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Equivariant rigidity theorems

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Abstract. Let Γ be a discrete group which is a split extension of a group Δ by a Coxeter group W, with Δ acting on W by Coxeter graph automorphisms with kernel Δ_0. Let M_i, $i = 1, 2$, be two Γ-manifolds (possibly with boundary) such that the isotropy groups are finite and the fixed point sets are contractible and W acts by reflections. Let f be a Γ-homotopy equivalence between them that it is a homeomorphism outside the orbit of a compact subset. Then f is Γ-homotopic to a Γ-homeomorphism, provided that certain finite extensions of Δ_0 that fix the faces of the fundamental domains are topologically rigid groups.

Contents

1. Introduction

Let Γ be a discrete group. A *manifold of type* $\mathcal{E}\Gamma$ is a manifold, without boundary, on which Γ acts properly discontinuously, and so that the fixed point sets of finite subgroups are contractible ([10]). If the manifold has boundary, then we call it a *manifold with boundary of type* $\mathcal{E}\Gamma$. Such manifolds are unique, up to Γ-homotopy

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([10], [16, Appendix]). We call a group Γ *topologically rigid* if any two manifolds, of type $\mathcal{E}\Gamma$ that are Γ-homotopy equivalent, with a map that it is the identity outside the orbit of a compact subset, are Γ-homeomorphic. This definition is the analogue of the assumptions given in [17]. In our case, Γ is not necessarily torsion-free. The purpose of this paper is to show that a class of geometrically interesting groups is topologically rigid.

The rigidity problem of group actions has already appeared in topology in different forms. For Γ a torsion-free group, this is a classical problem in geometric topology. It is associated with two conjectures. The first, Wall's Conjecture, states that every Poincaré Duality group is the fundamental group of a closed aspherical manifold. The second, Borel's Conjecture, asserts that the fundamental groups of closed aspherical manifolds are topologically rigid. The status of these conjectures is reported in the papers of Farrell–Jones ([15], [16], [17]).

When Γ has torsion then the characterization of topologically rigid groups (at least for manifolds without boundaries) becomes a question in equivariant topology. As such, it is known as the Borel–Quinn Conjecture, stated explicitly in [11]. There are examples of discrete groups (which are crystallographic) that are not topologically rigid ([12], [28]).

In [23], it was shown that Coxeter groups are topologically rigid, if certain low dimensional conditions are satisfied, which are not needed if the three-dimensional Poincaré Conjecture is true ([22]). We will extend the rigidity result to certain group extensions of Coxeter groups. Let (W, S) be a Coxeter group such that the simplicial complex of the poset of its finite parabolic subgroups is an orientable pseudomanifold ([9]). Let $\Gamma = W \rtimes \Delta$ where W is a Coxeter group as before, $\mathcal{C}(W, S)$ the Coxeter graph, and Δ acts on W by automorphisms of $\mathcal{C}(W, S)$. Thus there is an exact sequence

$$1 \to \Delta_0 \to \Delta \xrightarrow{\alpha} \mathrm{Aut}(\mathcal{C}(W, S)).$$

We assume that:

- There is a manifold X of type $\mathcal{E}\Gamma$ on which W acts by reflections.
- For each subgroup H of $\mathrm{Aut}(\mathcal{C}(W, S))$, the group $\alpha^{-1}(H)$ is topologically rigid.

Theorem (Main Theorem)**.** *Let Γ be a virtually torsion-free group as above. Then any two manifolds, possibly with boundary, properly Γ-homotopy equivalent to X are Γ-homeomorphic* (*provided the homotopy equivalence may be taken to be a homeomorphism on the boundary*)*.*

Groups that satisfy the conditions (except the rigidity) of the Main Theorem appear as subgroups of Coxeter groups. Let (V, T) be a Coxeter system such that V admits a manifold of type $\mathcal{E}V$. Let V_J be a finite parabolic subgroup, i.e., a finite subgroup generated by a subset $J \subset T$. Then the Weyl group $N_V(V_J)/V_J$ is isomorphic to subgroup of V that satisfies the conditions of the theorem ([3], [7]). In this case, Δ_0 is topologically rigid because it is a torsion-free ([3]) subgroup of $GL(n, \mathbb{R})$ ([17]). It should be noticed that the Weyl groups $N_V(V_J)/V_J$ are nonpositively curved groups. That follows from the fact that Coxeter groups are nonpositively curved ([21]).

The result of the Main Theorem generalizes the result in [23], where it was shown that Coxeter groups are topologically rigid, under certain low dimensional assumptions. The methods used for the proof of the main result are equivariant analogues of the methods used in [23]. Let M_i, $i = 1, 2$, be manifolds of type $\mathcal{E}\Gamma$, possibly with boundary. The model X guarantees that W acts by reflections on M_i. Thus there are fundamental domains $(Q_i, (Q_{is})_{s\in S})$, $i = 1, 2$, which are panel spaces and Δ acts on them by homeomorphisms that preserve the panel structure. Here we use the assumption on W to get that they are both S-panel spaces ([23], [9]). As in [23], we construct an S-panel Δ-homotopy equivalence ϕ between Q_1 and Q_2. The topological rigidity assumption on extensions of Δ_0 implies that ϕ is Δ-homotopic to a Δ-homeomorphism χ which preserves panels. The construction of the homeomorphism is done inductively on the panels as in [23]. Then the map induced by χ on M_1 is a Γ-homeomorphism. The general rigidity assumption is necessary because of the counterexamples in [12] and [28]. The main result is stated as Theorem 5.6.

