewicz fibrations with fibres of the homotopy type of $K(A, 1)$ and base spaces of the homotopy type of a CW complex. Thus, over such a base space B, the fibre-homotopy equivalence classes of $K(A, 1)$-fibrations correspond bijectively to homotopy classes of maps $B \to \hat{K}(A, 2)$. Throughout this paper, we use the 'based' convention for equivalences, whereby each base space has a basepoint and each fibre has a fixed identification with a given space. See [9] and [2, 16.7].

Remark. By Proposition A.1, BDiff(T^2) classifies T^2-fibrations, which are the same as $K(\mathbb{Z}^2, 1)$-fibrations. So BDiff(T^2) is homotopy equivalent to $\hat{K}(\mathbb{Z}^2, 2)$, implying that it too fibres over $K(\mathrm{GL}(2, \mathbb{Z}^2), 1)$ with fibre $K(\mathbb{Z}^2, 2)$. This fact is well-known, but we mention it to connect the two constructions of classifying spaces. It gives one way of seeing why, for a simply-connected base space, there is a bijective correspondence between equivalence classes of torus bundles and equivalence classes of principal torus bundles. A similar comment applies to BSymp(T^2, ω), which fibres over $K(\mathrm{SL}(2, \mathbb{Z}^2), 1)$ with fibre $K(\mathbb{Z}^2, 2)$.

As usual, each $K(A, 1)$-fibration admits a representation $\rho : \pi_1(B) \to \mathrm{Aut}(A) = \pi_0(\mathcal{E}(K(A, 1)))$. We are interested in the finer classification that fixes such a ρ. Robinson derives this from his construction as follows. The fibration $p : \hat{K}(A, 2) \to Q$ admits a canonical section $s_0 : Q \to \hat{K}(A, 2)$ defined by the rule $s_0[q] = [\vartheta, q]$, where here [ ] refers to the Aut(A)-orbit and ϑ denotes the identity element of the abelian group $K(A, 2)$ Clearly, representations $\rho : \pi_1(B) \to \mathrm{Aut}(A)$ correspond to homotopy classes of maps $r : B \to Q$, whereas $K(A, 1)$-fibrations over B with associated representation ρ correspond to homotopy classes of maps $f : B \to \hat{K}(A, 2)$ for which pf induces ρ. In fact, as Robinson shows, if we fix ρ (and r inducing ρ), the foregoing set of homotopy classes may be described as the set of homotopy classes of lifts f of r to $\hat{K}(A, 2)$. Let f_0 denote the lift $s_0 r$.

Given two lifts f and g of r, classical obstruction theory produces a so-called primary obstruction class $d(f, g) \in H^2(B; \pi_2(K(A, 2))) = H^2(B; A_\rho)$ whose vanishing is a necessary condition for the existence of a homotopy of lifts between f and g. In this context, the condition is also sufficient. Moreover, given any g and any class $d \in H^2(B; A_\rho)$, there is a unique homotopy class of lifts f such that $d(f, g) = d$. We now set $d(f, f_0) = c(f)$, where f_0 is the lift described above. A standard additivity formula yields $d(f, g) = c(f) - c(g)$. If f classifies a $K(A, 1)$-fibration η, we may write $c(f) = c(\eta)$. This is the so-called characteristic class of η that we have been using. It follows that the rule $\eta \mapsto c(\eta)$ gives a bijection between equivalence classes of fibrations and $H^2(B, A_\rho)$, as stated earlier.

There remains one further observation about the classes $c(\eta)$ which we have used in an important way. Start by considering a homotopy h_t between the lifts f and f_0 of r as above. Then, for any $b \in B$, $h_t(b)$ is a path from $f(b)$ to $f_0(b)$ lying completely in a fibre of $p : \hat{K}(A, 2) \to Q$. This motivates the following construction given by Robinson: let P denote the space of all paths γ in $\hat{K}(A, 2)$ that begin in $s_0(Q)$ and lie completely in a fibre of p. The rule $\gamma \mapsto \gamma(1)$ defines a fibration $P \to \hat{K}(A, 2)$ with fibre of type $K(A, 1)$. Clearly the partial homotopies between any lift f of r and the lift f_0 correspond to partial lifts of f to P. It follows easily that we can interpret $c(f)$ as the primary obstruction to lifting f to P.

Now, Robinson shows that P is, in fact, a *universal* $K(A, 1)$-fibration over $\hat{K}(A, 2)$, so that $f^*(P)$ is equivalent to η. This implies that $c(\eta)$ may be interpreted directly as the primary obstruction to a section of η, which is the interpretation we have used.

