New York Journal of Mathematics

New York J. Math. **11** (2005) 171–203.

On the asymptotics of certain Wiener–Hopf-plus-Hankel determinants

Estelle L. Basor and Torsten Ehrhardt

Abstract. In this paper we determine the asymptotics of the determinants of truncated Wiener–Hopf plus Hankel operators $\det(W_R(a) \pm H_R(a))$ as $R \to \infty$ for symbols $a(x) = (x^2/(1+x^2))^\beta$ with the parameter β being of small size.

Contents

1. Introduction

For a function a defined on the real line $\mathbb{R}$ such that $a-1 \in L^1(\mathbb{R})$ the truncated Wiener–Hopf and Hankel operators acting on $L^2[0,R]$ with symbol a are defined by

$$W_R(a) \quad : \quad f(x) \mapsto g(x) = f(x) + \int_0^R k(x-y)f(y)\,dy, \tag{1}$$

$$H_R(a) \quad : \quad f(x) \mapsto g(x) = \int_0^R k(x+y)f(y)\,dy, \tag{2}$$

Received February 14, 2005.
Mathematics Subject Classification. 47B35.
Key words and phrases. Wiener–Hopf operator, Hankel operator, determinant, asymptotics.
The first author was supported in part by NSF Grant DMS-0200167.

ISSN 1076-9803/05

where k is the Fourier transform of $a-1$,

$$k(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}(a(\xi)-1)e^{-i\xi x}\,d\xi. \tag{3}$$

It is well-known that under the above assumption the operators $W_R(a)-I$ and $H_R(a)$ are trace class operators. Hence the determinants

$$\det(W_R(a)\pm H_R(a))$$

are well-defined.

The purpose of this paper is to determine the asymptotics of these determinants as $R\to\infty$ for a particular class of even generating functions which have a single singularity at $x=0$. Before explaining the scope of this paper in more detail, let us briefly review related problems.

The asymptotics of Wiener–Hopf determinants $\det W_R(a)$ as $R\to\infty$ for sufficiently smooth nonvanishing functions a with winding number zero are described by the Akhiezer–Kac formula (see, e.g., [14] and the references therein). A more complicated situation occurs when the symbol a possesses singularities such as jumps, zeros, or poles. Let $\hat{u}_\beta$ and $\hat{v}_\beta$ be the functions

$$\hat{v}_\beta(x):=\left(\frac{x^2}{x^2+1}\right)^{\beta},\qquad \hat{u}_\beta(x):=\left(\frac{x-0i}{x-i}\right)^{-\beta}\left(\frac{x+0i}{x+i}\right)^{\beta}. \tag{4}$$

Notice that $\hat{v}_\beta$ has a zero or a pole at $x=0$, (except in the case where the real part of β is zero and the singularity is of an oscillating type) while $\hat{u}_\beta$ has a jump discontinuity at $x=0$ whose size is determined by the parameter β. If the symbol is of the Fisher–Hartwig form,

$$\hat{a}(x)=b(x)\prod_{r=1}^{R}\hat{v}_{\alpha_r}(x-x_r)\hat{u}_{\beta_r}(x-x_r), \tag{5}$$

where $|\mathrm{Re}\,\alpha_r|<1/2$, $|\mathrm{Re}\,\beta_r|<1/2$, $x_1,\dots,x_R\in\mathbb{R}$ are distinct, and b is a sufficiently smooth function satisfying the assumptions of the Akhiezer–Kac formula, then the asymptotics of the determinants are described by the continuous analogue of the Fisher–Hartwig conjecture. One minor complication is encountered since, except in special cases, the symbol $\hat{a}$ does not belong to $L^1(\mathbb{R})$, but only to $L^2(\mathbb{R})$. Because then the above operators are only Hilbert–Schmidt one has to consider regularized determinants $\det_2(I+K)=\det(I+K)e^{-K}$. The conjectured asymptotic formula for such Wiener–Hopf determinants reads

$$\det{}_2 W_R(\hat{a})\sim G_2[\hat{a}]^R\,R^{\Omega}E,\qquad R\to\infty, \tag{6}$$

where $\Omega=\sum_{r=1}^{R}(\alpha_R^2-\beta_R^2)$, $G_2[\hat{a}]$ is a regularized version of the geometric means of $\hat{a}$, and E is a complicated constant.

Formula (6) has not yet been proved in general (see [15] for the proof in the special case where $\alpha_r=0$ for all r), but it is very likely that such a proof can be accomplished with the help of two main ingredients. One of these is a localization theorem for Wiener–Hopf determinants, which had to be analogous to a corresponding (well-known) localization theorem for Toeplitz determinants with Fisher–Hartwig symbols [14]. (The outline of a possible proof of such a theorem was communicated to us by A. Böttcher [11], but the details still need to be verified.)

The localization reduces the problem to symbols that are "pure" Fisher–Hartwig symbols, that is where $R = 1$ and $b(x) \equiv 1$. This last problem was outstanding for a long time and was recently solved by one of the authors and Widom [8]. They made use of the so-called Borodin–Okounkov formula [9] (see also [7, 10]) to compare the asymptotics of $\det_2 W_R(\hat{a})$ with the (known) asymptotics of a Toeplitz determinant $\det T_n(a)$ where $R \sim 2n$ and $n, R \to \infty$. The Borodin–Okounkov identity is an exact identity for both the Toeplitz and Wiener–Hopf determinants and made the comparisons possible.

We will do something very similar in this paper, in the sense that we will also make a comparison to already known asymptotics. These will involve the discrete analogue of the sum of the finite Wiener–Hopf and Hankel operators. The discrete analogues are the Toeplitz and Hankel matrices,

$$T_n(a) = (a_{j-k})_{j,k=0}^{n-1}, \qquad H_n(a) = (a_{j+k+1})_{j,k=0}^{n-1}. \tag{7}$$

Here $a \in L^1(\mathbb{T})$ is a function defined on the unit circle $\mathbb{T} = \{t \in \mathbb{C} \,:\, |t| = 1\}$ with Fourier coefficients

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} a(e^{i\theta})e^{-ik\theta}\,d\theta, \qquad k \in \mathbb{Z}.$$

The asymptotics of Toeplitz determinants have a long and interesting history. For the latest results and more information we refer to [17].

The study of the asymptotics of Toeplitz-plus-Hankel determinants $\det(T_n(a) \pm H_n(a))$ was begun recently. The main interest is in even symbols (i.e., $a(t) = a(t^{-1})$, $t \in \mathbb{T}$). In this case the results for Fisher–Hartwig type symbols are nearly complete [4]. Some results were obtained also for noneven symbols [3].

Let us now return to the topic of this paper, namely the asymptotics of Wiener–Hopf-plus-Hankel determinants,

$$\det(W_R(\hat{a}) \pm H_R(\hat{a})).$$

First of all, the case of smooth, nonvanishing and even functions (i.e, $\hat{a}(x) = \hat{a}(-x)$, $x \in \mathbb{R}$) follows from (more general) results in [5]. In regard to Fisher–Hartwig type symbols only the case of a function $\hat{a}(x) = \hat{u}_\beta(x-1)\hat{u}_{-\beta}(x+1)$ (which is a even piecewise constant function with two jump discontinuities) was treated recently in [6].

In this paper we consider the case of a function $\hat{a}(x) = \hat{v}_\beta(x)$, which is an even function having a zero, a pole, or a singularity of oscillating type at $x = 0$. In order to state the main result we introduce

$$\hat{D}_R^+(\beta) := \det\left[W_R(\hat{v}_\beta) + H_R(\hat{v}_\beta)\right], \qquad \hat{D}_R^-(\beta) := \det\left[W_R(\hat{v}_\beta) - H_R(\hat{v}_\beta)\right]. \tag{8}$$

The natural assumption on β is that $\operatorname{Re}\beta > -1/2$ since then $\hat{v}_\beta \in L^1(\mathbb{R})$. Moreover, because $W_R(\hat{v}_\beta)$ and $H_R(\hat{v}_\beta)$ are analytic operator valued functions with respect to β, the functions $\hat{D}_R^\pm(\beta)$ are analytic on the set of all $\beta \in \mathbb{C}$ for which $\operatorname{Re}\beta > -1/2$.

Theorem 1.1.

(a) *If* $-1/2 < \operatorname{Re}\beta < 3/2$, *then*

$$\hat{D}_R^+(\beta) \sim e^{-\beta R} R^{\beta^2/2-\beta/2}(2\pi)^{\beta/2} 2^{-\beta^2+\beta/2}\frac{G(1/2)}{G(1/2+\beta)}, \quad R \to \infty. \tag{9}$$

(b) *The function $\hat{D}_R^-(\beta)$ admits an analytic continuation onto the set of all $\beta \in \mathbb{C}$ for which $\operatorname{Re}\beta > -3/2$. Moreover, if $-1 < \operatorname{Re}\beta < 1/2$, then*

$$\hat{D}_R^-(\beta) \sim e^{-\beta R} R^{\beta^2/2+\beta/2} (2\pi)^{\beta/2} 2^{-\beta^2-\beta/2} \frac{G(3/2)}{G(3/2+\beta)}, \quad R \to \infty. \tag{10}$$

Therein $G(z)$ is the Barnes G-function [1], which is an entire function defined by

$$G(1+z) = (2\pi)^{z/2} e^{-(z+1)z/2-\gamma_E z^2/2} \prod_{k=1}^{\infty} \Big((1+z/k)^k e^{-z+z^2/(2k)} \Big) \tag{11}$$

with γ_E equal to Euler's constant. Notice that the Barnes function has the remarkable property that $G(1+z) = \Gamma(z)G(z)$.

The proof of the above theorem will be given in Section 3.6. However, the first statement in part (b) concerning the analytic continuablity will already follow from Proposition 3.14 in Section 3.4.

The assumption $-1 < \operatorname{Re}\beta < 1/2$ rather than $-3/2 < \operatorname{Re}\beta < 1/2$ in part (b) seems to be too restrictive. Unfortunately, we have not been able to remove it.

It is interesting to observe that Theorem 1.1 implies the asymptotics for Wiener–Hopf determinants

$$\det W_{2R}(\hat{v}_\beta) \sim e^{-2\beta R} R^{\beta^2} \frac{G(1+\beta)^2}{G(1+2\beta)}, \qquad R \to \infty, \tag{12}$$

which was proved in [8]. To see this one has to use the formula

$$\det W_{2R}(\hat{v}_\beta) = \det \big[W_R(\hat{v}_\beta) + H_R(\hat{v}_\beta)\big] \cdot \det \big[W_R(\hat{v}_\beta) - H_R(\hat{v}_\beta)\big], \tag{13}$$

which can be easily proved by observing that $W_{2R}(\hat{v}_\beta)$ can be identified with the block operator

$$\begin{pmatrix} W_R(\hat{v}_\beta) & H_R(\hat{v}_\beta) \\ H_R(\hat{v}_\beta) & W_R(\hat{v}_\beta) \end{pmatrix}.$$

One has also to make use of a consequence of the duplication formula for the Barnes G-function, which will be stated below in (20).

An outline of the paper is as follows. In the next section, we review the asymptotics in the discrete case. The final section (Section 3) is divided into several subsections and contains the proof of Theorem 1.1. We first (Sections 3.1 and 3.2) review the basic operator theory facts that are needed and proceed then (Sections 3.3 and 3.4) with identifying the determinants $\hat{D}_n^{\pm}(\beta)$ and their discrete analogues with different kinds of determinants. In Section 3.5 we prove some theorems concerning the strong and trace class convergence of certain operators that naturally occur in our proof. Lemma 3.15 is really the basic formula that allows us to compare the desired determinants with those of the discrete analogues. Finally, in Section 3.6, we will complete the proof of the main results, and, for completeness sake, we will compute some Fredholm determinants that occur and may be interesting in other settings.

We end this introduction by pointing out that the asymptotics of the determinants of Wiener–Hopf-plus-Hankel operators with the type of discontinuity considered have important applications. The computation of such asymptotics is a crucial step in the work of the second author's recent proof of the complete asymptotics of the Fredholm determinant of the sine kernel [18] (see also [16, 23]). It is also true that such operators occur in the Laguerre random matrix ensemble in a very natural way when one considers special parameters ($\nu = \pm 1/2$ for Bessel operators; see [5]). Computing the asymptotics for singular symbols yields information about certain discontinuous random variables.

2. The asymptotics in the discrete case

In what follows we are going to recall the results about the asymptotics of the determinants

$$D_n^+(\beta) := \det\big[T_n(v_\beta) + H_n(v_\beta)\big], \qquad D_n^-(\beta) := \det\big[T_n(v_\beta) - H_n(v_\beta)\big], \tag{14}$$

as $n \to \infty$, which were established in [3]. Therein v_β is the function

$$v_\beta(e^{i\theta}) := (2 - 2\cos\theta)^\beta, \qquad \operatorname{Re}\beta > -1/2. \tag{15}$$

Let us also introduce the function

$$u_\beta(e^{i\theta}) := e^{i\beta(\theta-\pi)}, \qquad 0 < \theta < 2\pi. \tag{16}$$

It is easily seen that $D_n^\pm(\beta)$ are analytic in β for $\operatorname{Re}\beta > -1/2$. In the following theorem we will provide some information about the analytic continuability of $D_n^\pm(\beta)$ with respect to β and about the asymptotics of $D_n^\pm(\beta)$ as $n \to \infty$ for fixed β.

Theorem 2.1.

(a) *For each $n \geq 1$ the function $D_n^+(\beta)$ is analytic in β on*

$$U_+ := \mathbb{C} \setminus \{-1/2, -3/2, -5/2, \dots\}.$$

Moreover, for $\beta \in U_+$,

$$D_n^+(\beta) \sim n^{\beta^2/2-\beta/2}\,(2\pi)^{\beta/2}\,2^{-\beta^2/2}\,\frac{G(1/2)}{G(1/2+\beta)}, \qquad n \to \infty. \tag{17}$$

(b) *For each $n \geq 1$ the function $D_n^-(\beta)$ is analytic in β on*

$$U_- := \mathbb{C} \setminus \{-3/2, -5/2, -7/2, \dots\}.$$

Moreover, for $\beta \in U_-$,

$$D_n^-(\beta) \sim n^{\beta^2/2+\beta/2}(2\pi)^{\beta/2}\,2^{-\beta^2/2}\,\frac{G(3/2)}{G(3/2+\beta)}, \qquad n \to \infty. \tag{18}$$

Proof. From the proof of Thm. 7.7 in [3], it follows that

$$\frac{\det\big[T_n(v_\beta) \pm H_n(v_\beta)\big]}{\det\big[T_n(u_{-\beta}) \mp H_n(u_{-\beta})\big]} = \prod_{k=0}^{n-1} \frac{\Gamma(1+2\beta+k)\Gamma(1-\beta+k)}{k!\,\Gamma(1+\beta+k)}$$

$$= \frac{G(1+2\beta+n)G(1-\beta+n)}{G(1+n)G(1+\beta+n)}\,\frac{G(1+\beta)}{G(1+2\beta)G(1-\beta)}.$$

(Notice the different meaning of the notation u_β used there.) Furthermore, the proof of Thms. 6.2 and 6.3 in [3] implies that $\det\big[T_n(u_\alpha) + H_n(u_\alpha)\big]$ is equal to

$$(2\pi)^{\alpha/2}\,2^{\alpha^2/2+1}\,\frac{G(1/2-\alpha)G(1+\alpha)G(1-\alpha)}{G(1/2)}\,\frac{G(2n)G(2n-2\alpha)}{G(2n+1-\alpha)G(2n-1-\alpha)}$$

$$\times \frac{G(n+3/2-\alpha)G(n+1)G(n+1-\alpha)G(n-1/2-\alpha/2)G(n-\alpha/2)^2G(n+1/2-\alpha/2)}{G(n+1/2-\alpha)^2G(n+1/2)G(n)G(n+1-2\alpha)G(n-\alpha)G(n+\alpha+1)}$$

and $\det\big[T_n(u_\alpha) - H_n(u_\alpha)\big]$ is equal to

$$(2\pi)^{\alpha/2}\,2^{\alpha^2/2+1}\,\frac{G(3/2-\alpha)G(1+\alpha)G(1-\alpha)}{G(3/2)}\,\frac{G(2n)G(2n-2\alpha)}{G(2n+1-\alpha)G(2n-1-\alpha)}$$

$$\times\,\frac{G(n+3/2)G(n+1)G(n+1-\alpha)G(n-1/2-\alpha/2)G(n-\alpha/2)^2G(n+1/2-\alpha/2)}{G(n+1/2-\alpha)G(n+1/2)^2G(n)G(n+1-2\alpha)G(n-\alpha)G(1+\alpha+n)}.$$