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2. Preliminaries

We review the basic properties of Coxeter groups. References are [4], [19], and [18], [8] for a more geometric approach.

A *Coxeter system* (W, S) is a pair where W is a group generated by the elements of the set S and admits a presentation:

$$W = \left\langle s{\in}S : s^2 = (ss')^{m_{ss'}} = 1,\ s{\neq}s',\ m_{s,s'}{\in}\{2, 3, \ldots, \infty\} \right\rangle .$$

In other words W is generated by a set of reflections and the only relations in W come from the angle between the hyperplanes corresponding to the reflections. The group W is called a *Coxeter group* and the elements of S are called *simple reflections*. We will consider finitely generated (and therefore finitely presented) Coxeter groups.

The *Coxeter graph*, $\mathcal{C}(W, S)$, associated to a Coxeter System is the weighted graph with vertices elements of S. Two vertices s and s' are connected if $m_{s,s'} \geq 3$. The edge $\{s, s'\}$ is marked by $m_{s,s'}$ if $m_{s,s'} \geq 4$.

Let $J{\subset}S$. Let W_J be the subgroup of W generated by J. The the pair (W_J, J) is again a Coxeter system ([4], [19]). The subgroups of W of this form are called *parabolic subgroups*. We write

$$\mathcal{F}(W, S) = \{J{\subset}S : W_J \text{ is finite}\}$$

for the poset of the subsets of S that generate finite subgroups. Denote by $\mathcal{F}_k(W, S)$ the subset of $\mathcal{F}(W, S)$ consisting of all elements containing k elements ($k \geq 0$) and $\mathcal{F}_{>0}(W, S) = \mathcal{F}(W, S) - \{\emptyset\}$.

Definition 2.1. A panel structure on a topological space Q is a locally finite family of closed subspaces $(Q_s)_{s\in S}$, indexed by a set S. The subsets Q_s are called the panels of S. A pair $(Q, (Q_s)_{s\in S})$ consisting of a space together with a panel structure is called an S-paneled space.

For each $q{\in}Q$, we define $S(q) = \{s{\in}S : q{\in}Q_s\}$. For each nonempty subset $J{\subset}S$, set

$$Q_J = \{q{\in}Q : J{\subset}S(q)\} = \bigcap_{s\in J} Q_s.$$

By convention, $Q_\emptyset = Q$. The formal boundary of an S-paneled space is the union of all panels:

$$\mathcal{D}Q = \bigcup_{s\in S} Q_s.$$

The subspaces Q_J are called *faces* of Q. We will consider panel spaces with finitely many panels.

Let (W, S) be a Coxeter system. The S-paneled structure on Q is called W-finite if $S(q) \in \mathcal{F}(W, S)$ for each $q{\in}Q$. There is a natural W-finite S-paneled complex associated to it. We write $K_0(W, S)$ for the abstract simplicial set with vertex set S and with simplices $J{\in}\mathcal{F}_{>0}(W, S)$. Denote by $K(W, S)$ the cone of $K_0(W, S)$, i.e., $K(W, S) = K_0(W, S) \cup \{\emptyset\}$. There is a natural S-panel structure on the geometric realization of $K(W, S)$ ([13]).

Let Q admit a W-finite S-panel structure. Then we define a relation between the elements of the product $W \times Q$:

$$(w, q) \sim (w', q'), \text{ if and only if } q = q', \text{ and } w^{-1}w'{\in}W_{S(q)}.$$

The quotient space

$$\mathcal{U}(W, Q) = W \times Q/{\sim}$$

is called the *universal space* of (W, Q). We denote the elements of $\mathcal{U}(W, Q)$ by $[w, q]$. There is a natural embedding

$$i : Q \to \mathcal{U}(W, Q), \quad q \mapsto [e, q].$$

The group W acts on $\mathcal{U}(W, Q)$ by left multiplication on the first coordinate. The action is by reflections in the sense that the fixed point sets of the generators separate $\mathcal{U}(W, Q)$ into two components interchanged by the action. The isotropy group of the point $[e, q]$ is $W_{S(q)}$ because only the generators in $S(q)$ fix $[e, q]$. Therefore the isotropy group of a general element $[w, q]$ is $wW_{S(q)}w^{-1}$.

We will also need an equivariant analogue of the above construction.

Definition 2.2. Let Δ be a discrete group equipped with a homomorphism $\alpha : \Delta \to \mathrm{Aut}(\mathcal{C}(W, S))$ where $\mathcal{C}(W, S)$ be the Coxeter graph of W. Let $(Q, (Q_s)_{s\in S})$ be an W-finite S-paneled space. Then an action of Δ on Q by panel maps is called *compatible with* α if $\delta(Q_J) = Q_{\delta(J)}$ for all $J{\in}\mathcal{F}(W, S)$, $\delta \in \Delta$.

Remark 2.3. (i) It is immediate from the definition that the action is compatible with α if and only if $\delta Q_s = Q_{\delta(s)}$, for $s{\in}S$, $\delta{\in}\Delta$.
(ii) It follows from the definition that $\delta Q_{S(q)} = Q_{S(\delta q)}$, for all $\delta{\in}\Delta$, $q{\in}Q$.