Appendix C. Extensions by an abelian group

Let $\mathbb{S} : A \rightarrowtail G \twoheadrightarrow \pi$ be an extension of a group π by the abelian group A. There is a corresponding $K(A, 1)$-fibration, written $\eta : K(A, 1) \xrightarrow{i} K(G, 1) \xrightarrow{p} K(\pi, 1)$. Of course, the homotopy exact sequence of η collapses to $\mathbb{S}$. We use i_* and p_* to denote the corresponding homomorphisms in $\mathbb{S}$. The representation ρ corresponding to η is the same as that induced by inner automorphisms of G in $\mathbb{S}$. Let us hold this fixed.

Let $f : \pi\times\pi \to A$ be the normalized 2-cocycle of $\mathbb{S}$. In terms of the bar resolution of π, we may write f as the (possibly infinite) formal sum $\Sigma f(x, y)[x|y]$, where x, y range over π. In this appendix we show how this sum can be recognized as the primary obstruction to sectioning η. This establishes the identification $c(\mathbb{S}) = c(\eta)$, which we have been using throughout the paper. This fact is certainly part of the classical folklore of the subject, but I have been unable to find an explicit reference.

The description of the primary obstruction can be conveniently simplified in this case by using the following observation, which follows almost immediately from definitions.

Lemma 12. *Let $\zeta : F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration, with F connected and B a connected* CW *complex, and assume that $i_* : \pi_{m-1}(F) \to \pi_{m-1}(E)$ is injective. Let $\sigma : B^{m-1} \to E$ be a section of ζ over the $(m-1)$-skeleton of B, and let $o(\sigma)$ denote the obstruction cocycle to extending σ over the m-skeleton. Finally, suppose that if $\chi : D^m \to B$ is the characteristic map of an m-cell e of B, then $\sigma\circ\chi|\partial D^m : \partial D^m \to E$ represents $i_*(\alpha) \in \pi_{m-1}(E)$. Then*

$$o(\sigma)(e) = \alpha.$$

The best framework for recognizing $\Sigma f(x, y)[x|y]$ as the desired obstruction cocycle is that of semisimplicial topology, as in ([7, Chapters 1–3]). Thus, for example, we can describe $K(\pi, 1)$ semisimplicially as consisting of one 0-simplex, denoted $[\ \ ]$, and a k-simplex for each integer $k \geq 1$ and each symbol $[x_1|\dots|x_k]$, where $x_1, \dots, x_k$ range over π, with the well-known face and degeneracy maps. Similarly for $K(G, 1)$. The surjection $p_* : G \twoheadrightarrow \pi$ shows how to map $K(G, 1)$ onto $K(\pi, 1)$. This map is a minimal Kan fibration, say κ [7, page 64]. We shall define an obstruction to sectioning κ.

Let $s : \pi \to G$ be a function that is a right inverse of p_* and is related to the 2-cocycle $f : \pi \times \pi \to\ G$ by Equation (14). Use s to define a (semisimplicial) section σ of κ over the 1-skeleton of $K(\pi, 1)$: this is determined by $\sigma[\ \ ] = [\ \ ]$ and $\sigma[x] = [s(x)]$. Note that each 1-simplex $[s(x)]$ determines a directed loop in $K(G, 1)$, say $\langle s(x)\rangle$; these may be concatenated. Now consider any 2-simplex $[x|y]$ of $K(\pi, 1)$. Its boundary consists of the 1-simplexes $\partial_0[x|y] = [y]$, $\partial_1[x|y] = [xy]$, and $\partial_2[x|y] = [x]$, with corresponding loops concatenated as $\langle x\rangle\langle y\rangle\langle xy\rangle^{-1}$. Therefore, the loop obtained by applying σ to the boundary is $\langle s(x)\rangle\langle s(y)\rangle\langle s(xy)\rangle^{-1}$. Using the semisimplicial homotopy relation in $K(G, 1)$ and the definition of $f(x, y)$, this loop is easily shown to be homotopic to $\langle i_*(f(x, y))\rangle$ in $K(G, 1)$. Thus, it represents $i_*(f(x, y))$. It now follows from the semisimplicial analog of the above lemma that the obstruction to extending σ over the 2-skeleton is precisely the cocycle $\Sigma f(x, y)[x|y]$, as desired.

The foregoing can be translated to the more conventional topological obstruction theory by applying the geometric realization functor. This transforms κ into a topological fibration equivalent to η and σ into a partial section producing the same obstruction. Thus $c(\eta) = c(\mathbb{S})$.

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