Combining these results we obtain

$$D_n^+(\beta) = (2\pi)^{-\beta/2}\,2^{\beta^2/2+1}\,\frac{G(3/2+\beta)G(1+\beta)^2}{G(3/2)G(1+2\beta)}\,\frac{G(2n)G(2n+2\beta)}{G(2n+1+\beta)G(2n-1+\beta)}$$
$$\times\,\frac{G(n+3/2)G(n-1/2+\beta/2)G(n+\beta/2)^2G(n+1/2+\beta/2)}{G(n)G(n+1/2)^2G(n+1/2+\beta)G(n+\beta)}$$

and

$$D_n^-(\beta) = (2\pi)^{-\beta/2}\,2^{\beta^2/2+1}\,\frac{G(1/2+\beta)G(1+\beta)^2}{G(1/2)G(1+2\beta)}\,\frac{G(2n)G(2n+2\beta)}{G(2n+1+\beta)G(2n-1+\beta)}$$
$$\times\,\frac{G(n+3/2+\beta)G(n-1/2+\beta/2)G(n+\beta/2)^2G(n+1/2+\beta/2)}{G(n)G(n+1/2)G(n+1/2+\beta)^2G(n+\beta)}.$$

Using the duplication formula for the G-function [1, p. 291],

$$G(z)G(z+1/2)^2G(z+1) = G(1/2)^2\pi^z 2^{-2z^2+3z-1}G(2z), \tag{19}$$

it now follows that

$$\frac{G(1/2+\beta)G(1+\beta)^2G(3/2+\beta)}{G(1+2\beta)} = (2\pi)^\beta 2^{-2\beta^2}G(1/2)G(3/2) \tag{20}$$

and

$$\frac{G(2n)G(2n+2\beta)}{G(2n+1+\beta)G(2n-1+\beta)} = 2^{\beta^2-1}\,\times$$
$$\frac{G(n)G(n+1/2)^2G(n+1)G(n+\beta)G(n+1/2+\beta)^2G(n+1+\beta)}{G(n-1/2+\beta/2)G(n+\beta/2)^2G(n+1/2+\beta/2)^2G(n+1+\beta/2)^2G(n+3/2+\beta/2)}.$$

Hence

$$D_n^+(\beta) = (2\pi)^{\beta/2}\,2^{-\beta^2/2}\,\frac{G(1/2)}{G(1/2+\beta)}$$
$$\times\,\frac{G(n+3/2)G(n+1)G(n+1+\beta)G(n+1/2+\beta)}{G(n+1/2+\beta/2)G(n+1+\beta/2)^2G(n+3/2+\beta/2)}$$

and

$$D_n^-(\beta) = (2\pi)^{\beta/2}\,2^{-\beta^2/2}\,\frac{G(3/2)}{G(3/2+\beta)}$$
$$\times\,\frac{G(n+1/2)G(n+1)G(n+1+\beta)G(n+3/2+\beta)}{G(n+1/2+\beta/2)G(n+1+\beta/2)^2G(n+3/2+\beta/2)}.$$

From these identities it can be concluded that $D_n^{\pm}(\beta)$ can be continued analytically to all of $U_{\pm}$. Notice that the zeros in the denominator cancel with the zeros of the term $G(n+1+\beta)$ in the numerator.

In order to obtain the asymtotic result, we apply the formula

$$\prod_{r=1}^{R} \frac{G(1+x_r+n)}{G(1+y_r+n)} \sim n^{\omega/2}, \qquad n \to \infty, \tag{21}$$

which holds under the assumption $x_1 + \cdots + x_R = y_1 + \cdots + y_R$ and with the constant $\omega = x_1^2 + \cdots + x_R^2 - y_1^2 - \cdots - y_R^2$. This asymptotic formula was proved, e.g., in Lemma 6.1 of [3]. □

Once again it is interesting to remark that from the asymptotics established in the previous theorem and from the identity

$$\det T_{2n}(v_\beta) = D_n^{+}(\beta) D_n^{-}(\beta) \tag{22}$$

the well-known asymptotics

$$\det T_{2n}(v_\beta) \sim (2n)^{\beta^2} \frac{G(1+\beta)^2}{G(1+2\beta)} \tag{23}$$

follow by using the consequence (20) of the duplication formula for the Barnes function. As in the case of (13) formula (22) can be proved by identifying the symmetric matrix $T_{2n}(v_\beta)$ with a two-by-two block matrix having the entries $T_n(v_\beta)$ and $H_n(v_\beta)$.

3. Proof of the main results

3.1. Preliminary facts. An operator A acting on a Hilbert space H is called a trace class operator if it is compact and if the series consisting of the singular values $s_n(A)$ (i.e., the eigenvalues of $(A^*A)^{1/2}$ taking multiplicities into account) converges. The norm

$$\|A\|_1 = \sum_{n \geq 1} s_n(A) \tag{24}$$

makes the set of all trace class operators into a Banach space, which forms also a two-sided ideal in the algebra of all bounded linear operators on H. Moreover, the estimates $\|AB\|_1 \leq \|A\|_1 \|B\|$ and $\|BA\|_1 \leq \|A\|_1 \|B\|$ hold, where A is a trace class operator and B is a bounded operator with the operator norm $\|B\|$.

A sequence of bounded linear operators A_n on a Hilbert space H is said to converge strongly on H to an operator A if $A_n x \to Ax$ for all $x \in H$.

A useful property is that if B is a trace class operator, if $A_n \to A$ and $C_n^* \to C^*$ strongly on H, then $A_n B C_n \to ABC$ in the trace norm. Therein C^* stands for the Hilbert space adjoint of the operator C.

If A is a trace class operator, then the operator trace "trace(A)" and the operator determinant "det($I+A$)" are well-defined. For more information concerning these concepts we refer to [20].

A sequence of bounded linear operators A_n defined on a Hilbert space H is called stable if the operators A_n are invertible for all sufficiently large n and if

$$\sup_{n \geq n_0} \|A_n^{-1}\| < \infty.$$

Lemma 3.1. *Let A_n be a sequence of bounded linear operators on a Hilbert space such that $A_n \to A$ strongly, and assume that A is invertible. Then $A_n^{-1} \to A^{-1}$ strongly if and only if the sequence A_n is stable.*

Proof. The "if" part follows easily from the estimate

$$\|A_n^{-1}x - A^{-1}x\| \le \|A_n^{-1}\| \cdot \|(A - A_n)A^{-1}x\|.$$

The "only if" part follows from the Banach–Steinhaus Theorem. □

In what follows we consider some concrete classes of bounded linear operators.

For a function $a \in L^\infty(\mathbb{T})$ with Fourier coefficients $\{a_n\}_{n=-\infty}^{\infty}$, the Toeplitz and Hankel operators are bounded linear operators acting on $\ell^2 = \ell^2(\mathbb{Z}_+)$ defined by the infinite matrices

$$T(a) = (a_{j-k})_{j,k=0}^{\infty}, \qquad H(a) = (a_{j+k+1})_{j,k=0}^{\infty}. \tag{25}$$

The connection to $n \times n$ Toeplitz and Hankel matrices is given by

$$P_n T(a) P_n \cong T_n(a), \qquad P_n H(a) P_n \cong H_n(a), \tag{26}$$

where P_n is the finite rank projection operator defined on ℓ^2 by

$$P_n : (x_0, x_1, \dots) \in \ell^2 \mapsto (x_0, \dots, x_{n-1}, 0, \dots) \in \ell^2. \tag{27}$$

Toeplitz and Hankel operators satisfy the following well-known formulas:

$$T(ab) = T(a)T(b) + H(a)H(\widetilde{b}), \tag{28}$$

$$H(ab) = T(a)H(b) + H(a)T(\widetilde{b}), \tag{29}$$

where $\widetilde{b}(t) := b(t^{-1})$, $t \in \mathbb{T}$.

For a functions $a \in L^\infty(\mathbb{R})$ the Wiener–Hopf operator and the Hankel operator acting on $L^2(\mathbb{R}_+)$ are defined by

$$W(a) = P_+ \mathcal{F} M(a) \mathcal{F}^{-1} P_+|_{L^2(\mathbb{R}_+)}, \tag{30}$$

$$H(a) = P_+ \mathcal{F} M(a) \mathcal{F}^{-1} \hat{J} P_+|_{L^2(\mathbb{R}_+)}, \tag{31}$$

where $\mathcal{F}$ stands for the Fourier transform acting on $L^2(\mathbb{R})$, $M(a)$ stands for the multiplication operator on $L^2(\mathbb{R})$, $P_+ = M(\chi_{\mathbb{R}_+})$, and $(\hat{J}f)(x) = f(-x)$. If $a \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then $W(a)$ and $H(a)$ are integal operators on $L^2(\mathbb{R})$ with the kernels $k(x-y)$ and $k(x+y)$, respectively, where $k(x)$ is the Fourier transform (3) of a. We remark that

$$W(ab) = W(a)W(b) + H(a)H(\widetilde{b}), \tag{32}$$

$$H(ab) = W(a)H(b) + H(a)W(\widetilde{b}), \tag{33}$$

where $\widetilde{b}(x) := b(-x)$, $x \in \mathbb{R}$. Moreover,

$$W_R(a) = P_R W(a) P_R|_{L^2[0,R]}, \qquad H_R(a) = P_R H(a) P_R|_{L^2[0,R]}, \tag{34}$$

where $P_R = M(\chi_{[0,R]})$.

It is important to note that Wiener–Hopf and Hankel operators are related to their discrete analogues by a unitary transform $S : \ell^2 \to L^2(\mathbb{R}_+)$,

$$T(a) = S^* W(\hat{a}) S, \qquad H(a) = S^* H(\hat{a}) S, \tag{35}$$

where the symbols are related by

$$\hat{a}(x) = a\left(\frac{1+ix}{1-ix}\right). \tag{36}$$

(The use of the same notation for the continuous and the discrete Hankel operators should not cause confusion.) For sake of further reference, let us introduce the mapping

$$\Phi : A \in \mathcal{L}(L^2(\mathbb{R}_+)) \mapsto S^*AS \in \mathcal{L}(\ell^2). \tag{37}$$

The unitary transform S is given explicitly by the composition $S = \mathcal{F}U\mathcal{F}_{\mathrm{d}}^{-1}$, where

$$\ell^2 \xrightarrow{\mathcal{F}_{\mathrm{d}}^{-1}} H^2(\mathbb{T}) \xrightarrow{U} H^2(\mathbb{R}) \xrightarrow{\mathcal{F}} L^2(\mathbb{R}_+).$$

Therein $H^2(\mathbb{T})$ and $H^2(\mathbb{R})$ are the Hardy spaces with respect to $\mathbb{T}$ and $\mathbb{R}$,

$$H^2(\mathbb{T}) = \Big\{ f \in L^2(\mathbb{T}) \ :\ f_k = 0 \text{ for all } k < 0 \Big\},$$
$$H^2(\mathbb{R}) = \Big\{ f \in L^2(\mathbb{R}) \ :\ (\mathcal{F}f)(x) = 0 \text{ for all } x < 0 \Big\},$$

$\mathcal{F}_{\mathrm{d}}^{-1} : \{x_n\}_{n=0}^{\infty} \mapsto f(t) = \sum_{n=0}^{\infty} x_n t^n$ is the inverse discrete Fourier transform, and

$$(Uf)(x) = \frac{1}{\sqrt{\pi}(1-ix)} f\left(\frac{1+ix}{1-ix}\right).$$

Under the unitary transform $\mathcal{F}_{\mathrm{d}} : H^2(\mathbb{T}) \to \ell^2$, the Toeplitz and Hankel operators can be identified with operators acting on $H^2(\mathbb{T})$,

$$T(a) \cong PM(a)P|_{H^2(\mathbb{T})}, \qquad H(a) \cong PM(a)JP|_{H^2(\mathbb{T})}, \tag{38}$$

where P is the Riesz projection on $L^2(\mathbb{T})$, $M(a)$ is the multiplication operator on $L^2(\mathbb{T})$, and $(Jf)(t) = t^{-1}f(t)$, $t \in \mathbb{T}$.

A sequence of functions $a_n \in L^\infty(\mathbb{T})$ is said to converge to $a \in L^\infty(\mathbb{T})$ in measure if for each $\varepsilon > 0$ the Lebesgue measure of the set

$$\Big\{ t \in \mathbb{T} \ :\ |a_n(t) - a(t)| \geq \varepsilon \Big\}$$

converges to zero.

Lemma 3.2. *Assume that $a_n \in L^\infty(\mathbb{T})$ are uniformly bounded and converge to $a \in L^\infty(\mathbb{T})$ in measure. Then*

$$T(a_n) \to T(a) \quad \text{ and } \quad H(a_n) \to H(a)$$

strongly on ℓ^2, and the same holds for the adjoints.

Proof. We use the identification (38). If a_n converges in measure to a and is uniformly bounded, then a_n also converges to a in the L^2-norm. Hence for all $f \in L^\infty$, we have $a_n f \to af$ in the L^2-norm. Using an approximation argument and the uniform boundedness of a_n, it follows that $M(a_n) \to M(a)$ strongly on $L^2(\mathbb{T})$. Hence the corresponding Toeplitz and Hankel operators converge strongly on $H^2(\mathbb{T})$, too. Since $T(a_n)^* = T(a_n^*)$ and $H(a_n)^* = H(\widetilde{a}_n^*)$, this holds also for the adjoints. □

3.2. Invertibility of operators $I \pm H(u_\beta)$. In this section we prove that operators of the form $I \pm H(u_\beta)$ are invertible for certain β. We think of the Hankel operators as discrete ones acting on ℓ^2 (or, equivalently, on $H^2(\mathbb{T})$). Obviously, these invertibility results can be extended with the help of (36) and (37) to operators $I \pm H(\hat{u}_\beta)$ where continuous Hankel operators acting on $L^2(\mathbb{R}_+)$ are involved.

For $\tau \in \mathbb{T}$ and $\beta \in \mathbb{C}$ we introduce the functions

$$\eta_\beta(t) = (1-t)^\beta, \qquad \xi_\beta(t) = (1-1/t)^\beta, \tag{39}$$

where these functions are analytic in an open neighborhood of

$$\{z \in \mathbb{C} \,:\, |z| \le 1,\ z \ne 1\} \quad \text{and} \quad \{z \in \mathbb{C} \,:\, |z| \ge 1,\ z \ne 1\} \cup \{\infty\}, \quad \text{resp.},$$

and the branch of the power function is chosen in such a way that $\eta_\beta(0) = 1$ and $\xi_\beta(\infty) = 1$. Notice that

$$v_\beta(t) = \eta_\beta(t)\xi_\beta(t), \qquad u_\beta(t) = \eta_\beta(t)\xi_{-\beta}(t), \qquad u_{\beta+n}(t) = (-t)^n u_\beta(t). \tag{40}$$

The essential spectrum $\mathrm{sp}_{\mathrm{ess}} A$ of a bounded linear operator A defined on a Hilbert space is the set of all $\lambda \in \mathbb{C}$ for which $A - \lambda I$ is not a Fredholm operator, that is, not invertible modulo the compact operators.

Proposition 3.3. *Let $\beta \in \mathbb{C}$. Then:*

(a) *$I + H(u_\beta)$ is Fredholm on ℓ^2 if and only if $\mathrm{Re}\,\beta \notin \frac{1}{2} + 2\mathbb{Z}$.*
(b) *$I - H(u_\beta)$ is Fredholm on ℓ^2 if and only if $\mathrm{Re}\,\beta \notin -\frac{1}{2} + 2\mathbb{Z}$.*

Proof. We use a result of Power [21] in order to determine the essential spectrum of the Hankel operator operators $H(u_\beta)$. It says that the essential spectrum is a union of line segments in the complex plane, namely

$$\mathrm{sp}_{\mathrm{ess}} H(b) = [0, ib_{-1}] \cup [0, -ib_1] \cup \bigcup_{\substack{\tau \in \mathbb{T} \\ \mathrm{Im}\tau > 0}} \Big[-i\sqrt{b_\tau b_{\bar\tau}}, i\sqrt{b_\tau b_{\bar\tau}}\Big]. \tag{41}$$

Therein we use the notation

$$b_\tau = (b(\tau+0) - b(\tau-0))/2 \quad \text{with} \quad b(\tau \pm 0) = \lim_{\varepsilon \to \pm 0} b(\tau e^{i\varepsilon}).$$

This result can also be obtained from the more general results contained in [22] and [14, Sect. 4.95–102].