Lemma 2.4. *Let $\alpha : \Delta \to Aut(\mathcal{C}(W, S))$ be a homomorphism. Let Δ admit a panel action on an S-paneled space Q compatible with α. Set $\Gamma = W \rtimes \Delta$, where Δ acts on W through α. Then there is an action of Γ on $\mathcal{U}(W, Q)$ extending the natural action of W.*

Proof. Let $[w, q]$ represent an element of $\mathcal{U}(W, Q)$ and $\gamma = (w', \delta)$ be an element of Γ. Define

$$\gamma[w, q] = (w', \delta)[w, q] = [w'\delta(w), \delta q].$$

The action is well-defined: Let $[w_1, q] = [w_2, q]$ in $\mathcal{U}(W, Q)$. Then $w_1^{-1} w_2 \in W_{S(q)}$. Also,

$$(w', \delta)[w_i, q] = [w'\delta(w_i), \delta q], \quad i = 1, 2.$$

Then

$$\delta(w_1^{-1})(w')^{-1}w'\delta(w_2) = \delta(w_1^{-1}w_2) \in \delta W_{S(q)} = W_{S(\delta q)}$$

by Remark 2.3, Part (ii). The other properties of the action are immediate. □

Proposition 2.5. *Let Δ be an in Lemma* 2.4 *and $\Gamma = W \rtimes \Delta$. Let $(Q_1, (Q_{1s})_{s\in S})$ be a panel space that admits a panel action of Δ compatible with α.*

1. *Let M be a Γ-space and $f : Q_1 \to M$ a Δ-map such that $f(q) \in M^s$ for each $q \in Q_{1s}$, $s \in S$. Then there is a unique Γ-map $\hat{f} : \mathcal{U}(W, Q_1) \to M$ extending f.*
2. *Let $(Q_2, (Q_{2s})_{s\in S})$ be a second panel space that admits a panel action of Δ compatible with α. Let $\phi : (Q_1, (Q_{1s})_{s\in S}) \to (Q_2, (Q_{2s})_{s\in S})$ be a Δ-map such that $\phi(Q_{1s}) \subset Q_{2s}$. Then there is a unique Γ-map $\mathcal{U}(W, \phi) : \mathcal{U}(W, Q_1) \to \mathcal{U}(W, Q_2)$ extending ϕ.*

Proof. This is a direct generalization of the classical case ([26]). For (1), define

$$\hat{f}([w, q]) = wf(q), \quad [w, q] \in \mathcal{U}(W, Q_1).$$

Part (2) follows from (1). □

We summarize the naturality properties of the universal construction. Let (W, S) be a Coxeter system and Δ a group equipped with a homomorphism $\alpha : \Delta \to \mathrm{Aut}(\mathcal{C}(W, S))$. We define a category, $\mathcal{PS}_\Delta(W, S)$, with objects W-finite S-paneled spaces, $(Q, (Q_s)_{s\in S})$, on which Δ acts by panel maps compatible with α. Morphisms are panel maps:

$$f : (Q, (Q_s)_{s\in S}) \;\to\; (Q', (Q'_s)_{s\in S})$$

such that for each $s \in S$, $f(Q_s) \subset Q'_s$, which are Δ-equivariant. An isomorphism in the category $\mathcal{PS}_\Delta(W, S)$ is called an S-paneled Δ-homeomorphism. Notice that an S-paneled Δ-homeomorphism induces a Γ-homeomorphism on the universal spaces, where $\Gamma = W \rtimes \Delta$, with Δ acting on W through α. An S-paneled Δ-homotopy is a homotopy in $\mathcal{PS}_\Delta(W, S)$, i.e., an S-paneled Δ-map:

$$F : (Q \times I, (Q_s \times I)_{s\in S}) \;\to\; (Q', (Q'_s)_{s\in S})$$

An S-paneled homotopy induces a Γ-homotopy on the corresponding universal spaces.

3. Reflections

A *reflection* on a manifold M is a locally linear involution with fixed point set M^r such that $M \setminus M^r$ has two components. A discrete group generated by a set of reflections on a manifold is a Coxeter group ([13]).

Definition 3.1. A Coxeter group W is called a *manifold-reflection* group if there is a cocompact manifold of type $\mathcal{E}W$ on which W acts by reflections.

Remark 3.2. The following are well-known about manifold-reflection groups:

(i) Manifold-reflection groups are virtual Poincaré Duality Coxeter groups.

(ii) A Coxeter group W is a manifold-reflection group if and only if, for some Coxeter system (W, S), the geometric realization of $\mathcal{F}_{>0}(W, S)$ is a homology manifold that is a homology sphere ([13]).
(iii) For a manifold-reflection group W, the classifying manifold $\mathcal{E}W$ is not necessarily homeomorphic to a Euclidean space ([13]).
(iv) Let W be a manifold-reflection Coxeter group. Then for any two Coxeter systems (W, S) and (W, T), there is an inner automorphism of W that maps S to T ([9], [23, Proposition 4.7]).

Actually, there is a complete characterization of virtual Poincaré Duality Coxeter groups ([14]):

Proposition 3.3. *A Coxeter group W is a virtual Poincaré Duality group if and only if $W = W_1 \times W_2$ where W_1 is a manifold-reflection group and W_2 is a finite group.*

There is broader class of Coxeter groups that satisfies Property (iv) above. A Coxeter system (W, S) is type PM_n if $|\mathcal{F}_{>0}(W, S)|$ is an orientable pseudo-$(n-1)$-manifold whose $(n-1)$-homology groups is isomorphic to $\mathbb{Z}$ ([9]). The following is restatement of Theorem 5.10 in [9]:

Proposition 3.4. *Let W be a Coxeter group of type PM_n. Then any two sets of Coxeter generators are conjugate.*

The next result follows as in Lemma 4.1 in [23]. It shows that the action by reflections is invariant under proper homotopy equivalences.

Lemma 3.5. *Let M and M' be locally linear $\mathbb{Z}/2\mathbb{Z}$-manifolds. Assume that:*
1. *The nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acts as a reflection on M.*
2. *$f : (M', \partial M') \to (M, \partial M)$ is a proper $\mathbb{Z}/2\mathbb{Z}$-homotopy equivalence such that $f|\partial M'$ is a $\mathbb{Z}/2\mathbb{Z}$-homeomorphism (we allow $\partial M = \partial M' = \emptyset$).*

Then the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acts on M' as a reflection.

Proposition 3.6. *Let (W, S) be a Coxeter system and M and M' be locally linear W-manifolds with boundary such that W acts on M' by reflections. Let*

$$f : (M', \partial M') \to (M, \partial M)$$

be a W-homotopy equivalence such that $f|\partial M'$ is a W-homeomorphism.
1. *If f is a proper W-homotopy equivalence, then W acts on M by reflections.*
2. *If the W-action on M and M' is cocompact, then W acts on M by reflections.*

Proof. For Part (1), we use Lemma 3.5 to show that every element of S acts on M as a reflection. For Part (2), notice that the cocompactness assumption implies that the map f is a proper W-homotopy equivalence, and then use Part (1). □

4. Classifying spaces

We define the universal complexes of discrete group actions that have finite isotropy groups.

Definition 4.1. Let Γ be a discrete group. A complex of type $\mathcal{E}\Gamma$ is a Γ-CW-complex on which Γ acts cellularly, with finite isotropy groups such that the fixed point sets are contractible. If the action is cocompact we call the complex a cocompact complex of type $\mathcal{E}\Gamma$.

The basic properties of the construction are shown in [10], [25] (for cocompact complexes) and [16, Appendix] (where it is defined as the classifying space for the class of finite subgroups of Γ). Spaces of type $\mathcal{E}\Gamma$ are universal for actions with finite isotropy groups. More precisely, any Γ-space with finite isotropy groups admits a unique (up to Γ-homotopy) map to a space of type $\mathcal{E}\Gamma$. In particular, spaces of type $\mathcal{E}\Gamma$ are unique up to Γ-homotopy. If Γ admits a cocompact complex of type $\mathcal{E}\Gamma$ and it is virtually torsion-free, then Γ has finite virtual cohomological dimension. For groups of finite virtual dimension cocompact complexes of type $\mathcal{E}\Gamma$ exist ([10], [25]). If there is a manifold without boundary which is a cocompact space of type $\mathcal{E}\Gamma$ and Γ is a virtually torsion-free group, then Γ is a virtual Poincaré Duality group, i.e., it contains a subgroup of finite index that is Poincaré Duality.

Coxeter groups admit finite dimensional linear representations ([26]) and they have finite virtual cohomological dimension ([2], [13]). Let (W,S) be a Coxeter system and $(Q,(Q_s)_{s\in S})$ a W-finite S-paneled complex.

Definition 4.2. Let (W,S) be a Coxeter system and $(Q,(Q_s)_{s\in S})$ a W-finite S-paneled complex. A W-finite S-paneled space is called *admissible* if Q_J is contractible for all $J\in\mathcal{F}(W,S)$.

In [23, Proposition 3.6], it was shown that the classifying space $\mathcal{U}(W,Q)$ is a cocompact space of type $\mathcal{E}W$ if and only if Q_J is contractible for each $J\in\mathcal{F}(W,S)$, i.e., if the S-paneled structure is admissible.

Let (W,S) be a Coxeter system with Coxeter graph $\mathcal{C}(W,S)$. Let Δ be a group that admits a homomorphism (possibly trivial) to $\mathrm{Aut}(\mathcal{C}(W,S))$. More precisely, there is an exact sequence

$$1\to\Delta_0\to\Delta\xrightarrow{\alpha}\mathrm{Aut}(\mathcal{C}(\mathrm{W},\mathrm{S})).$$

This action induces an action of Δ on W, denoted also α. From now on, by an action of a group on a manifold we will mean a locally linear action. Let $\Gamma=W\rtimes\Delta$ where Δ acts on W by α.

Lemma 4.3. *Γ is a virtual Poincaré Duality group if and only if both W and Δ are.*

Proof. Let Δ' be a subgroup of Δ of finite index that is a Poincaré Duality group. Then $\Delta''=\Delta_0\cap\Delta'$ has finite index in Δ and Δ' and thus it is a Poincaré Duality group. Also, Δ'' acts trivially on W. Let W' be a Poincaré Duality subgroup of finite index of W. Then $W'\times\Delta''$ is a Poincaré subgroup of finite index of Γ.