Clearly, $b_\tau = 0$ for $b = u_\beta$ if $\tau \ne 1$. In the case $\tau = 1$ we have $b_1 = -i\sin(\beta\pi)$. Hence

$$\mathrm{sp}_{\mathrm{ess}} H(u_\beta) = [0, -\sin(\pi\beta)],$$

from which the assertion is easy to conclude. □

Lemma 3.4. *Let $\beta \in \mathbb{C}$ and $\mathrm{Re}\,\beta > -1/2$. Then $\det T_n(v_\beta) \ne 0$ for all $n \ge 1$.*

Proof. This follows from the formula

$$\det T_n(v_\beta) = \frac{G(1+\beta)^2}{G(1+2\beta)} \cdot \frac{G(1+n)\,G(1+2\beta+n)}{G(1+\beta+n)^2},$$

which was proved, e.g., in [14] (see also [12] and [2]). □

Let $\mathcal{P}_{n,m}$ $(n \le m)$ stand for the set of all trigonometric polynomials of the form

$$p(t) = \sum_{k=n}^{m} p_k t^k. \tag{42}$$

We also introduce the Hardy space

$$\overline{H^2(\mathbb{T})} = \{ f \in L^2(\mathbb{T}) \;:\; f_k = 0 \text{ for all } k > 0 \}. \tag{43}$$

which consists of those functions f for which $\bar{f} \in H^2(\mathbb{T})$. Notice that $f \in H^2(\mathbb{T})$ if and only if $\widetilde{f} \in \overline{H^2(\mathbb{T})}$. In the proof of the following we will use the identification (38).

Proposition 3.5. *Let $\beta \in \mathbb{C}$ and $n \in \mathbb{Z}$. Then:*

(a) *If* $\operatorname{Re}\beta \in (-3/2, 1/2]$, *then*

$$\dim\ker(I + H(u_{\beta-2n})) = \max\{0, -n\}. \tag{44}$$

(b) *If* $\operatorname{Re}\beta \in (-1/2, 3/2]$, *then*

$$\dim\ker(I - H(u_{\beta-2n})) = \max\{0, -n\}. \tag{45}$$

Proof. We will treat the operators $A = I + H(u_{\beta-2n})$ and $B = I - H(u_{\beta-2n})$ simultaneously. For the sake of easy reference we will speak of Case A and Case B, respectively.

Assume that $\operatorname{Re}\beta \in (-3/2, 3/2)$ and let f_+ be in the kernel of $\ker(I \pm H(u_{\beta-2n}))$. Then

$$f_+ \mp t^{-2n-2} u_{\beta+1} \widetilde{f}_+ = t^{-1} f_- \in t^{-1}\overline{H^2(\mathbb{T})}.$$

We multiply with $\xi_{\beta+1} t^{n+1}$ and it follows that

$$f_0 := \xi_{\beta+1} t^{n+1} f_+ \mp t^{-n-1} \eta_{\beta+1} \widetilde{f}_+ = t^n \xi_{\beta+1} f_- \in t^n \overline{H^1(\mathbb{T})}.$$

Notice that $\xi_{\beta+1} \in \overline{H^2(\mathbb{T})}$. Obviously, $\widetilde{f}_0 = \mp f_0$. Comparing the Fourier coefficients, it follows that the right-hand side is zero if $n < 0$.

We claim that the right-hand side is also zero in the case $n \ge 0$. In this case it follows first that $f_0 = q_n$, where $q_n \in \mathcal{P}_{-n,n}$ and $\widetilde{q}_n = \mp q_n$. Hence

$$t^{n+1} f_+ \mp t^{-n-1} u_{\beta+1} \widetilde{f}_+ = \xi_{-\beta-1} q_n. \tag{46}$$

We distinghish three cases.

Case 1: $\operatorname{Re}\beta \in (-3/2, -1/2)$. The last equation implies

$$\eta_{-\beta-1} t^{n+1} f_+ \mp \xi_{-\beta-1} t^{-n-1} \widetilde{f}_+ = \xi_{-\beta-1} \eta_{-\beta-1} q_n, \tag{47}$$

where $\xi_{-\beta-1} \in \overline{H^2(\mathbb{T})}$, $\eta_{-\beta-1} \in H^2(\mathbb{T})$. For $k = -n, \dots, n$, the k-th Fourier coefficient of $\xi_{-\beta-1}\eta_{-\beta-1} q_n$ is zero. This condition is equivalent to an equation $T_{2n+1}(\xi_{-\beta-1}\eta_{-\beta-1})\hat{q}_n = 0$, where $\hat{q}_n$ is the vector consisting of the Fourier coefficients of q_n. From Lemma 3.4 it follows that $q_n = 0$.

Case 2: $\operatorname{Re}\beta \in [-1/2, 1/2)$. Since $\xi_{-\beta-1} \notin L^2(\mathbb{T})$, Equation (46) implies that $q_n(1) = 0$. Write $q_n(t) = (1-t) q_{n-1}(t)$ with $q_{n-1} \in \mathcal{P}_{-n,n-1}$. Multiplying (47) with $(1 - t^{-1})$, we obtain

$$-\eta_{-\beta} t^n f_+ \mp \xi_{-\beta} t^{-n-1} \widetilde{f}_+ = \xi_{-\beta} \eta_{-\beta} q_{n-1}. \tag{48}$$

Since $\xi_{-\beta} \in \overline{H^2(\mathbb{T})}$ and $\eta_{-\beta} \in H^2(\mathbb{T})$, for $k = -n, \dots, n-1$ the k-th Fourier coefficient of $\xi_{-\beta}\eta_{-\beta}q_{n-1}$ is zero. This condition leads us to $T_n(\xi_{-\beta}\eta_{-\beta})\hat{q}_{n-1} = 0$ with $\hat{q}_{n-1}$ consisting of the Fourier coefficients of q_{n-1}. Again Lemma 3.4 implies that $q_{n-1} = 0$. Hence $q_n = 0$ as desired.

Case 3: $\operatorname{Re}\beta \in [1/2, 3/2)$. Since $\xi_{-\beta} \notin L^2(\mathbb{T})$, Equation (46) implies that $q_n(1) = q_n'(1) = 0$. Write $q_n(t) = (1-t)(1-t^{-1})q_{n-1}$ with $\mathcal{P}_{-n+1,n-1}$. Multipliying (47) with $(1-t)(1-t^{-1})$ it follows that

$$-\eta_{-\beta+1}t^n f_+ \pm \xi_{-\beta+1}t^{-n}\widetilde{f}_+ = \xi_{-\beta+1}\eta_{-\beta+1}q_{n-1}. \tag{49}$$

Notice that $\xi_{-\beta+1} \in \overline{H^2(\mathbb{T})}$ and $\eta_{-\beta+1} \in H^2(\mathbb{T})$. As before, but now with the Toeplitz matrix $T_{2n-1}(\xi_{-\beta+1}\eta_{-\beta+1})$, we obtain $q_n = 0$.

After having proved that $q_n = 0$ in all cases we can conclude that Equations (47), (48) and (49) hold in all cases with the right-hand side equal to zero. It is now appropriate to distinguish again between several cases, but in a different way.

Case (i): $-1/2 < \operatorname{Re}\beta < 1/2$. From (48), i.e., $-\eta_{-\beta}t^n f_+ \mp \xi_{-\beta}t^{-n-1}\widetilde{f}_+ = 0$, we obtain $f_+ = 0$ if $n \geq 0$. If $n < 0$, then the general solution is $f_+ = \eta_\beta p_n$ with $p_n \in \mathcal{P}_{0,-2n-1}$ and $p_n(t) = \pm t^{-2n-1}\widetilde{p}_n(t)$. Notice that $\eta_\beta \in H^2(\mathbb{T})$ and that the set of those polynomials p_n is a linear space of dimension $-n$.

Case (ii): $-3/2 < \operatorname{Re}\beta < -1/2$ and Case A. From (47), i.e.,

$$\eta_{-\beta-1}t^{n+1}f_+ - \xi_{-\beta-1}t^{-n-1}\widetilde{f}_+ = 0,$$

it follows $f_+ = 0$ if $n \geq 0$. If $n < 0$, then the general solution is given by $f_+ = \eta_{\beta+1}p_n$ with $p_n \in \mathcal{P}_{0,-2n-2}$ and $p_n(t) = t^{-2n-2}\widetilde{p}_n(t)$. The dimension of the space consisting of those polynomials p_n is $-n$.

Case (iii): $1/2 < \operatorname{Re}\beta < 3/2$ and Case B. From (49), i.e.,

$$\eta_{-\beta+1}t^n f_+ + \xi_{-\beta+1}t^{-n}\widetilde{f}_+ = 0,$$

we obtain $f_+ = 0$ in case $n \geq 0$. If $n < 0$, then the general solution is given by $f_+ = \eta_{\beta-1}p_n$ with $p_n \in \mathcal{P}_{0,-2n}$ and $p_n(t) = -t^{-2n}\widetilde{p}_n(t)$. The space of those polynomials is $-n$.

Case (iv): $\operatorname{Re}\beta = -1/2$. Here we proceed as in Case (i) and obtain that the solution is of the form $f_+ = \eta_\beta p_n$ if $n < 0$. However, $\eta_\beta \notin L^2(\mathbb{T})$, which implies that $p_n(t) = (1-t)p_{n-1}(t)$. Hence the general solution is $f_+ = \eta_{\beta+1}p_{n-1}$ with $p_{n-1} \in \mathcal{P}_{0,-2n-2}$ and $p_{n-1}(t) = \mp t^{-2n-2}\widetilde{p}_{n-1}(t)$. The dimension of the space of all solutions is $-n$ in Case A and $-n-1$ in Case B.

Case (v): $\operatorname{Re}\beta = 1/2$. Here we proceed as in Case (iii) and obtain that the solution is of the form $f_+ = \eta_{\beta-1}p_n$ if $n < 0$. Since $\eta_{\beta-1} \notin L^2(\mathbb{T})$, we can write $p_n(t) = (1-t)p_{n-1}(t)$. Hence the general solution is $f_+ = \eta_\beta p_{n-1}$ with $p_{n-1} \in \mathcal{P}_{0,-2n-1}$ and $p_{n-1}(t) = \pm t^{-2n-1}\widetilde{p}_{n-1}(t)$. The dimension of the space of all solutions is $-n$ both in Case A and in Case B.

The statement of the proposition follows easily from Cases (i)–(v). □

Theorem 3.6. *Let $\beta \in \mathbb{C}$. Then:*

(a) *$I + H(u_\beta)$ is invertible of ℓ^2 if and only if $\operatorname{Re}\beta < 1/2$ and $\operatorname{Re}\beta \notin \frac{1}{2} + 2\mathbb{Z}$.*

(b) *$I - H(u_\beta)$ is invertible of ℓ^2 if and only if $\operatorname{Re}\beta < 3/2$ and $\operatorname{Re}\beta \notin \frac{3}{2} + 2\mathbb{Z}$.*

Proof. The Fredholm criteria are contained in Proposition 3.3. The dimension of the kernel and cokernel of $I \pm H(u_\beta)$ is given by Proposition 3.5. Notice that $H(u_\beta)^* = H(u_{\overline{\beta}})$. □

3.3. The determinants of the discrete operators. For $\beta \in \mathbb{C}$ and $r \in [0,1)$ we introduce the functions

$$v_{\beta,r}(t) := (1-r/t)^{\beta}(1-rt)^{\beta}, \quad u_{\beta,r}(t) := (1-r/t)^{-\beta}(1-rt)^{\beta}, \quad t \in \mathbb{T}. \tag{50}$$

We will use these functions as approximations of the functions v_β and u_β. Recalling (40) notice that we can write

$$v_\beta(t) = (1-1/t)^{\beta}(1-t)^{\beta}, \qquad u_\beta(t) = (1-1/t)^{-\beta}(1-t)^{\beta}, \qquad t \in \mathbb{T}.$$

Lemma 3.7.

(a) *If* $-3/2 < \operatorname{Re}\beta < 1/2$*, then* $(I + H(u_{\beta,r}))^{-1} \to (I + H(u_\beta))^{-1}$ *strongly on* ℓ^2 *as* $r \to 1$.

(b) *If* $-1/2 < \operatorname{Re}\beta < 3/2$*, then* $(I - H(u_{\beta,r}))^{-1} \to (I - H(u_\beta))^{-1}$ *strongly on* ℓ^2 *as* $r \to 1$.

Proof. We first remark that $u_{\beta,r}$ is bounded in the L^∞-norm with respect to r and converges in measure to u_β as $r \to 1$. Hence $H(u_{\beta,r}) \to H(u_\beta)$ strongly on ℓ^2 as $r \to 1$. In order to prove the strong convergence of the inverses of $I \pm H(u_{\beta,r})$ it is thus necessary and sufficient to show that the sequence $I \pm H(u_{\beta,r})$ is stable.

Using the results of [19] one can prove that $I \pm H(u_{\beta,r})$ is stable if and only if the operators

$$I \pm H(u_\beta) \quad \text{and} \quad I \pm H(u_{-\beta,-1})$$

are invertible, where $u_{-\beta,-1}(t) := u_{-\beta}(-t)$. Introducing the operator

$$W : \{x_n\}_{n=0}^{\infty} \in \ell^2 \mapsto \{(-1)^n x_n\}_{n=0}^{\infty} \in \ell^2$$

and noting that

$$W^2 = I \quad \text{and} \quad WH(a)W = -H(b) \tag{51}$$

with $b(t) := a(-t)$, $t \in \mathbb{T}$, it follows that $I \pm H(u_{-\beta,-1})$ is invertible if and only if the operator $I \mp H(u_{-\beta})$ is invertible. Applying Theorem 3.6 completes the proof.

In order to give some more details about the derivation of the above stability criterion from [19] we rely on the notation introduced there. What we have to do is to apply Theorem 2.1 and Theorem 2.2 of [19] in the setting where k_λ equals the harmonic extension (1.8). The corresponding functions $K(x)$ and $f(e^{i\theta})$ (see (2.4)) evaluate to $K(x) = 1/(\pi(1+x^2))$ and $f(e^{i\theta}) = (\theta+\pi)/(2\pi)$, $|\theta| < \pi$. Hence the functions a_τ ($\tau \in \mathbb{T}$) that are associated to each function $a \in PC$ (see (2.5)) are given by

$$a_\tau(e^{i\theta}) = a(\tau+0)\frac{\pi+\theta}{2\pi} + a(\tau-0)\frac{\pi-\theta}{2\pi}, \qquad -\pi < \theta < \pi.$$

We need those functions in the case $a = \log u_\beta$.

The setting of Theorem 2.2 is with

$$A_\lambda = I \pm H(\exp(k_\lambda(\log u_\beta)))$$

(since $k_\lambda(\log u_\beta) = \log u_{\beta,r}$ with $\lambda = -1/\log r$). The homomorphisms evaluate with the help of Theorem 2.1 to:

(i) $\Psi_0[A_\lambda] = I \pm H(\exp(\log u_\beta)) = I \pm H(u_\beta)$.

(ii) $\Psi_1[A_\lambda] = I \pm H(\exp(\log u_\beta)_1) = I \pm H(u_{-\beta,-1})$, and
$\Psi_{-1}[A_\lambda] = I \mp H(\exp(\log u_\beta)_{-1}) = I$.