Let Γ' be the subgroup of Γ of finite index that is Poincaré Duality. Let W_0 be a torsion-free subgroup of W of finite index. Then $\Gamma'\cap(W_0\times\Delta_0)$ has finite index in Γ' and thus it is a Poincaré Duality group. But $\Gamma'\cap(W\times\Delta_0)$ has also finite index in $W_0\times\Delta_0$, which implies that $W_0\times\Delta_0$ is also a Poincaré Duality Group. By assumption both W_0 and Δ_0 are groups of finite cohomological dimension. Since the product is a Poincaré Duality group, each factor must be a Poincaré Duality group [27, Theorem 2.5, (ii)]. Thus W and Δ are virtual Poincaré Duality groups. □

Proposition 4.4. *Let $\Gamma=W\rtimes\Delta$ as before, with Δ a virtually torsion-free group. Let M be a cocompact manifold, without boundary, of type $\mathcal{E}\Gamma$. Then:*

(i) *$W=W_1\times W_2$ where W_1 is a manifold reflection group and W_2 is a finite Coxeter group.*

(ii) *The action of* Δ *on* W *restricts to an action on* W_i, $i = 1, 2$.
(iii) *There is a manifold, without boundary, of type* $\mathcal{E}\Delta_0$.
(iv) W_1 *acts on* $\mathcal{E}\Gamma$ *by reflections and* W_2 *acts trivially on* M.

Proof. (i) Since M is a cocompact manifold of type $\mathcal{E}\Gamma$, and Γ is a virtually torsion-free group (since Δ and W are) Γ is a virtual Poincaré Duality group. Lemma 4.3 implies that W is a virtual Poincaré Duality group. By [14] (also Remark 3.2), $W \cong W_1 \times W_2$ where W_1 is a manifold reflection group and W_2 is a finite Coxeter group.

(ii) Manifold reflection groups cannot be decomposed in nontrivial products of two Coxeter groups with one of them finite ([23, Corollary 4.3]). So W_1 cannot contain other finite Coxeter groups factors. Since Δ acts on W by Coxeter graph automorphisms, it must fix W_2.

(iii) It is clear that M is a manifold of type $\mathcal{E}(W \times \Delta_0)$ with the restriction of the action. Let H be a maximal finite parabolic subgroup of W. Then

$$N_W(H)/H = \{e\} \;\Rightarrow\; N_{W\times\Delta_0}(H)/H \cong \Delta_0$$

(the first equality follows from [3] and [7]). Thus there is a natural action of Δ_0 on the fixed point set M^H. It is immediate that this action makes M^H a manifold of type $\mathcal{E}\Delta_0$. Furthermore, the natural map

$$M^H/\Delta_0 \to M/\Gamma$$

embeds the quotient as a closed subset of M/Γ, which is compact. Thus M^H is a cocompact manifold of type $\mathcal{E}\Delta_0$.

(iv) Since W_1 is a manifold reflection group, there is a manifold N of type $\mathcal{E}W_1$ on which W_1 acts by reflections. By (iii), there is a cocompact manifold N' of type $\mathcal{E}\Delta_0$. Thus $N \times N'$ is a cocompact manifold of type $\Gamma' = W_1 \times \Delta_0$ on which W_1 acts by reflections. Since both spaces M and $N \times N'$ are of type $\mathcal{E}\Gamma'$, there is a Γ'-homotopy equivalence $f : M \to N \times N'$. Since the actions are cocompact, f is a Γ'-proper homotopy equivalence and thus it is a proper W_1-homotopy equivalence. Proposition 3.6 (also the argument in Lemma 4.1 in [23]) shows that W_1 acts by reflections on M. For the W_2 action, notice that the group $N_\Gamma(W_2)/W_2 \cong W_1 \rtimes \Delta$ acts on the contractible manifold M^{W_2}. The action is cocompact since $M^{W_2}/W_2 \rtimes \Delta$ is homeomorphic to a closed subset of the compact space M/Γ. Thus

$$\dim(M^{W_2}) = \mathrm{vcd}(W_1 \rtimes \Delta) = \mathrm{vcd}(\Gamma) = \dim(M).$$

Therefore M^{W_2} is a closed submanifold in M of the same dimension. The invariance of domain implies that $M^{W_2} = M$. Thus W_2 acts trivially on M (for more details, see Lemma 4.2 in [23]). □

Corollary 4.5. *Let* M *be a cocompact manifold, without boundary, of type* $\mathcal{E}\Gamma$ *with* Γ *virtually torsion-free. Then there is:*

(i) *a* W*-finite admissible* S*-paneled manifold* $(Q, (Q_s)_{s\in S})$,
(ii) *a panel action of* Δ *on* Q, *compatible with* α.

Furthermore, M *is* Γ*-homeomorphic to* $\mathcal{U}(W, Q)$.

Proof. Proposition 4.4 implies that W acts on M by reflections. Thus there is a W-finite admissible S-paneled manifold $(Q, (Q_s)_{s\in S})$ such that $M \cong_W \mathcal{U}(W, Q)$. Since M/W can be also identified with Q, there is an action of Δ on Q. To show that the action is compatible with α, it is enough to show that $\delta Q_s = Q_{\delta(s)}$, for $\delta \in \Delta$,

$s \in S$ (Remark 2.3). If $x \in Q_s$, then $x \in Q \cap M^s$ and it is fixed by s. Therefore δx is fixed by $(1,\delta)(s,1)(1,\delta^{-1})$ which is equal to $(\delta(s),1)$. Thus $\delta x \in M^{\delta(s)} \cap Q = Q_{\delta(s)}$. Therefore, $\delta Q_s \subset Q_{\delta(s)}$. A similar argument shows the other inclusion.