(iii)
$$\Psi_{\tau,\bar\tau}[A_\lambda] = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \pm \begin{pmatrix} 0 & PM(\exp(\log u_\beta)_\tau)Q \\ QM(\exp\widetilde{(\log u_\beta)_\tau})P & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Observe that $(\log u_\beta)_\tau$ is a constant except for $\tau \neq 1$ since u_β has only a discontinuity at 1. For $\tau = 1$ we have

$$(\log u_\beta)_1(e^{i\theta}) = -i\beta\pi\frac{\pi+\theta}{2\pi} + i\beta\pi\frac{\pi-\theta}{2\pi} = -i\beta\theta, \qquad -\pi < \theta < \pi.$$

Hence $\exp((\log u_\beta)_1(e^{i\theta})) = e^{-i\beta\theta} = u_{-\beta}(e^{i(\theta+\pi)}) = u_{-\beta,-1}(e^{i\theta})$, $|\theta| < \pi$, which settles (ii).

The invertibility of the operators in (i)–(iii) (which is necessary and sufficient for the stability of the sequence $A_\lambda = I \pm H(u_{\beta,r})$) is nothing else than what was stated above. This completes the derivation. □

A simple conclusion of the previous lemma is the following result:

Proposition 3.8. *Let $n \geq 1$ be fixed.*

(a) *If $-3/2 < \operatorname{Re}\beta < 1/2$, then*
$$\det\big[P_n(I + H(u_{\beta,r}))^{-1}P_n\big] \to \det\big[P_n(I + H(u_\beta))^{-1}P_n\big], \qquad r \to 1.$$

(b) *If $-1/2 < \operatorname{Re}\beta < 3/2$, then*
$$\det\big[P_n(I - H(u_{\beta,r}))^{-1}P_n\big] \to \det\big[P_n(I - H(u_\beta))^{-1}P_n\big], \qquad r \to 1.$$

Let $\mathcal{W}(\mathbb{T})$ denote the Wiener algebra, which is the Banach algebra of all functions defined on the unit circle with Fourier coefficients satisfying

$$\sum_{n=-\infty}^{\infty} |a_n| < \infty. \tag{52}$$

A canonical Wiener–Hopf factorization in $\mathcal{W}(\mathbb{T})$ is a representation of the form

$$a(t) = a_-(t)a_+(t), \qquad t \in \mathbb{T}, \tag{53}$$

such that the functions $a_\pm$ and their inverses belong to $\mathcal{W}(\mathbb{T})$ and such that the n-th Fourier coefficients of the functions $a_+^{\pm1}(t)$ and $a_-^{\pm1}(t^{-1})$ vanish for each $n < 0$. It is well-known [14] that a function $a \in \mathcal{W}(\mathbb{T})$ possesses a canonical Wiener–Hopf factorization in $\mathcal{W}(\mathbb{T})$ if and only if a is nonzero on all of $\mathbb{T}$ and has winding number zero. This is equivalent to the condition that a possesses a logarithm $\log a \in \mathcal{W}(\mathbb{T})$.

Under this last condition we can define

$$G[a] := \exp\Big(\frac{1}{2\pi}\int_0^{2\pi} \log a(e^{i\theta})\,d\theta\Big) \tag{54}$$

as the geometric mean of a.

Proposition 3.9. *Let $a \in \mathcal{W}(\mathbb{T})$ be an even function which possesses a canonical Wiener–Hopf factorization $a(t) = a_-(t)a_+(t)$. Define $\psi(t) = \widetilde{a}_+(t)a_+^{-1}(t)$. Then $I \pm H(\psi)$ is invertible on ℓ^2 and*

$$\det\big[T_n(a) \pm H_n(a)\big] = G[a]^n \det\big[P_n(I \pm H(\psi))^{-1}P_n\big]. \tag{55}$$

Proof. First of all notice that from (28)

$$\Big(T(a) \pm H(a)\Big)\Big(T(a^{-1}) \pm H(a^{-1})\Big) = \Big(T(a^{-1}) \pm H(a^{-1})\Big)\Big(T(a) \pm H(a)\Big) = I.$$

Moreover, using (29) it follows that

$$T(a^{-1}) \pm H(a^{-1}) = T(a_-^{-1})(I \pm H(\psi))T(a_+^{-1}).$$

Notice that also from (28)

$$T(a_\pm)T(a_\pm^{-1}) = T(a_\pm^{-1})T(a_\pm) = I.$$

Hence $I \pm H(\psi)$ is invertible and

$$T(a) \pm H(a) = T(a_+)(I \pm H(\psi))^{-1}T(a_-).$$

Now we multiply from the left and right with P_n, and observing that $T(a_+)$ and $T(a_-)$ are lower and upper triangular matrices we obtain

$$T_n(a) \pm H_n(a) = T_n(a_+)\Big(P_n(I \pm H(\psi))^{-1}P_n\Big)T_n(a_-).$$

Since $\det T_n(a_\pm) = ([a_\pm]_0)^n$ and $[a_+]_0[a_-]_0 = \exp([\log a_+]_0 + [\log a_-]_0) = G[a]$ we conclude the desired assertion. □

Proposition 3.10. *Let $n \geq 1$ be fixed.*

(a) *If $-1/2 < \operatorname{Re}\beta < 3/2$, then*

$$D_n^+(\beta) = \det\big[P_n(I + H(u_{-\beta}))^{-1}P_n\big]. \tag{56}$$

(b) *If $-3/2 < \operatorname{Re}\beta < 1/2$, then*

$$D_n^-(\beta) = \det\big[P_n(I - H(u_{-\beta}))^{-1}P_n\big]. \tag{57}$$

Proof. We apply the previous proposition with $a(t) = v_{\beta,r}(t)$ (where $0 \leq r < 1$). Noting that $a_+(t) = (1 - rt)^\beta$, $\psi(t) = u_{-\beta,r}(t)$ and $G[a] = 1$ we obtain

$$\det\big[T_n(v_{\beta,r}) \pm H_n(v_{\beta,r})\big] = \det\big[P_n(I \pm H(u_{-\beta,r}))^{-1}P_n\big]$$

for all $\beta \in \mathbb{C}$. For $\operatorname{Re}\beta > -1/2$, we have that $v_{\beta,r} \to v_\beta$ in the L^1-norm. Hence the limit as $r \to 1$ of the left-hand side of the previous identity is (for n fixed) equal to $\det[T_n(v_\beta) \pm H_n(v_\beta)]$. From Proposition 3.8 we obtain the limit of the right-hand side and thus the identities (56) and (57).

In order to justify identity (57) in the case $-3/2 < \operatorname{Re}\beta \leq -1/2$ in (b) we argue by analyticity (see Theorem 2.1). Notice that the analyticity of the determinant of the right-hand side follows essentially from the fact that the mapping $\beta \in \mathbb{C} \mapsto H(u_\beta) \in \mathcal{L}(\ell^2)$ is an analytic operator-valued function. □

Obviously, the previous result in connection with Theorem 2.1 allows us to determine the asymptotics of the determinants $\det[P_n(I \pm H(u_{-\beta}))^{-1}P_n]$ as $n \to \infty$ in the case where $-1/2 < \pm\operatorname{Re}\beta < 3/2$. This will be one of the cornerstones in the proof of the main result (see Section 3.6).

3.4. The determinants of the continuous operators. In this subsection we will establish the continuous analogues to the results of the previous subsection. We introduce the functions

$$\hat{v}_{\beta,\varepsilon}(x) := \left(\frac{x^2+\varepsilon^2}{x^2+1}\right)^{\beta}, \qquad \hat{u}_{\beta,\varepsilon}(x) := \left(\frac{x-\varepsilon i}{x-i}\right)^{-\beta}\left(\frac{x+\varepsilon i}{x+i}\right)^{\beta}, \tag{58}$$

where $\beta \in \mathbb{C}$ and $\varepsilon \in (0,1]$. These functions will approximate the functions $\hat{v}_\beta$ and $\hat{u}_\beta$, respectively, which were defined in (4).

Proposition 3.11. *Let $R > 0$ be fixed.*

(a) *If $-3/2 < \operatorname{Re}\beta < 1/2$, then*

$$\det\big[P_R(I + H(\hat{u}_{\beta,\varepsilon}))^{-1}P_R\big] \to \det\big[P_R(I + H(\hat{u}_{\beta}))^{-1}P_R\big], \qquad \varepsilon \to 0.$$

(b) *If $-1/2 < \operatorname{Re}\beta < 3/2$, then*

$$\det\big[P_R(I - H(\hat{u}_{\beta,\varepsilon}))^{-1}P_R\big] \to \det\big[P_R(I - H(\hat{u}_{\beta}))^{-1}P_R\big], \qquad \varepsilon \to 0.$$

Proof. We first write

$$P_R(I \pm H(\hat{u}_{\beta,\varepsilon}))^{-1}P_R = P_R \mp P_R(I \pm H(\hat{u}_{\beta,\varepsilon}))^{-1}H(\hat{u}_{\beta,\varepsilon})P_R.$$

Noting that $W_R^2 = P_R$, $W_R P_R = P_R W_R = W_R$, where $W_R := H(e^{ixR})$, it follows that

$$\det\big[P_R(I \pm H(\hat{u}_{\beta,\varepsilon}))^{-1}P_R\big] = \det[P_R \mp A_\varepsilon],$$

where

$$A_\varepsilon := W_R(I \pm H(\hat{u}_{\beta,\varepsilon}))^{-1}H(\hat{u}_{\beta,\varepsilon})W_R.$$

We claim that $H(\hat{u}_{\beta,\varepsilon})W_R$ is a trace class operator, which converges in the trace norm to $H(\hat{u}_\beta)W_R$ as $\varepsilon \to 0$. In order to see this we apply the transform Φ (see (37)) and obtain

$$\Phi[H(\hat{u}_{\beta,\varepsilon})W_R] = H(u_{\beta,r})H(h_R), \qquad \Phi[H(\hat{u}_\beta)W_R] = H(u_\beta)H(h_R),$$

where $h_R(t) := \exp(R(t-1)/(t+1))$, $r = (1-\varepsilon)/(1+\varepsilon)$. Let f_1, f_2 be smooth functions on $\mathbb{T}$ satisfying $f_1 + f_2 = 1$ such that $f_1(t)$ vanishes on a neighborhood of $t = 1$ and $f_2(t)$ vanishes on a neighborhood of $t = -1$. By applying (29) we decompose

$$\begin{aligned} H(u_{\beta,r})H(h_R) &= H(u_{\beta,r})\big[T(f_1) + T(f_2)\big]H(h_R) \\ &= \big[H(u_{\beta,r}\widetilde{f}_1) - T(u_{\beta,r})H(\widetilde{f}_1)\big]H(h_R) \\ &\quad + H(u_{\beta,r})\big[H(f_2 h_R) - H(f_2)T(\widetilde{h}_R)\big]. \end{aligned} \tag{59}$$

A similar identity where $u_{\beta,r}$ is replaced with u_β can also be established. Notice that $f_2 h_R$ is a smooth function because h_R has its only singularity at $t = -1$. Analogously, the functions $u_{\beta,r}\widetilde{f}_1$ and $u_\beta \widetilde{f}_1$ are smooth, and the derivative of $u_{\beta,r}\widetilde{f}_1$ converges uniformly to the derivative of $u_\beta \widetilde{f}_1$ as $r \to 1$.

Using, for instance, the estimate

$$\|H(a)\|_1 \;\le\; \sum_{n=1}^{\infty} |a_n| \cdot \|H(t^n)\|_1 \;=\; \sum_{n=1}^{\infty} n\,|a_n| \;\le\; C \cdot \|a'\|_\infty,$$

which follows from determining the trace norm of $H(t^n)$ and from partial integration, it is easily seen that all Hankel operators appearing within the brackets of (59) are trace class and that $H(u_{\beta,r}\widetilde{f}_1) \to H(u_\beta \widetilde{f}_1)$ in the trace norm. Since

$H(u_{\beta,r}) \to H(u_\beta)$ and $T(u_{\beta,r}) \to T(u_\beta)$ strongly on ℓ^2 (see Lemma 3.2) we conclude that $H(u_{\beta,r})H(h_R) \to H(u_\beta)H(h_R)$ in the trace norm. Hence $H(\hat{u}_{\beta,\varepsilon})W_R$ is a trace class operator which converges in the trace norm to $H(\hat{u}_\beta)W_R$.

Using Lemma 3.7 and the transform Φ we conclude that $(I \pm H(\hat{u}_{\beta,\varepsilon}))^{-1}$ converges strongly to $(I \pm H(\hat{u}_\beta))^{-1}$ as $\varepsilon \to 0$. Hence $A_\varepsilon \to A$ in the trace norm where A is the trace class operator

$$A := W_R(I \pm H(\hat{u}_\beta))^{-1} H(\hat{u}_\beta) W_R.$$

Thus we obtain $\det[P_R \mp A_\varepsilon] \to \det[P_R \mp A]$, which proves the assertion. □

Let $\mathcal{W}(\mathbb{R})$ be the set of all functions $\hat{a}$ defined on $\mathbb{R}$ such that $a \in \mathcal{W}(\mathbb{T})$ where

$$a\left(\frac{1+ix}{1-ix}\right) := \hat{a}(x), \qquad x \in \mathbb{R}. \tag{60}$$

For a function $\hat{a} \in \mathcal{W}(\mathbb{R})$ which possesses a logarithm $\log \hat{a} \in L^1(\mathbb{R}) \cap \mathcal{W}(\mathbb{R})$, the geometric means is well-defined by

$$G[\hat{a}] := \exp\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \log \hat{a}(x)\, dx\right). \tag{61}$$

Notice that the logarithm is uniquely determined.

We say that $\hat{a}(x) = \hat{a}_-(x)\hat{a}_+(x)$, $x \in \mathbb{R}$, is a canonical Wiener–Hopf factorization in $\mathcal{W}(\mathbb{R})$ if $a(t) = a_-(t)a_+(t)$, $t \in \mathbb{T}$, is a canonical Wiener–Hopf factorization in $\mathcal{W}(\mathbb{T})$, where the functions a and $a_\pm$ are defined according to (60).

Lemma 3.12. *Let $a \in \mathcal{W}(\mathbb{R})$ be a function which possesses a canonical Wiener–Hopf factorization $a(x) = a_-(x)a_+(x)$ in $\mathcal{W}(\mathbb{R})$ and a logarithm $\log a \in L^1(\mathbb{R}) \cap \mathcal{W}(\mathbb{R})$. Then $W_R(a_-)W_R(a_+) - P_R$ is a trace class operator on $L^2[0,R]$ and*

$$\det\left[W_R(a_-)W_R(a_+)\right] = G[a]^R. \tag{62}$$

Proof. We can assume without loss of generality that the factors are normalized such that $(\log a_\pm)(\infty) = 0$. Obviously, $\log a \in L^1(\mathbb{R}) \cap \mathcal{W}(\mathbb{R})$ implies that $\log a \in L^2(\mathbb{R}) \cap \mathcal{W}(\mathbb{R})$. Notice that $L^p(\mathbb{R}) \cap \mathcal{W}(\mathbb{R})$ are Banach algebras without unit elements. Since the Riesz projection with respect to the upper half-plane is bounded on $L^2(\mathbb{R}) \cap \mathcal{W}(\mathbb{R})$ it follows

$$\log a_\pm \in L^2(\mathbb{R}) \cap \mathcal{W}(\mathbb{R}), \qquad a_\pm - 1 \in L^2(\mathbb{R}) \cap \mathcal{W}(\mathbb{R}).$$

Hence $W_R(a_\pm - 1)$ are Hilbert–Schmidt operators, while $W_R(a_\pm - 1 - \log a_\pm)$ are trace class operators. The latter is true since $a_\pm - 1 - \log a_\pm \in L^1(\mathbb{R})$. On the other hand $W_R(\log a) = W_R(\log a_+) + W_R(\log a_-)$ is also a trace class operator.

For Hilbert–Schmidt operators K, L for which $K + L$ is a trace class operator the identity

$$\det\left[(I+K)(I+L)\right] = \det\left[(I+K)e^{-K}\right]\det\left[(I+L)e^{-L}\right]\exp\left[\operatorname{trace}(K+L)\right]$$

holds, which can be proved by an approximation argument. We use this identity in the setting $K = W_R(a_-) - I$ and $L = W_R(a_+) - I$ and remark that the operator

$$\begin{aligned} K + L &= W_R(a_- - 1) + W_R(a_+ - 1) \\ &= W_R(a_- - 1 - \log a_-) + W_R(a_+ - 1 - \log a_+) + W_R(\log a) \end{aligned}$$

is trace class. Noting that $W_R(a_\pm)=e^{W_R(\log a_\pm)}$, which implies

$$\det\big[(I+K)e^{-K}\big]=\det W_R(a_-)e^{W_R(1-a_-)}=\exp\big[\operatorname{trace}W_R(\log a_-+1-a_-)\big],$$
$$\det\big[(I+L)e^{-L}\big]=\det W_R(a_+)e^{W_R(1-a_+)}=\exp\big[\operatorname{trace}W_R(\log a_++1-a_+)\big],$$

it follows that $\det\big[(I+K)(I+L)\big]$ equals the exponential of

$$\operatorname{trace}W_R(\log a_-+1-a_-)+\operatorname{trace}W_R(\log a_++1-a_+)+\operatorname{trace}(K+L)\\ =\operatorname{trace}W_R(\log a).$$

Since the trace of $W_R(\log a)$ is equal to R times the Fourier transform of $\log a$ evaluated at the point $\xi=0$, the assertion follows easily. □

In regard to the proof of the previous lemma we remark that in general $\log a_\pm\notin L^1(\mathbb{R})$. In particular, $W_R(a_\pm)-P_R$ need not be trace class operators.

Proposition 3.13. *Let $a\in\mathcal{W}(\mathbb{R})$ be an even function which possesses a canonical Wiener–Hopf factorization $a(x)=a_-(x)a_+(x)$ in $\mathcal{W}(\mathbb{R})$. Suppose that $\log a\in L^1(\mathbb{R})\cap\mathcal{W}(\mathbb{R})$ and define $\psi(x)=\widetilde{a}_+(x)a_+^{-1}(x)$. Then $I\pm H(\psi)$ is invertible on $L^2(\mathbb{R}_+)$, the operator $P_R(I\pm H(\psi))^{-1}P_R-P_R$ is trace class on $L^2[0,R]$, and*

$$\det\big[W_R(a)\pm H_R(a)\big]=G[a]^R\det\big[P_R(I\pm H(\psi))^{-1}P_R\big].$$

Proof. We can prove in the same way as in Proposition 3.9 that $I\pm H(\psi)$ is invertible, and we derive the identity

$$W(a)\pm H(a)=W(a_+)(I\pm H(\psi))^{-1}W(a_-).$$

Since $a_\pm$ are appropriate Wiener–Hopf factors, we have $P_RW(a_+)=W_R(a_+)$ and $W(a_-)P_R=W_R(a_-)$. Hence

$$W_R(a)\pm H_R(a)=W_R(a_+)\Big(P_R(I\pm H(\psi))^{-1}P_R\Big)W_R(a_-).$$

Because the operators $W_R(a_\pm)$ are invertible and since both $W_R(a)+H_R(a)$ and $W_R(a_-)W_R(a_+)$ are identity plus trace class, it is easy to conclude (for instance by moving $W_R(a_-)$ to the left-hand side) that $P_R(I\pm H(\psi))^{-1}P_R$ is identity plus trace class, too. In particular, we obtain

$$\det\big[W_R(a)\pm H_R(a)\big]=\det\big[W_R(a_-)W_R(a_+)P_R(I\pm H(\psi))^{-1}P_R\big]\\ =\det\big[W_R(a_-)W_R(a_+)\big]\cdot\det\big[P_R(I\pm H(\psi))^{-1}P_R\big],$$

which implies the assertion by employing Lemma 3.12. □

Proposition 3.14.

(a) *If $-1/2<\operatorname{Re}\beta<3/2$, then*

$$\hat{D}_R^+(\beta)=e^{-\beta R}\det\big[P_R(I+H(\hat{u}_{-\beta}))^{-1}P_R\big]. \tag{63}$$

(b) *The function $\hat{D}_R^-(\beta)$ admits an analytic continuation onto the set of all $\beta\in\mathbb{C}$ for which $\operatorname{Re}\beta>-3/2$. Moreover, if $-3/2<\operatorname{Re}\beta<1/2$, then*

$$\hat{D}_R^-(\beta)=e^{-\beta R}\det\big[P_R(I-H(\hat{u}_{-\beta}))^{-1}P_R\big]. \tag{64}$$

Proof. We already know that $\hat{D}_R^{\pm}(\beta)$ are analytic functions on the set of all $\beta \in \mathbb{C}$ for which $\operatorname{Re}\beta > -1/2$. Moreover, the right-hand side in (64) is analytic for $-3/2 < \operatorname{Re}\beta < 1/2$. This follows from the fact that $H(\hat{u}_\beta)$ is an operator-valued analytic function in $\beta \in \mathbb{C}$ and that the inverses of $I - H(\hat{u}_{-\beta})$ exist for $-3/2 < \operatorname{Re}\beta < 1/2$. Hence in order to prove statement (b) it suffices to prove the identity (64) for $-1/2 < \operatorname{Re}\beta < 1/2$.

We apply Proposition 3.13 with $a(x) = \hat{v}_{\beta,\varepsilon}(x)$. The corresponding Wiener–Hopf factors are

$$a_\pm(x) = \left(\frac{x \pm \varepsilon i}{x \pm i}\right)^\beta,$$

whence we obtain $\psi(x) = \widetilde{a}_+(x)a_+^{-1}(x) = \hat{u}_{-\beta,\varepsilon}(x)$. Noting that $G[a] = e^{-\beta(1-\varepsilon)}$ it follows that

$$\det\left[W_R(\hat{v}_{\beta,\varepsilon}) \pm H_R(\hat{v}_{\beta,\varepsilon})\right] = e^{-\beta R(1-\varepsilon)} \det\left[P_R(I \pm H(\hat{u}_{\beta,\varepsilon}))^{-1}P_R\right].$$

Passing to the limit $\varepsilon \to 0$ and applying Proposition 3.11 the assertion follows. □

It is obvious from the previous proposition that we can determine the asymptotics of $\hat{D}_R^{\pm}(\beta)$ from the asymptotics of the determinant $\det\left[P_R(I \pm H(\hat{u}_{-\beta}))^{-1}P_R\right]$ and vice versa. This is the second ingredient in the proof of the main result (see Section 3.6).

3.5. Asymptotic relation between discrete and continuous operators. In this section we are going to prove that (for certain fixed β)

$$\det\left[P_n(I \pm H(u_\beta))^{-1}P_n\right] \sim \det\left[P_R(I \pm H(\hat{u}_\beta))^{-1}P_R\right]$$

as $n \to \infty$, $R \to \infty$ and $R = 2n + O(1)$.

We start with a couple of auxiliary results. The first result is one of the ingredients to the proof of the Borodin–Okounkov identity as given in [7, 10].

Lemma 3.15. *Let A be a trace class operator on a Hilbert space H and assume that $I + A$ is invertible. Let P be a projection on H and let $Q = I - P$. Then*

$$\det\left[P(I + A)^{-1}P\right] = \frac{\det(I + QAQ)}{\det(I + A)}. \qquad (65)$$

Proof. We write $(I + A)^{-1} = I - (I + A)^{-1}A$ and extend the operator appearing on the left-hand side in the operator determinant by the projection Q,

$$P(I + A)^{-1}P + Q = I - P(I + A)^{-1}AP.$$

It follows that

$$\begin{aligned}\det\left[P(I + A)^{-1}P\right] &= \det\left[I - P(I + A)^{-1}AP\right]\\ &= \det\left[I - (I + A)^{-1}AP\right]\\ &= \det(I + A)^{-1} \cdot \det\left[I + A - AP\right]\\ &= \det(I + A)^{-1} \cdot \det\left[I + QAQ\right],\end{aligned}$$

which is the desired assertion. □

Lemma 3.16. *For $-1 < \sigma < 1$, the trace norm of the integral operator with the kernel*

$$k(x, y) = \frac{f_1(x)f_2(y)}{x + y}$$

on $L^2(M)$, where $M \subset \mathbb{R}_+$, is at most a constant times the square root of

$$\int_M |f_1(x)|^2 \frac{dx}{x^{1+\sigma}} \cdot \int_M |f_2(x)|^2 \frac{dx}{x^{1-\sigma}}.$$

Proof. We can write this operator as a product $K_1 K_2$ where $K_1 : L^2(\mathbb{R}_+) \to L^2(M)$ and $K_2 : L^2(M) \to L^2(\mathbb{R}_+)$ have the kernels

$$k_1(x,\eta) = f_1(x) e^{-x\eta} \eta^{\sigma/2}, \qquad k_2(\xi, y) = f_2(y) e^{-y\xi} \xi^{-\sigma/2}.$$

The operators K_1 and K_2 are Hilbert–Schmidt and their norms can be estimated appropriately. □

Let $K_{\beta,\varepsilon,n}$ and $\hat{K}_{\beta,\varepsilon,R}$ be the integral operators on $L^2[\varepsilon,1]$ with the kernels

$$K_{\beta,\varepsilon,n}(x,y) = -\frac{\sin(\pi\beta)}{\pi} \left(\frac{(1+x)(x-\varepsilon)}{(1-x)(x+\varepsilon)} \frac{(1+y)(y-\varepsilon)}{(1-y)(y+\varepsilon)} \right)^{\beta/2} \times \left(\frac{1-x}{1+x}\right)^{2n} \frac{1}{x+y}, \tag{66}$$

$$\hat{K}_{\beta,\varepsilon,R}(x,y) = -\frac{\sin(\pi\beta)}{\pi} \left(\frac{(1+x)(x-\varepsilon)}{(1-x)(x+\varepsilon)} \frac{(1+y)(y-\varepsilon)}{(1-y)(y+\varepsilon)} \right)^{\beta/2} \frac{e^{-2Rx}}{x+y}. \tag{67}$$

Proposition 3.17. *Let $-1 < \operatorname{Re}\beta < 1$. Then $K_{\beta,\varepsilon,n}$ and $\hat{K}_{\beta,\varepsilon,R}$ are trace class operators on $L^2[\varepsilon,1]$ and*

$$\det(I \pm Q_n H(u_{\beta,r}) Q_n) = \det(I \pm K_{\beta,\varepsilon,n}), \quad r = \tfrac{1-\varepsilon}{1+\varepsilon}, \tag{68}$$

$$\det(I \pm Q_R H(\hat{u}_{\beta,\varepsilon}) Q_R) = \det(I \pm \hat{K}_{\beta,\varepsilon,R}), \tag{69}$$

where $Q_n = I - P_n$ and $Q_R = I - P_R$.

Proof. The fact that $K_{\beta,\varepsilon,n}$ and $\hat{K}_{\beta,\varepsilon,R}$ are trace class operators follows from Lemma 3.16 (with $\sigma = 0$).

Let us first prove identity (68). The operator $Q_n H(u_{\beta,r}) Q_n$ can be identified with the matrix kernel

$$\begin{aligned} k(j,k) &= \frac{1}{2\pi i} \int_{\mathbb{T}} \left(\frac{1-rt}{1-rt^{-1}}\right)^{\beta} t^{-2-j-k-2n}\, dt \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} \left(\frac{1-rt^{-1}}{1-rt}\right)^{\beta} t^{j+k+2n}\, dt. \\ &= -\frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{(\xi+i)(\xi-i\varepsilon)}{(\xi-i)(\xi+i\varepsilon)}\right)^{\beta} \left(\frac{i-\xi}{i+\xi}\right)^{j+k+2n} \frac{d\xi}{(i+\xi)^2}. \end{aligned}$$

Therein we have employed first the substitution $t \mapsto t^{-1}$ and then $t = \frac{i-\xi}{i+\xi}$, $r = \frac{1-\varepsilon}{1+\varepsilon}$. The integrand is analytic in the upper half-plane cut along the segment $[i\varepsilon, i]$. We deform the path of integration to this segment described back and forth. The expression in parentheses is real and negative. The limit of its argument from the left equals $-\pi$ and from right equals π. We obtain (with the substitution $\xi = i\eta$)

$$k(j,k) = -\frac{2\sin(\pi\beta)}{\pi} \int_{\varepsilon}^{1} \left(\frac{(1+\eta)(\eta-\varepsilon)}{(1-\eta)(\eta+\varepsilon)}\right)^{\beta} \left(\frac{1-\eta}{1+\eta}\right)^{j+k+2n} \frac{d\eta}{(1+\eta)^2}.$$

This operator can be written as UV where $U : L^2[\varepsilon, 1] \to \ell^2(\mathbb{Z}_+)$ and $V : \ell^2(\mathbb{Z}_+) \to L^2[\varepsilon, 1]$ are given by

$$U(j, \xi) = -\frac{2\sin(\pi\beta)}{\pi}\left(\frac{1-\xi}{1+\xi}\right)^{j-\beta/2}\left(\frac{\xi-\varepsilon}{\xi+\varepsilon}\right)^{\beta/2}\frac{1}{(1+\xi)},$$
$$V(\eta, k) = \left(\frac{1-\eta}{1+\eta}\right)^{2n+k-\beta/2}\left(\frac{\eta-\varepsilon}{\eta+\varepsilon}\right)^{\beta/2}\frac{1}{(1+\eta)}.$$

Under the assumption $-1 < \operatorname{Re}\beta < 1$, the operators U and V are Hilbert–Schmidt. The operator VU is the integral operator with the kernel

$$\begin{aligned} h(\eta, \xi) &= -\frac{2\sin(\pi\beta)}{\pi}\left(\frac{(\eta-\varepsilon)(\xi-\varepsilon)}{(\eta+\varepsilon)(\xi+\varepsilon)}\right)^{\beta/2} \\ &\quad\times \frac{1}{(1+\xi)(1+\eta)}\sum_{k=0}^{\infty}\left(\frac{1-\eta}{1+\eta}\right)^{k+2n-\beta/2}\left(\frac{1-\xi}{1+\xi}\right)^{k-\beta/2} \\ &= -\frac{\sin(\pi\beta)}{\pi}\left(\frac{(\eta-\varepsilon)(\xi-\varepsilon)}{(\eta+\varepsilon)(\xi+\varepsilon)}\frac{(1-\eta)(1-\xi)}{(1+\eta)(1+\xi)}\right)^{\beta/2}\left(\frac{1-\eta}{1+\eta}\right)^{2n}\frac{1}{\xi+\eta}. \end{aligned}$$

Hence $VU = K_{\beta,\varepsilon,n}$.