Lemma 2.4 implies that there is a Γ action on $\mathcal{U}(W,Q)$. In [13] (also [26]), it was shown that the map

$$\mathcal{U}(W,Q) \to M, \quad [w,q] \mapsto wq$$

is a W-homeomorphism. We will show that it is actually a Γ-map and thus a Γ-homeomorphism:

$$\begin{aligned} f((w',\delta)[w,q]) &= f([w'\delta(w),\delta q]) \\ &= (w'\delta(w))\delta q \\ &= (w',\delta)(w,1)q \\ &= (w',\delta)wq \\ &= (w',\delta)f([w,q]) \end{aligned}$$

□

Corollary 4.6. *With the assumption of Corollary* 4.5*, set* $m = dim(M)$*. Then for each* $J \in \mathcal{F}(W,S)$*,* Q_J *is a manifold of dimension* $m - |J|$ (*with boundary unless* J *generates a maximal finite parabolic subgroup*).

Proof. Let $J \in \mathcal{F}_k(W,S)$. Then, by construction, $Q_J = Q \cap M^{W_J}$ and Q_J has the same dimension of M^{W_J}. The dimension of M^{W_J} is equal to the dimension of the intersection

$$\dim(M^{W_J}) = \dim\left(\bigcap_{s\in J} M^s\right)$$

which has codimension k, by transversality. The result follows.

If J is a maximal subset in $\mathcal{F}(W,S)$ then Q_J is a manifold, without boundary, of dimension $m-n$, where $n = |J|$. In this case, $Q_J = M^{W_J}$, which is a manifold of type $\mathcal{E}\Delta_0$. □

5. Topological rigidity

We start by stating the rigidity assumption for certain subgroups of Γ: Let G be a discrete group.

Assumption (R) for *G*: Let N and N' be two G-manifolds without boundary that are spaces of type $\mathcal{E}G$. Let $f : N' \to N$ be a G-homotopy equivalence which is a homeomorphism outside the G-orbit of a compact subset of N'. Then f is G-homotopic to a G-homeomorphism, which agrees with f outside the orbit of a compact set.

Remark 5.1. Groups that satisfy Assumption (R) above are groups given in the work of Farrell–Jones. They proved that if G is torsion-free subgroup of $GL(n,\mathbb{R})$

and of cohomological dimension greater than or equal to 5 then G satisfies Assumption (R). Also, all the subgroups of G of cohomological dimension greater than or equal to 5 satisfy Assumption (R). ([17]).

The following is immediate from the rigidity assumption:

Lemma 5.2. *Let G satisfy Assumption* (R) *and N, N' two manifolds with boundary, of type $\mathcal{E}G$, and of the same dimension. Let $f : (N', \partial N') \to (N, \partial N)$ be a Δ_H-homotopy equivalence such that:*

1. *f is homeomorphism outside the G-orbit of a compact subset of N'.*
2. *$f|\partial N'$ is a G-homeomorphism.*

Then f is G-homotopic to a G-homeomorphism which agrees with f on the boundary.

Proof. By attaching a collar if necessary, we assume that f is a G-homeomorphism on a collar of $\partial N'$. Actually we arrange that $f' = f|N' - \partial N'$ is a G-homeomorphism on the complement of a closed subcollar C. Since f' is a G-homeomorphism outside the orbit of a compact subset, Assumption (R) implies that $f' \simeq_G g'$ where g' is a G-homeomorphism that agrees with f' on a complement of a larger subcollar. Thus g' extends to a G-homeomorphism $g : N' \to N$ such that $g|\partial N' = f|\partial N'$. □

If the action in cocompact we get a stronger result:

Lemma 5.3. *Let G satisfy Assumption* (R) *and N, N' two manifolds with boundary, of the same dimension, that are cocompact manifolds of type $\mathcal{E}G$. Let*

$$f : (N', \partial N') \to (N, \partial N)$$

be a G-homotopy equivalence such that $f|\partial N'$ is a G-homeomorphism. Then f is G-homotopic to a G-homeomorphism which agrees with f on the boundary.

Proof. As before, we define $f' : N' - \partial N' \to N - \partial N$ to be a G-homeomorphism on the complement of a closed collar C of $\partial N'$. Since N'/G is compact, there is a compact subset K of N' such that the complement of the orbit of K is contained in C. Thus f' is a G-homeomorphism outside the orbit of a compact subset. The rest of the proof follows as in Lemma 5.2. □

Let $\Gamma = W \rtimes \Delta$, with (W, S) a Coxeter system, of type PM, with S finite and Δ acting on W through a map to the automorphisms of the Coxeter graph $\mathcal{C}(W, S)$:

$$1 \to \Delta_0 \to \Delta \xrightarrow{\alpha} \mathrm{Aut}(\mathcal{C}(W, S)).$$

For $J \subset S$, let H_J be the subgroup of $\mathrm{Aut}(\mathcal{C}(W, S))$ that fixes J. We write $\Delta_J = \alpha^{-1}(H_J)$.

We start with setting up the assumptions.

Assumptions: Let $(M_i, \partial M_i)$, $i = 1, 2$, be two manifolds that are spaces of type $\mathcal{E}\Gamma$ and

$$f : (M_1, \partial M_1) \to (M_2, \partial M_2)$$

a Γ-homotopy equivalence that is a Γ-homeomorphism outside the orbit of a compact subset and when restricted to ∂M_1. We also assume that:

(i) There is a manifold X, with boundary, of type $\mathcal{E}\Gamma$ such that:
  1. W acts on X by reflections.
  2. X is properly Γ-homotopy equivalent to M_i with a homotopy equivalence that restricts to a Γ-homeomorphism on the boundary.

(ii) Δ_J satisfies Condition (R), for each $J \subset S$.

By Proposition 3.6, because of Assumption (i.1), we derive that W acts by reflections on M_1 and M_2. By Proposition 3.4 ([9, Theorem 5.10]), there are admissible S-paneled manifolds $(Q_i, (Q_{i,s})_{s\in S})$ such that $\mathcal{U}(W, Q_i) \cong_\Gamma M_i$, $i = 1, 2$ (Corollary 4.5). We can choose the same S-paneled structure on the two fundamental domains because of Proposition 3.4. Each face Q_{iJ}, $i = 1, 2$, is a manifold with boundary unless J is maximal and $\partial M_i = \emptyset$, $i = 1, 2$ (Corollary 4.6). Each face is a space of type $\mathcal{E}\Delta_0$. Actually, the face Q_{iJ} is a space of type $\mathcal{E}\Delta_J$, for each $J \in \mathcal{F}(W, S)$.

Lemma 5.4. *With the above notation, for each $J \in \mathcal{F}(W, S)$, the restriction $f_J = f|Q_{1J}$ is a Δ_J-homotopy equivalence that is a homeomorphism outside the orbit of a compact subset. The same conclusion follows if f_J is just considered as a Δ_0-homotopy equivalence.*

Proof. Let K be a compact subset of M_1 such that f is a homeomorphism outside the Γ-orbit of K. Notice that $Q_{1J} \cap wQ_{1J} \subset Q_{1J}$, for all $w \in W$. Then

$$Q_{1J} \cap \Gamma K = Q_{1J} \cap (W \rtimes \Delta)K \subset Q_{1J} \cap \Delta K \subset Q_{1J} \bigcap \Delta_J \left(\bigcup_{j=1}^{n} \delta_j K \right)$$

where $\{\delta_j\}_{j=1}^n$ is a complete set of right coset representatives in Δ/Δ_J. So if we set

$$L = Q_{iJ} \cap \left(\bigcup_{j=1}^{n} \delta_j K \right)$$

then f_J is a homeomorphism outside the Δ_J-orbit of L.

The same method proves the second assertion. □

Lemma 5.5. *Let Q_i be the W-fundamental domain of M_i that determines the Coxeter generating set S of W. There exists a compact subset $C \subset M_1$ such that*

$$f(Q_{1J} - \Gamma C) \subset Q_{2J} - f(\Gamma C), \quad f(Q_{1J} - \Delta C) \subset Q_{2J} - f(\Delta C),$$

for each $J \in \mathcal{F}(W, S)$.

Proof. There is a compact subset C of M_1 such that f is a Γ-homeomorphism outside the orbit of C. Thus, outside the orbit of C, f is an isovariant W-homeomorphism. The first result follows. For the second result, we choose the compact subset C such that $sC \subset C$ for all $s \in S$. To achieve that, any choice of C can be enlarged by defining:

$$C \cup \left(\bigcup_{s\in S} sC \right).$$

Since, for all $w \in W$, $wQ_1 \cap Q_1 \subset sQ_1$, for some $s \in S$, and Δ acts on the faces by permuting them, the second relation follows. □

Theorem 5.6. *With the above notation, f is Γ-homotopic to a Γ-homeomorphism which agrees with f on ∂M_1 and in the complement of the Γ-orbit of a compact subset of M_1.*

Proof. The group Δ_0 preserves each face of Q_i $(i = 1, 2)$ because it acts trivially on the Coxeter graph. For each face Q_{iJ}, $J \in \mathcal{F}(W, S)$, we write

$$\partial Q_{iJ} = \bigcup_{T \supsetneq J} Q_{iT}$$

which is the boundary of Q_{iJ}. We fix a complete set of right cosets representatives of Δ/Δ_0, $\{\delta_1, \dots \delta_r\}$. In other words,

$$\Delta/\Delta_0 = \{\delta_1, \dots \delta_r\}.$$

Claim 1. There is a Δ-paneled homeomorphism $\phi : Q_1 \to Q_2$ that agrees with f on the complement, in Q_1, of the Γ-orbit of a compact subset.

Proof. We construct the map inductively as in [23, Theorem 5.3].