Now we turn to the proof of (69). Since $\hat{u}_{\beta,\varepsilon} - 1 \in L^2(\mathbb{R})$, the operator $Q_R H(\hat{u}_{\beta,\varepsilon}) Q_R$ can be identified with an integral operator with the kernel

$$\begin{aligned} k(x, y) &= \lim_{M\to\infty}\frac{1}{2\pi}\int_{-M}^{M}\left[\left(\frac{(\xi-i)(\xi+\varepsilon i)}{(\xi+i)(\xi-\varepsilon i)}\right)^{\beta} - 1\right]e^{-i\xi(2R+x+y)}\,d\xi \\ &= \lim_{M\to\infty}\frac{1}{2\pi}\int_{-M}^{M}\left[\left(\frac{(\xi+i)(\xi-\varepsilon i)}{(\xi-i)(\xi+\varepsilon i)}\right)^{\beta} - 1\right]e^{i\xi(2R+x+y)}\,d\xi. \end{aligned}$$

The integrand is analytic in the upper half-plane cut along the segment $[i\varepsilon, i]$ and decays as $O(\xi^{-1}e^{-2R\operatorname{Im}\xi})$ as $\xi \to \infty$, $\operatorname{Im}\xi \geq 0$. We deform the path of integration to this segment described back and forth. The expression in parentheses is real and negative. The limit of its argument from the left equals $-\pi$ and from right equals π. Hence we obtain

$$k(x, y) = -\frac{\sin(\pi\beta)}{\pi}\int_{\varepsilon}^{1}\left(\frac{(1+\eta)(\eta-\varepsilon)}{(1-\eta)(\eta+\varepsilon)}\right)^{\beta}e^{-(2R+x+y)\eta}\,d\eta.$$

This operator can be written as a product UV, where $U : L^2[\varepsilon, 1] \to L^2(\mathbb{R}_+)$ and $V : L^2(\mathbb{R}_+) \to L^2[\varepsilon, 1]$ are given by

$$U(x, \xi) = -\frac{\sin(\pi\beta)}{\pi}\left(\frac{(1+\xi)(\xi-\varepsilon)}{(1-\xi)(\xi+\varepsilon)}\right)^{\beta/2}e^{-x\xi},$$
$$V(\eta, y) = \left(\frac{(1+\eta)(\eta-\varepsilon)}{(1-\eta)(\eta+\varepsilon)}\right)^{\beta/2}e^{-(2R+y)\eta}.$$

Under the assumption $-1 < \operatorname{Re}\beta < 1$, the operators U and V are Hilbert–Schmidt operators. The operator VU has the kernel

$$\begin{aligned} h(\eta,\xi) &= -\frac{\sin(\pi\beta)}{\pi}\left(\frac{(1+\eta)(\eta-\varepsilon)}{(1-\eta)(\eta+\varepsilon)}\frac{(1+\xi)(\xi-\varepsilon)}{(1-\xi)(\xi+\varepsilon)}\right)^{\beta/2}\int_0^\infty e^{-(2R+x)\eta-x\xi}\,dx \\ &= -\frac{\sin(\pi\beta)}{\pi}\left(\frac{(1+\eta)(\eta-\varepsilon)}{(1-\eta)(\eta+\varepsilon)}\frac{(1+\xi)(\xi-\varepsilon)}{(1-\xi)(\xi+\varepsilon)}\right)^{\beta/2}\frac{e^{-2R\eta}}{\eta+\xi}, \end{aligned}$$

which is the operator $\hat{K}_{\beta,\varepsilon,R}$. □

Proposition 3.18. *Let* $-1 < \pm\operatorname{Re}\beta < 1/2$. *Then*

$$\frac{\det\left[P_R(I\pm H(\hat{u}_\beta))^{-1}P_R\right]}{\det\left[P_n(I\pm H(u_\beta))^{-1}P_n\right]} = \lim_{\varepsilon\to 0}\frac{\det(I\pm\hat{K}_{\beta,\varepsilon,R})}{\det(I\pm K_{\beta,\varepsilon,n})}. \tag{70}$$

Proof. Applying Lemma 3.15 with $P = P_n$, $A = \pm H(u_{\beta,r})$, and $P = P_R$, $A = \pm H(\hat{u}_{\beta,\varepsilon})$, respectively, and Proposition 3.17, it follows that

$$\det\left[P_n(I\pm H(u_{\beta,r}))^{-1}P_n\right] = \frac{\det(I\pm K_{\beta,\varepsilon,n})}{\det(I\pm H(u_{\beta,r}))}, \tag{71}$$

$$\det\left[P_R(I\pm H(\hat{u}_{\beta,\varepsilon}))^{-1}P_R\right] = \frac{\det(I\pm \hat{K}_{\beta,\varepsilon,R})}{\det(I\pm H(\hat{u}_{\beta,\varepsilon}))}, \tag{72}$$

where $r = \frac{1-\varepsilon}{1+\varepsilon}$. By (35) and (36) the operators $H(u_{\beta,r})$ and $H(\hat{u}_{\beta,\varepsilon})$ are unitarily equivalent. The invertibility of $I \pm H(u_{\beta,r})$ for r sufficiently close to 1 follows from Lemma 3.7. Hence the fractions on the right-hand side of (71) and (72) are well-defined for $r \to 1$ and $\varepsilon \to 0$.

In fact, one can even say more. From Proposition 3.9 with ψ chosen as in the proof Proposition 3.10, it follows that $I \pm H(u_{\beta,r})$ is invertible for all $r \in [0,1)$. Similarly, from Proposition 3.13 with ψ chosen as in in proof of Proposition 3.14, it follows that $I \pm H(\hat{u}_{\beta,\varepsilon})$ is invertible for all $\varepsilon > 0$.

Taking the quotient of (71) and (72) and passing to the limit $\varepsilon \to 0$ we obtain the desired assertion by using Propositions 3.8 and 3.11. □

One remark is in order concerning the nonvanishing of the denominators of the fractions in (70). First of all, a careful examination of the expression for $D_n^{\pm}(\beta)$ as stated in the proof of Theorem 2.1 combined with the exact formulas of Proposition 3.10 imply that the determinants $\det\left[P_n(I\pm H(u_\beta))^{-1}P_n\right]$ are nonzero for all $n \geq 1$ and β satisfying $-3/2 < \pm\operatorname{Re}\beta < 1/2$. From Proposition 3.8 we can conclude that $\det\left[P_n(I\pm H(u_{\beta,r}))^{-1}P_n\right]$ are nonzero for r sufficiently close to 1. Formula (71) now implies that also $\det(I \pm K_{\beta,\varepsilon,n})$ is nonzero for $\varepsilon \to 0$.

Our next step is to determine the limit $\varepsilon \to 0$ on the right-hand side of (70). Before we are able to do this, we have to establish a couple of auxiliary results. Some of them will be needed only later on in order to analyze the expression which is obtained for the limit.

Let K_β^0, $K_{\beta,n}$ and $\hat{K}_{\beta,R}$ stand for the integral operators on $L^2[0,1]$ with the following kernels:

$$K_\beta^0(x,y) = -\frac{\sin(\pi\beta)}{\pi}\frac{1}{x+y}, \tag{73}$$

$$K_{\beta,n}(x,y) = -\frac{\sin(\pi\beta)}{\pi}\left(\frac{1-x}{1+x}\right)^{2n-\beta/2}\left(\frac{1-y}{1+y}\right)^{-\beta/2}\frac{1}{x+y}, \tag{74}$$

$$\hat{K}_{\beta,R}(x,y) = -\frac{\sin(\pi\beta)}{\pi}\left(\frac{1-x}{1+x}\right)^{-\beta/2}\left(\frac{1-y}{1+y}\right)^{-\beta/2}\frac{e^{-2Rx}}{x+y}. \tag{75}$$

Moreover, let H_β^0 and H_β stand for the integral operators with the following kernels on $L^2[1,\infty)$,

$$H_\beta^0(x,y) = -\frac{\sin(\pi\beta)}{\pi}\frac{1}{x+y}, \tag{76}$$

$$H_\beta(x,y) = -\frac{\sin(\pi\beta)}{\pi}\left(\frac{x-1}{x+1}\right)^{\beta/2}\left(\frac{y-1}{y+1}\right)^{\beta/2}\frac{1}{x+y}. \tag{77}$$

Finally, let Y_ε stand for the unitary operator

$$f(x) \in L^2[\varepsilon,1] \mapsto \sqrt{\varepsilon}f(\varepsilon x) \in L^2[1,\varepsilon^{-1}], \tag{78}$$

and let $\Pi_{[a,b]}$ stand for the projections operator $f(x) \mapsto \chi_{[a,b]}(x)f(x)$, which is thought of acting on appropriate spaces $L^2(M)$, $M \subset \mathbb{R}$.

In what follows we will prove that the above integral operators are bounded and that certain differences between them are even trace class. Moreover, certain invertibility results will be established, too.

Lemma 3.19. *The operators K_β^0 and H_β^0 are bounded. Moreover, for $\operatorname{Re}\beta \notin \pm 1/2 + 2\mathbb{Z}$, the operators $I \pm K_\beta^0$ and $I \pm H_\beta^0$ are invertible,*

$$\left(I \pm \Pi_{[\varepsilon,1]}K_\beta^0\Pi_{[\varepsilon,1]}\right)^{-1} \to (I \pm K_\beta^0)^{-1}, \qquad \varepsilon \to 0,$$

strongly on $L^2[0,1]$, and

$$\left(I \pm Y_\varepsilon\Pi_{[\varepsilon,1]}K_\beta^0\Pi_{[\varepsilon,1]}Y_\varepsilon^*\right)^{-1} \to (I \pm H_\beta^0)^{-1}, \qquad \varepsilon \to 0,$$

strongly on $L^2[1,\infty)$.

Proof. The operator on $L^2(\mathbb{R}_+)$ with the kernel $\pi^{-1}(x+y)^{-1}$ is a bounded selfadjoint operator with spectrum equal to $[0,1]$. Indeed, by a substitution $x \mapsto e^{-x}$, $y \mapsto e^{-y}$ it is easily seen that this operator is unitary equivalent to the integral operator on $L^2(\mathbb{R})$ with the kernel $(2\pi)^{-1}\operatorname{sech}((x-y)/2)$. This is a convolution operator with the symbol $\operatorname{sech}(\pi\xi)$, $\xi \in \mathbb{R}$, and thus its spectrum equals $[0,1]$.

The restrictions of this operator onto the spaces $L^2[0,1]$, $L^2[\varepsilon,1]$, $L^2[1,\varepsilon^{-1}]$ and $L^2[1,\infty)$ are also bounded selfadjoint operators with spectrum contained in (in fact, equal to) the interval $[0,1]$.

Hence under the above conditions on the parameter β, the operators

$$I \pm K_\beta^0, \quad I \pm H_\beta^0, \quad I \pm \Pi_{[\varepsilon,1]}K_\beta^0\Pi_{[\varepsilon,1]}, \quad I \pm Y_\varepsilon\Pi_{[\varepsilon,1]}K_\beta^0\Pi_{[\varepsilon,1]}Y_\varepsilon^*$$

are all bounded, and (in the last two cases) the norms of their inverses do not depend on ε. It remains to observe that

$$\Pi_{[\varepsilon,1]}K^0_\beta\Pi_{[\varepsilon,1]} \to K^0_\beta \quad \text{and} \quad Y_\varepsilon\Pi_{[\varepsilon,1]}K^0_\beta\Pi_{[\varepsilon,1]}Y^*_\varepsilon = \Pi_{[1,\varepsilon^{-1}]}H^0_\beta\Pi_{[1,\varepsilon^{-1}]} \to H^0_\beta$$

strongly on $L^2[0,1]$ and $L^2[1,\infty)$, respectively, as $\varepsilon \to 0$. □

The following lemma shows, in particular, that the operators $K_{\beta,n}$, $\hat{K}_{\beta,R}$ and H_β are bounded for certain β.

Lemma 3.20. *If* $\operatorname{Re}\beta < 1$, *then the operators*

$$K_{\beta,n} - K^0_\beta \quad \text{and} \quad \hat{K}_{\beta,R} - K^0_\beta$$

are trace class operators and

$$K_{\beta,n} - \hat{K}_{\beta,R} \to 0$$

in the trace norm as $R \to \infty$, $n \to \infty$, $R = 2n + O(1)$. *If* $\operatorname{Re}\beta > -1$, *then*

$$H_\beta - H^0_\beta$$

is a trace class operator.

Proof. The assertion that $K_{\beta,n} - K^0_\beta$ and $K_{\beta,R} - K^0_\beta$ are trace class operators can be proved by considering "intermediate" operators with the kernel

$$-\frac{\sin(\pi\beta)}{\pi}\left(\frac{1-y}{1+y}\right)^{-\beta/2}\frac{1}{x+y},$$

and by applying Lemma 3.16. Similarly, the fact that $H_\beta - H^0_\beta$ is trace class can be proved by introducing the operator with the kernel

$$-\frac{\sin(\pi\beta)}{\pi}\left(\frac{1-y}{1+y}\right)^{\beta/2}\frac{1}{x+y}.$$

Finally, the trace norm of $K_{\beta,n} - \hat{K}_{\beta,R}$ can be estimated by a constant times the square root of

$$\int_0^1 \left(\frac{1-y}{1+y}\right)^{-\operatorname{Re}\beta}\frac{dy}{y^{1/2}} \cdot \int_0^1 \left(\frac{1-x}{1+x}\right)^{-\operatorname{Re}\beta}\left|\left(\frac{1-x}{1+x}\right)^{2n} - e^{-2Rx}\right|^2\frac{dx}{x^{3/2}}.$$

The first integral is finite, and the second one can be split (for each $0 < \delta < 1$) into an integral from 0 to δ and an integral from δ to 1. The integral from δ to 1 is finite and converges to zero as $n, R \to \infty$. In the integral from 0 to δ we estimate the first term in the integrand by a constant (depending on δ) and make a substitution $x \mapsto x/(4n)$ to obtain an upper estimate

$$C_\delta n^{1/2}\int_0^{4n\delta}\left|\left(\frac{1-x/(4n)}{1+x/(4n)}\right)^{2n} - e^{-xR/(2n)}\right|^2\frac{dx}{x^{3/2}}.$$

This equals

$$C_\delta n^{1/2}\int_0^{4n\delta}\left|e^{-x+O(x^2/n)} - e^{-x+O(x/n)}\right|^2\frac{dx}{x^{3/2}}.$$

Omitting the constant C_δ, we split this integral into

$$n^{1/2}\int_0^{n^{1/3}} \left|e^{-x+O(x^2/n)} - e^{-x+O(x/n)}\right|^2 \frac{dx}{x^{3/2}} \\ = n^{1/2}\int_0^{n^{1/3}} e^{-2x} O\left(\frac{x^2}{n^{4/3}}\right)\frac{dx}{x^{3/2}} = O(n^{-5/6})$$

and

$$n^{1/2}\int_{n^{1/3}}^{4n\delta} \left|e^{-x+O(x^2/n)} - e^{-x+O(x/n)}\right|^2 \frac{dx}{x^{3/2}} \\ = n^{1/2}\int_{n^{1/3}}^{4n\delta} \left|e^{-x+O(\delta x)} - e^{-x+O(\delta)}\right|^2 \frac{dx}{x^{3/2}} = O(e^{-n^{1/3}}),$$

where the last estimate holds under the assumption that δ is chosen small enough to guarantee that $O(x\delta) \le x/2$. Collecting all terms, this proves the convergence of $K_{\beta,n} - \hat{K}_{\beta,R}$ in the trace norm. □

Lemma 3.21. *If* $-3/2 < \operatorname{Re}\beta < 1/2$*, then the inverses of*

$$I + K_{\beta,n} \quad \text{and} \quad I + \hat{K}_{\beta,R}$$

exist for sufficiently large n *and* R*, respectively, and are uniformly bounded. If* $-1/2 < \operatorname{Re}\beta < 1$*, then the inverses of*

$$I - K_{\beta,n} \quad \text{and} \quad I - \hat{K}_{\beta,R}$$

exist for sufficiently large n *and* R*, respectively, and are uniformly bounded.*

Proof. We prove the statements only for the case of the operators $\hat{K}_{\beta,R}$. The proof in the case of $K_{\beta,n}$ is analogous. Introduce the operator $\hat{K}'_{\beta,R}$ with the kernel

$$\hat{K}'_{\beta,R}(x,y) = -\frac{\sin(\pi\beta)}{\pi}\frac{e^{-2Rx}}{x+y}.$$

Using Lemma 3.16, the difference $\hat{K}_{\beta,R} - \hat{K}'_{\beta,R}$ can be estimated in the trace norm by a constant times the sum of the square roots of the integrals

$$\int_0^1 \left|\left(\frac{1-x}{1+x}\right)^{-\beta/2} - 1\right|^2 e^{-4Rx}\frac{dx}{x^{3/2}} \cdot \int_0^1 \left(\frac{1-y}{1+y}\right)^{-\operatorname{Re}\beta}\frac{dy}{y^{1/2}} \quad \text{and}$$

$$\int_0^1 e^{-4Rx}\frac{dx}{x^{1/2}} \cdot \int_0^1 \left|\left(\frac{1-y}{1+y}\right)^{-\beta/2} - 1\right|^2 \frac{dy}{y^{3/2}}.$$

These terms converge to zero as $R \to \infty$ (under the assumption $\operatorname{Re}\beta < 1$). Thus it is sufficient to prove that the inverses of $I \pm \hat{K}'_{\beta,R}$ are uniformly bounded. Now notice that $\hat{K}'_{\beta,R} = A_R^2 K_\beta^0$, where A_R is the multiplication operator with the symbol e^{-Rx}. Since A_R is uniformly bounded, the well-known relationship between the inverses of $I \pm AB$ and $I \pm BA$ implies that the remaining problem is reduced to showing that the inverses of $I \pm A_R K_\beta^0 A_R$ are uniformly bounded. It remains to observe that $A_R = A_R^*$, $A_R A_R^* \le I$ and that the operator with the kernel $1/(\pi(x+y))$ (i.e., K_β^0 without the sine-factor) is selfadjoint with its spectrum contained in $[0,1]$. The proof can now be completed as in Lemma 3.19. □

Lemma 3.22. *Let* $-1 < \operatorname{Re}\beta < 1$. *Then*

$$\Pi_{[\sqrt{\varepsilon},1]}K_{\beta,\varepsilon,n} = \Pi_{[\varepsilon,1]}K_{\beta,n}\Pi_{[\varepsilon,1]} + o_1(1), \tag{79}$$

$$\Pi_{[\sqrt{\varepsilon},1]}\hat{K}_{\beta,\varepsilon,R} = \Pi_{[\varepsilon,1]}\hat{K}_{\beta,R}\Pi_{[\varepsilon,1]} + o_1(1), \tag{80}$$

$$Y_\varepsilon\Pi_{[\varepsilon,\sqrt{\varepsilon}]}K_{\beta,\varepsilon,n}Y_\varepsilon^* = \Pi_{[1,\varepsilon^{-1}]}H_\beta\Pi_{[1,\varepsilon^{-1}]} + o_1(1), \tag{81}$$

$$Y_\varepsilon\Pi_{[\varepsilon,\sqrt{\varepsilon}]}\hat{K}_{\beta,\varepsilon,R}Y_\varepsilon^* = \Pi_{[1,\varepsilon^{-1}]}H_\beta\Pi_{[1,\varepsilon^{-1}]} + o_1(1) \tag{82}$$

as $\varepsilon \to 0$, *where* $o_1(1)$ *stands for a sequence of operators converging to zero in the trace norm.*

Proof. We are going to prove only the identities involving $\hat{K}_{\beta,\varepsilon,R}$. The assertions involving $K_{\beta,\varepsilon,n}$ can be proved analogously.

As to identity (80), we have to show that the integral operator on $L^2[\varepsilon,1]$ with the kernel

$$\chi_{[\sqrt{\varepsilon},1]}\left(\frac{(1-x)(1-y)}{(1+x)(1+y)}\right)^{-\beta/2}\left[\left(\frac{(x-\varepsilon)(y-\varepsilon)}{(x+\varepsilon)(y+\varepsilon)}\right)^{\beta/2}-1\right]\frac{e^{-2xR}}{x+y}$$

converges in the trace norm to zero. We split this kernel into the sum of the kernels

$$\chi_{[\sqrt{\varepsilon},1]}\left(\frac{(1-x)(1-y)}{(1+x)(1+y)}\right)^{-\beta/2}\left(\frac{x-\varepsilon}{x+\varepsilon}\right)^{\beta/2}\left[\left(\frac{y-\varepsilon}{y+\varepsilon}\right)^{\beta/2}-1\right]\frac{e^{-2xR}}{x+y}$$

and

$$\chi_{[\sqrt{\varepsilon},1]}\left(\frac{(1-x)(1-y)}{(1+x)(1+y)}\right)^{-\beta/2}\left[\left(\frac{x-\varepsilon}{x+\varepsilon}\right)^{\beta/2}-1\right]\frac{e^{-2xR}}{x+y}.$$

The first of these kernels can be estimated by

$$\int_{\sqrt{\varepsilon}}^1\left(\frac{1-x}{1+x}\right)^{-\operatorname{Re}\beta}\left(\frac{x-\varepsilon}{x+\varepsilon}\right)^{\operatorname{Re}\beta}\frac{dx}{x^{3/2}}\cdot\int_\varepsilon^1\left(\frac{1-y}{1+y}\right)^{-\operatorname{Re}\beta}\left|\left(\frac{y-\varepsilon}{y+\varepsilon}\right)^{\beta/2}-1\right|^2\frac{dy}{y^{1/2}},$$

and the second one can be estimated by

$$\int_{\sqrt{\varepsilon}}^1\left(\frac{1-x}{1+x}\right)^{-\operatorname{Re}\beta}\left|\left(\frac{x-\varepsilon}{x+\varepsilon}\right)^{\beta/2}-1\right|^2\frac{dx}{x^{3/2}}\cdot\int_\varepsilon^1\left(\frac{1-y}{1+y}\right)^{-\operatorname{Re}\beta}\frac{dy}{y^{1/2}}.$$

We split off from all these integrals integrals from $1/2$ to 1 in order to get rid of the singularity at 1. In the remaining integrals (from $\sqrt{\varepsilon}$ to $1/2$ and ε to $1/2$, resp.) we make a substitution $x = \sqrt{\varepsilon}z$ and $y = \varepsilon z$, respectively. Collecting all terms we obtain $(O(1)+O(\varepsilon^{-1/4}))(O(\varepsilon^2)+O(\varepsilon^{1/2})) = O(\varepsilon^{1/4})$ for the first expression and $(O(\varepsilon)+O(\varepsilon^{3/4}))O(1) = O(\varepsilon^{3/4})$ for the second expression. Hence both terms converge to zero as $\varepsilon \to 0$.

As to (82), we have to prove that the integral operator on $L^2[1,\varepsilon^{-1}]$ with the kernel

$$\chi_{[1,1/\sqrt{\varepsilon}]}(x)\left(\frac{(x-1)(y-1)}{(x+1)(y+1)}\right)^{\beta/2}\left[\left(\frac{(1-x\varepsilon)(1-y\varepsilon)}{(1+x\varepsilon)(1+y\varepsilon)}\right)^{-\beta/2}e^{-2Rx\varepsilon}-1\right]\frac{1}{x+y}$$

tends to zero in the trace norm. We split this kernel into

$$\chi_{[1,1/\sqrt{\varepsilon}]}(x)\left(\frac{(x-1)(y-1)}{(x+1)(y+1)}\right)^{\beta/2}\left(\frac{1-x\varepsilon}{1+x\varepsilon}\right)^{-\beta/2}e^{-2Rx\varepsilon}\left[\left(\frac{1-y\varepsilon}{1+y\varepsilon}\right)^{-\beta/2}-1\right]\frac{1}{x+y}$$

and

$$\chi_{[1,1/\sqrt{\varepsilon}]}(x)\left(\frac{(x-1)(y-1)}{(x+1)(y+1)}\right)^{\beta/2}\left[\left(\frac{1-x\varepsilon}{1+x\varepsilon}\right)^{-\beta/2}e^{-2Rx\varepsilon}-1\right]\frac{1}{x+y}.$$

These kernels can be estimated by

$$\int_1^{1/\sqrt{\varepsilon}}\left(\frac{x-1}{x+1}\right)^{\operatorname{Re}\beta}\left(\frac{1-x\varepsilon}{1+x\varepsilon}\right)^{-\operatorname{Re}\beta}\frac{dx}{x^{1/2}} \times\int_1^{1/\varepsilon}\left(\frac{y-1}{y+1}\right)^{\operatorname{Re}\beta}\left|\left(\frac{1-y\varepsilon}{1+y\varepsilon}\right)^{-\beta/2}-1\right|^2\frac{dy}{y^{3/2}}$$

and

$$\int_1^{1/\sqrt{\varepsilon}}\left(\frac{x-1}{x+1}\right)^{\operatorname{Re}\beta}\left|\left(\frac{1-x\varepsilon}{1+x\varepsilon}\right)^{-\beta/2}e^{-2Rx\varepsilon}-1\right|^2\frac{dx}{x^{1/2}}\cdot\int_1^{1/\varepsilon}\left(\frac{y-1}{y+1}\right)^{\operatorname{Re}\beta}\frac{dy}{y^{3/2}}.$$

By a subtitution $x\mapsto 1/x$, $y\mapsto 1/y$, these integrals become precisely the above integrals (with β replaced by $-\beta$) except that in one integral a term $e^{-2R\varepsilon/x}$ appears, which does affect not the argumentation. Hence also these terms converge to zero as $\varepsilon\to 0$. □

In view of the following proposition, let us make the following observations. From Lemma 3.19 and Lemma 3.20 it follows that

$$\det(I\pm K_\beta^0)^{-1}(I\pm K_{\beta,n})\quad\text{and}\quad\det(I\pm K_\beta^0)^{-1}(I\pm\hat{K}_{\beta,R})$$

are well-defined operator determinants for $-3/2<\operatorname{Re}\beta<1/2$ (in the "+"-case) and $-1/2<\operatorname{Re}\beta<1$ (in the "−"-case). Moreover, by Lemma 3.21 these operator determinants are nonzero for sufficiently large n and R.

Furthermore it follows that the operator determinant

$$\det(I\pm H_\beta^0)^{-1}(I\pm H_\beta)$$

is well-defined for $-1<\operatorname{Re}\beta<1/2$ (in the "+"-case) and $-1/2<\operatorname{Re}\beta<3/2$ (in the "−"-case), respectively. This operator determinant represents a not identically vanishing analytic function in β (since it equals 1 for $\beta=0$), and thus it is nonzero except possibly on a discrete set.

Finally, from Lemma 3.19 and its proof we can conclude that the determinants $\det(I\pm\Pi_{[\varepsilon,1]}K_\beta^0\Pi_{[\varepsilon,1]})$ are nonzero for all β satisfying $-3/2<\pm\operatorname{Re}\beta<1/2$ and all $\varepsilon>0$.

Proposition 3.23. *Let $-1 < \pm\mathrm{Re}\,\beta < 1/2$. Then for each $n \geq 1$ and $R > 0$ we have*

$$\lim_{\varepsilon\to 0}\frac{\det(I \pm K_{\beta,\varepsilon,n})}{\det(I \pm \Pi_{[\varepsilon,1]}K^0_\beta\Pi_{[\varepsilon,1]})} = \det(I \pm H^0_\beta)^{-1}(I \pm H_\beta)\cdot\det(I \pm K^0_\beta)^{-1}(I \pm K_{\beta,n}), \tag{83}$$

$$\lim_{\varepsilon\to 0}\frac{\det(I \pm \hat{K}_{\beta,\varepsilon,R})}{\det(I \pm \Pi_{[\varepsilon,1]}K^0_\beta\Pi_{[\varepsilon,1]})} = \det(I \pm H^0_\beta)^{-1}(I \pm H_\beta)\cdot\det(I \pm K^0_\beta)^{-1}(I \pm \hat{K}_{\beta,R}). \tag{84}$$

Proof. First of all we can write

$$\det(I \pm \hat{K}_{\beta,\varepsilon,R}) = \det(I \pm \Pi_{[\varepsilon,1]}K^0_\beta\Pi_{[\varepsilon,1]})\det(I \pm A_\varepsilon \pm B_\varepsilon),$$

where

$$A_\varepsilon := (I \pm \Pi_{[\varepsilon,1]}K^0_\beta\Pi_{[\varepsilon,1]})^{-1}\Pi_{[\varepsilon,\sqrt{\varepsilon}]}(\hat{K}_{\beta,\varepsilon,R} - \Pi_{[\varepsilon,1]}K^0_\beta\Pi_{[\varepsilon,1]}),$$

$$B_\varepsilon := (I \pm \Pi_{[\varepsilon,1]}K^0_\beta\Pi_{[\varepsilon,1]})^{-1}\Pi_{[\sqrt{\varepsilon},1]}(\hat{K}_{\beta,\varepsilon,R} - \Pi_{[\varepsilon,1]}K^0_\beta\Pi_{[\varepsilon,1]}).$$

Equation (80) along with the fact that $\hat{K}_{\beta,R} - K^0_\beta$ is trace class implies that

$$A_\varepsilon = A + o_1(1), \qquad A := (I \pm K^0_\beta)^{-1}(\hat{K}_{\beta,R} - K^0_\beta)$$

(see also Lemma 3.19). Similarly, Equation (82) implies

$$Y_\varepsilon B_\varepsilon Y^*_\varepsilon = B + o_1(1), \qquad B := (I \pm H^0_\beta)^{-1}(H_\beta - H^0_\beta).$$

Moreover,

$$B_\varepsilon A_\varepsilon = Y^*_\varepsilon B Y_\varepsilon A + o_1(1) = o_1(1)$$

since $Y_* \to 0$ weakly. Hence we can conclude that

$$\lim_{\varepsilon\to 0}\frac{\det(I \pm \hat{K}_{\beta,\varepsilon,R})}{\det(I \pm \Pi_{[\varepsilon,1]}K^0_\beta\Pi_{[\varepsilon,1]})} = \lim_{\varepsilon\to 0}\det(I \pm A_\varepsilon)\det(I \pm B_\varepsilon) = \det(I \pm A)\det(I \pm B),$$

which proves the assertion (84). The case of the determinant $\det(I \pm K_{\beta,\varepsilon,n})$ can be treated analogously. □

The previous proposition puts us in position to identify the limit on the right-hand side of (70).

Proposition 3.24. *Let $-3/2 < \mathrm{Re}\,\beta < 1/2$ (in the "+"-case) or $-1/2 < \mathrm{Re}\,\beta < 1$ (in the "−"-case), respectively. Then for all sufficiently large n and R,*

$$\frac{\det\big[P_R(I \pm H(\hat{u}_\beta))^{-1}P_R\big]}{\det\big[P_n(I \pm H(u_\beta))^{-1}P_n\big]} = \det(I \pm K_{\beta,n})^{-1}(I \pm \hat{K}_{\beta,R}). \tag{85}$$

Proof. For $-1 < \pm\mathrm{Re}\,\beta < 1/2$ and β not belonging to a certain discrete set (namely the set where $\det(I \pm H^0_\beta)^{-1}(I \pm H_\beta)$ is zero), we can take the quotient of (83) and (84) and obtain

$$\lim_{\varepsilon\to 0}\frac{\det(I \pm \hat{K}_{\beta,\varepsilon,R})}{\det(I \pm K_{\beta,\varepsilon,n})} = \det(I \pm K_{\beta,n})^{-1}(I \pm \hat{K}_{\beta,R}). \tag{86}$$

Recall that in a remark made after Proposition 3.18 we have observed that the determinant $\det(I \pm K_{\beta,\varepsilon,n})$ is nonzero for $\varepsilon > 0$ sufficiently small. Applying Proposition 3.18 we obtain identity (85) under the assumptions that $-1 < \pm\mathrm{Re}\,\beta < 1/2$ and that β does not belong to a certain discrete subset. We can remove this extra assumption since both sides of the equality are analytic in β. □

Theorem 3.25. *Let $-3/2 < \mathrm{Re}\,\beta < 1/2$ (in the "+"-case) or $-1/2 < \mathrm{Re}\,\beta < 1$ (in the "−"-case), respectively. Then*

$$\det\bigl[P_R(I \pm H(\hat{u}_\beta))^{-1}P_R\bigr] \sim \det\bigl[P_n(I \pm H(u_\beta))^{-1}P_n\bigr] \tag{87}$$

as $R, n \to \infty$ and $R = 2n + O(1)$.

Proof. This follows from the previous proposition in connection with Lemma 3.20 and Lemma 3.21. □

3.6. Proof of the main results and remarks. Now are able to prove the main results.

Proof of Theorem 1.1. We notice first that the proof of the first statement in Theorem 1.1(b) follows from Proposition 3.14(b).

From Theorem 2.1 and Proposition 3.10 it follows that

$$\det\left[P_n(I + H(u_\beta))^{-1}P_n\right] \sim n^{\beta^2/2+\beta/2}(2\pi)^{-\beta/2}2^{-\beta^2/2}\frac{G(1/2)}{G(1/2-\beta)}, \quad n \to \infty, \tag{88}$$

for $-3/2 < \mathrm{Re}\,\beta < 1/2$ and

$$\det\left[P_n(I - H(u_\beta))^{-1}P_n\right] \sim n^{\beta^2/2-\beta/2}(2\pi)^{-\beta/2}2^{-\beta^2/2}\frac{G(3/2)}{G(3/2-\beta)}, \quad n \to \infty, \tag{89}$$

for $-1/2 < \mathrm{Re}\,\beta < 3/2$. With $n = [R/2]$ we can apply Theorem 3.25, and we obtain

$$\det\left[P_R(I + H(\hat{u}_\beta))^{-1}P_R\right] \sim R^{\beta^2/2+\beta/2}(2\pi)^{-\beta/2}2^{-\beta^2-\beta/2}\frac{G(1/2)}{G(1/2-\beta)}, \quad R \to \infty, \tag{90}$$

for $-3/2 < \mathrm{Re}\,\beta < 1/2$ and

$$\det\left[P_R(I - H(\hat{u}_\beta))^{-1}P_R\right] \sim R^{\beta^2/2-\beta/2}(2\pi)^{-\beta/2}2^{-\beta^2+\beta/2}\frac{G(3/2)}{G(3/2-\beta)}, \quad R \to \infty, \tag{91}$$

for $-1/2 < \mathrm{Re}\,\beta < 1$. The proof now follows from Proposition 3.14. □

Let us conclude with some final observations. The results of the previous sections allow us to establish formulas for the determinants of $\det\bigl[P_n(I \pm H(u_\beta))^{-1}P_n\bigr]$ and $\det\bigl[P_R(I \pm H(\hat{u}_\beta))^{-1}P_R\bigr]$ in terms of certain operator determinants. We are able to evaluate some (but not all) of these determinants explicitly.

The formulas that we obtain might give rise to an alternative (perhaps clearer) proof of the main result in the sense that one avoids taking the quotient of the determinants corresponding to the discrete and continuous part right from the beginning. This would eliminate the annoying discussion of the nonvanishing of several determinants.

Before establishing the formulas for the determinants $\det\big[P_n(I\pm H(u_\beta))^{-1}P_n\big]$ and $\det\big[P_R(I\pm H(\hat{u}_\beta))^{-1}P_R\big]$, we are going to evaluate the asymptotics of a truncated Wiener–Hopf determinant with a specific, well-behaved symbol. The result might be of interest in its own since precisely this symbol appears also elsewhere.

Lemma 3.26. *Let* $\phi_\beta(\xi)=1-\sin(\pi\beta)\mathrm{sech}\,(\pi\xi)$, $\xi\in\mathbb{R}$, *and* $-3/2<\mathrm{Re}\,\beta<1/2$. *Then*

$$\det W_s(\phi_\beta)\sim e^{-s(\beta/2+\beta^2/2)}\frac{G^2(3/2+\beta/2)G^2(1+\beta/2)G^2(1-\beta/2)G^2(1/2-\beta/2)}{G(1/2)G(3/2)G(3/2+\beta)G(1/2-\beta)}$$

as $s\to\infty$.

Proof. Using the Akhiezer–Kac formula (see e.g., [14, Sect. 10.80]), we obtain

$$\det W_s(\phi_\beta)\sim G[\phi_\beta]^s E[\phi_\beta],\quad s\to\infty,$$

where $G[\phi_\beta]$ is given by (61) and evaluates to $\exp(-\beta/2-\beta^2/2)$. The constant $E[\phi_\beta]$ is given by

$$\begin{aligned}E[\phi_\beta]&=\exp\left(\int_0^\infty x\,(\mathcal{F}(\log\phi_\beta)(x))(\mathcal{F}(\log\phi_\beta)(-x))\,dx\right)\\&=\exp\left(\frac{-i}{2\pi}\int_{-\infty}^{\infty}(\log\phi_{\beta,+})'(x)(\log\phi_{\beta,-})(x)\,dx\right),\end{aligned}$$

where $\mathcal{F}$ is the Fourier transform (3). The functions $\phi_{\beta,\pm}$ stand for the factors of the Wiener–Hopf factorization ϕ_β. It is possible to compute these factors explicitly, and one obtains $\phi_{\beta,\pm}(x)=\psi_\beta(\mp ix/2)$ with

$$\psi_\beta(z)=\frac{\Gamma(3/4+z)\Gamma(1/4+z)}{\Gamma(3/4+\beta/2+z)\Gamma(1/4-\beta/2+z)}.$$

This function is analytic in the right-half plane and has the appropriate behavior at infinity. Replacing $\phi_{\beta,\pm}$ by ψ_β and making a change of variables $z=ix$ gives

$$E[\phi_\beta]=\exp\left(\frac{1}{2\pi i}\int_{+i\infty}^{-i\infty}\frac{\psi_\beta'(-z)}{\psi_\beta(-z)}\,\log\psi_\beta(z)\,dz\right).$$

A complex function argument implies that this equals the exponential of the residues of the expression under the integral in the right half plane. Notice that due to the logarithmic derivative only simple poles are involved. Thus $E[\phi_\beta]$ equals the exponential of the sum $(n=0,1,\dots)$ of

$$\log\psi_\beta(n+3/4)+\log\psi_\beta(n+1/4)-\log\psi_\beta(n+3/4+\beta/2)-\log\psi_\beta(n+1/4-\beta/2).$$

A straightforward computation now gives

$$\begin{aligned}E[\phi_\beta]=\prod_{n=0}^{\infty}&\frac{\Gamma(3/2+n)\Gamma(1+n)}{\Gamma(3/2+\beta/2+n)\Gamma(1-\beta/2+n)}\cdot\frac{\Gamma(1+n)\Gamma(1/2+n)}{\Gamma(1+\beta/2+n)\Gamma(1/2-\beta/2+n)}\\&\times\frac{\Gamma(3/2+\beta+n)\Gamma(1+n)}{\Gamma(3/2+\beta/2+n)\Gamma(1+\beta/2+n)}\cdot\frac{\Gamma(1+n)\Gamma(1/2-\beta+n)}{\Gamma(1-\beta/2+n)\Gamma(1/2-\beta/2+n)}.\end{aligned}$$

Using the recursion relation for the Barnes G-function we obtain that

$$E[\phi_\beta]=\frac{G^2(3/2+\beta/2)G^2(1+\beta/2)G^2(1-\beta/2)G^2(1/2-\beta/2)}{G(1/2)G(3/2)G(3/2+\beta)G(1/2-\beta)}\cdot R$$

where

$$R = \lim_{n\to\infty} \frac{G(1/2+n)G(3/2+n)G(3/2+\beta+n)G(1/2-\beta+n)G^4(1+n)}{G^2(3/2+\beta/2+n)G^2(1+\beta/2+n)G^2(1-\beta/2+n)G^2(1/2-\beta/2+n)}.$$

Using (21) we conclude that $R = 1$, which settles the assertion. □

Theorem 3.27. *Let $-1 < \pm\mathrm{Re}\,\beta < 1/2$. Then for all $n \geq 1$ and $R > 0$ we have*

$$\begin{aligned}(92)\qquad & \det P_n(I \pm H(u_\beta))^{-1}P_n \\ &= C_{\pm\beta}\cdot \det(I\pm H_\beta^0)^{-1}(I\pm H_\beta)\cdot\det(I\pm K_\beta^0)^{-1}(I\pm K_{\beta,n}),\end{aligned}$$

$$\begin{aligned}(93)\qquad & \det P_R(I \pm H(u_\beta))^{-1}P_R \\ &= C_{\pm\beta}\cdot \det(I\pm H_\beta^0)^{-1}(I\pm H_\beta)\cdot\det(I\pm K_\beta^0)^{-1}(I\pm \hat{K}_{\beta,R}),\end{aligned}$$

where

$$C_\beta = 2^{\beta^2}\frac{G(1/2)G(3/2)G(3/2+\beta)G(1/2-\beta)}{G^2(3/2+\beta/2)G^2(1+\beta/2)G^2(1-\beta/2)G^2(1/2-\beta/2)}.$$

Proof. We obtain these fromulas from the identities (71) and (72), from Propositions 3.8 and 3.11 and from Proposition 3.23 with the constants

$$C_{\pm\beta} = \lim_{\varepsilon\to 0}\frac{\det(I\pm H(\hat{u}_{\beta,\varepsilon}))}{\det(I\pm \Pi_{[\varepsilon,1]}K_\beta^0\Pi_{[\varepsilon,1]})}.$$

Notice in this connection that $\det(I \pm H(\hat{u}_{\beta,\varepsilon})) = \det(I \pm H(u_{\beta,r}))$ for $\varepsilon = \frac{1-r}{1+r}$. It remains to evaluate these constants $C_{\pm\beta}$. This will be done in two steps by establishing an asymptotic formula for $\det(I \pm H(u_{\beta,r}))$ as $r \to 1$ and for the determinant $\det(I \pm \Pi_{[\varepsilon,1]}K_\beta^0\Pi_{[\varepsilon,1]})$ as $\varepsilon \to 0$.

For the evaluation of $\det(I \pm H(u_{\beta,r}))$ we rely on the results of [3] (see Theorem 2.5 and formulas (1.12) and (2.15) therein). These results say that for a sufficiently smooth nonvanishing function b on $\mathbb{T}$ with winding number zero the identity

$$\det(I + T^{-1}(b)H(b)) = \left(\frac{b_+(1)}{b_+(-1)}\right)^{1/2}\exp\left(-\frac{1}{2}\sum_{k=1}^{\infty}k[\log b]_k^2\right)$$

holds, where b_+ is the plus-factor of the Wiener–Hopf factorization of b. We apply this formula with $b(t) = b_+(t) = (1-rt)^\beta$ and $b(t) = b_+(t) = (1+rt)^\beta$, respectively. We notice that

$$\det(I + T^{-1}(b)H(b)) = \det(I + H(b_+)T(b_+^{-1})) = \det(I + H(b_+\widetilde{b}_+^{-1})),$$

which is equal to $\det(I \pm H(u_{\beta,r}))$. Notice that we rely on formula (51) in the "$-$"-case. The evaluation of the right-hand side gives

$$\det(I\pm H(u_{\beta,r})) = \left(\frac{1-r}{1+r}\right)^{\pm\beta/2}(1-r^2)^{\beta^2/2} \sim \varepsilon^{\pm\beta/2+\beta^2/2}\,2^{\beta^2},\quad \varepsilon\to 0.$$

The determinant $\det(I \pm \Pi_{[\varepsilon,1]}K_\beta^0\Pi_{[\varepsilon,1]})$ can be expressed as the determinant of a finite tructation of a Wiener–Hopf operator. Proceeding as in the proof of Lemma 3.19, the operator K_β^0 is unitarily equivalent to a Wiener–Hopf operator

$W(a_\beta)$ with the symbol $a_\beta(\xi) = -\sin(\pi\beta)\mathrm{sech}\,(\pi\xi)$, $\xi \in \mathbb{R}$, while the projections $\Pi_{[\varepsilon,1]}$ transform into $\Pi_{[0,-\log\varepsilon]}$. Thus

$$\det(I \pm \Pi_{[\varepsilon,1]} K^0_\beta \Pi_{[\varepsilon,1]}) = \det W_{-\log\varepsilon}(1 \pm a_\beta).$$

Applying Lemma 3.26 with $s = -\log\varepsilon$ and $\phi_{\pm\beta} = 1 \pm a_\beta$ we obtain that the determinant $\det(I \pm \Pi_{[\varepsilon,1]} K^0_\beta \Pi_{[\varepsilon,1]})$ is asymptotically equal to $\varepsilon^{\pm\beta/2+\beta^2/2}$ times the product of the Barnes functions (with β replaced by $-\beta$ in the "$-$"-case) appearing in Lemma 3.26.

Combining both asymptotics yields the desired expression of $C_{\pm\beta}$. □

References

[1] E.W. Barnes, *The theory of the G-function*, Quart. J. **31** (1899), 264–313, JFM 30.0389.02.

[2] E.L. Basor and Y. Chen, *Toeplitz determinants from compatibility conditions*, preprint arXiv math.CA/0402362, to appear in the Ramanujan Journal.

[3] E.L. Basor and T. Ehrhardt, *Asymptotic formulas for the determinants of a sum of finite Toeplitz and Hankel matrices*, Math. Nachr. **228** (2001), 5–45, MR 1845906 (2002d:47041), Zbl 1047.47020.

[4] E.L. Basor and T. Ehrhardt, *Asymptotic formulas for the determinants of symmetric Toeplitz plus Hankel matrices*, Toeplitz matrices and singular integral equations (Pobershau, 2001), Oper. Theory Adv. Appl., **135**, Birkhäuser, Basel, 2002, 61–90, MR 1935758 (2003h:47049), Zbl 1025.15011.

[5] E.L. Basor and T. Ehrhardt, *Asymptotics of determinants of Bessel operators*, Comm. Math. Phys. **234** (2003), 491–516, MR 1964380 (2004a:47056), Zbl 1022.47015.

[6] E.L. Basor, T. Ehrhardt and H. Widom, *On the determinant of a certain Wiener–Hopf + Hankel operator*, Integral Equations Operator Theory **47**, no. 3, (2003), 257–288, MR 2012839 (2004h:47041), Zbl 1042.47019.

[7] E.L. Basor and H. Widom, *On a Toeplitz determinant identity of Borodin and Okounkov*, Integral Equations Operator Theory **37**, no. 4, (2000), 397–401 MR 1780119 (2001g:47042b), Zbl 0967.47020.

[8] E.L. Basor and H. Widom, *Wiener–Hopf determinants with Fisher–Hartwig symbols*, Operator theoretical methods and applications to mathematical physics, Oper. Theory Adv. Appl., **147**, Birkhäuser, Basel, 2004, 131–149, MR 2053687 (2005c:47031).

[9] A. Borodin and A. Okounkov, *A Fredholm determinant formula for Toeplitz determinants*, Integral Equations Operator Theory **37**, no. 4, (2000), 386–396, MR 1780118 (2001g:47042a), Zbl 0970.47014.

[10] A. Böttcher, *On the determinant formulas by Borodin, Okounkov, Baik, Deift and Rains*, Toeplitz matrices and singular integral equations (Pobershau, 2001), Oper. Theory Adv. Appl., **135**, Birkhäuser, Basel, 2002, 91–99, MR 1935759 (2003j:47030), Zbl 1040.47016.

[11] A. Böttcher, personal communication, January 2005.

[12] A. Böttcher, *Two elementary derivations of the pure Fisher–Hartwig determinant*, preprint arXiv math.FA/0312198, to appear in Integral Equations Operator Theory.

[13] A. Böttcher and B. Silbermann, *Toeplitz matrices and determinants with Fisher–Hartwig symbols*, J. Funct. Anal. **63** (1985), 178–214, MR 0803092 (87b:47026), Zbl 0592.47016.

[14] A. Böttcher and B. Silbermann, *Analysis of Toeplitz operators*, Springer, Berlin 1990, MR 1071374 (92e:47001), Zbl 0732.47029.

[15] A. Böttcher, B. Silbermann and H. Widom, *Determinants of truncated Wiener–Hopf operators with Hilbert–Schmidt kernels and piecewise continuous symbols*, Arch. Math. (Basel) **63** (1994), 60–71, MR 1277912 (95e:47033), Zbl 0818.47019.

[16] F.J. Dyson, *Fredholm determinants and inverse scattering problems*, Comm. Math. Phys. **47** (1976), 171–183, MR 0406201 (53 #9993), Zbl 0323.33008.

[17] T. Ehrhardt, *A status report on the asymptotic behavior of Toeplitz determinants with Fisher–Hartwig singularities*, Recent advances in operator theory (Groningen, 1998), Oper. Theory Adv. Appl., **124**, Birkhäuser, Basel 2001, 217–241, MR 1839838 (2002e:47032), Zbl 0993.47028.

[18] T. Ehrhardt, *Dyson's constant in the asymptotics of the Fredholm determinant of the sine kernel*, preprint arXiv math.FA/0401205, submitted to Comm. Math. Phys.

[19] T. Ehrhardt and B. Silbermann, *Approximate identities and stability of discrete convolution operators with flip*, The Maz'ya anniversary collection, Vol. 2 (Rostock, 1998), Operator Theory Adv. Appl., **110**, Birkhäuser, Basel, 1999, 103–132, MR 1747891 (2001e:47049), Zbl 0938.45011.

[20] I. Gohberg and M.G. Krein, *Introduction to the theory of linear nonselfadjoint operators*, Transl. Math. Monographs, **18**, American Mathematical Society, Providence, R.I., 1969, MR 0246142 (39 #7447), Zbl 0181.13504.

[21] S. Power, *The essential spectrum of a Hankel operator with piecewise continuous symbol*, Michigan Math. J. **25** (1978), 117–121, MR 0487567 (58 #7186), Zbl 0365.47016.

[22] S. Power, *C*-algebras generated by Hankel operators and Toeplitz operators*, J. Funct. Anal. **31** (1979), 52–68, MR 0523113 (80d:47043), Zbl 0401.46036.

[23] H. Widom, *Asymptotics for the Fredholm determinant of the sine kernel on a union of intervals*, Comm. Math. Phys. **171** (1995), 159–180, MR 1341698 (96i:47050), Zbl 0839.47032.

Department of Mathematics, California Polytechnic State University, San Luis Obispo, CA 93407, USA
ebasor@calpoly.edu

Department of Mathematics, University of California, Santa Cruz, CA 95064, USA
ehrhardt@math.ucsc.edu

This paper is available via http://nyjm.albany.edu:8000/j/2005/11-10.html.