0-th step: Let W has rank m, i.e., $|S| =$ m. The smallest faces of Q_i are spaces of type $\mathcal{E}\Delta_0$ They correspond to the maximal finite parabolic subgroups of W. The panel structure of Q_i with panels "manifolds with corners" forces all the minimal fixed point sets have the same dimension, say n. They are transverse intersections of fixed point subsets of the maximal finite parabolic subgroups of W. Transversality means that all the maximal subgroups have rank $m - n$ (also Corollary 4.6). For each $J \in \mathcal{F}_{m-n}(W, S)$, the face Q_{iJ} is a space of type $\mathcal{E}\Delta_J$. The action of Δ on $\mathcal{C}(W, S)$ induces an action on $\mathcal{F}_{m-n}(W, S)$. Let $\{J_j : \ j = 1, \dots, s_{m-n}\}$ be a complete set of orbit representatives of the action. For each j, the map f restricts to a Δ_{J_j}-homotopy equivalence:

$$\psi_{J_j} : Q_{1J_j} = M_1^{W_{J_j}} \to M_2^{W_{J_j}} = Q_{2J_j}.$$

Lemma 5.4 implies that ψ_{J_j} is a homeomorphism outside the Δ_{J_j}-orbit of a compact subset, that agrees with f on the complement of a Γ-orbit of a compact subset. The rigidity assumption on Δ_{J_j} implies that there is a Δ_{J_j}-homeomorphism

$$\phi_{J_j} : Q_{1J_j} \to Q_{2J_j}.$$

that agrees with f in a complement of the Γ-orbit of a compact subset. Let $J \in \mathcal{F}_{m-n}(W, S)$. Then there is $\delta_{J_j} \in \Delta$ such that $J = \delta_{J_j}(J_j)$. Define:

$$\phi_J : Q_{1J} \to Q_{2J},$$
$$x \mapsto \delta_{J_j} \phi_{J_j}(\delta_{J_j}^{-1} x).$$

1. The definition of ϕ_J does not depend on the choice of δ_{J_j}: If δ'_{J_j} is another element such that $J = \delta'_{J_j}(J_j)$, then $\delta_{J_j}^{-1}\delta'_{J_j} \in \Delta_{J_j}$. Thus, there is $\delta \in \Delta_J$ such that $\delta'_{J_j} = \delta_{J_j}\delta$. Therefore

$$\delta'_{J_j}\phi_{J_j}((\delta'_{J_j})^{-1}x) = \delta_{J_j}\phi_{J_j}\delta(\delta^{-1}\delta_{J_j}^{-1}x) = \delta_{J_j}\phi_{J_j}(\delta_{J_j}^{-1}x)$$

   where the last equality follows from the fact that ϕ_{J_j} is Δ_{J_j}-equivariant.

2. ϕ_J is a Δ_J-homeomorphism: First of all, it is a homeomorphism. Let $\delta\in\Delta_J$ and $\delta_{J_j}(J_j) = J$. Then $\delta^{-1}\delta_{J_j}(J_j) = J$. By (1),
$$\phi_J(x) = \delta^{-1}\delta_{J_j}\phi_{J_j}(\delta_{J_j}^{-1}\delta x) \implies \delta\phi_J(x) = \delta_{J_j}\phi_{J_i}(\delta_{J_j}^{-1}\delta x) = \phi_J(\delta x).$$
3. Let $J\in\mathcal{F}_{m-n}$ and $\delta\in\Delta$. Then, for $x\in Q_{1J}$,
$$\phi_{\delta(J)}(\delta x) = \delta\phi_J(x).$$
That follows from the definition and (1).

We combine all the maps ϕ_J to get a map
$$\phi_{m-n} = \coprod_{J\in\mathcal{F}_{m-n}(W,S)} \phi_J : \coprod_{J\in\mathcal{F}_{m-n}(W,S)} Q_{1J} \longrightarrow \coprod_{J\in\mathcal{F}_{m-n}(W,S)} Q_{2J}.$$
Then ϕ_{m-n} is a homeomorphism that agrees with f outside the Γ-orbit of a compact subset. By (3), it follows that ϕ_{m-n} is a Δ-map and thus a Δ-homeomorphism.

Inductive step: We assume that ϕ_ℓ has been already defined for all $k < \ell < m-n$. We will construct ϕ_k as an extension of the map ϕ_{k+1}, already defined by the induction hypotheses. The procedure is similar to the previous case. Let $\{K_j : \; j = 1, \dots, s_k\}$ be a complete set of orbit representatives of the Γ action on $\mathcal{F}_k(W,S)$. The spaces Q_{1K_j} and Q_{2K_j} are manifolds with boundary of type $\mathcal{E}\Delta_{K_j}$. □

Claim 2. There is a panel Δ_{K_j}-homotopy equivalence $\psi_{K_j} : Q_{1K_j} \to Q_{2K_j}$, for $j = 1, \dots, s_k$ that extends the map on the boundaries and the map f on the complement of the Γ-orbit of a compact set.

Proof. For $j = 1, \dots, s_k$, let $\partial\phi_{K_j}$ for the Δ_{K_j}-homeomorphism defined on the boundary of Q_{1K_j}. Using the equivariant homotopy extension property, we extend $\partial\phi_{K_j}$ to a Δ_{K_j}-homotopy equivalence $\psi_{K_j} : Q_{1K_j} \to Q_{2K_j}$ which agrees with the restriction of f on a complement of the Γ-orbit of a compact subset (Lemma 5.5). □

By Lemma 5.2, there is a Δ_{K_j}-homeomorphism $\phi_{K_j} : Q_{1K_j} \to Q_{2K_j}$, Δ_{K_j}-homotopic to ψ_{K_j}, extending $\partial\phi_{K_j}$ and f. As before, for $K \in \mathcal{F}_K(W,S)$, with $K = \delta(K_j)$, define
$$\phi_K : Q_{1K} \to Q_{2K}, \quad x \mapsto \delta\phi_{K_j}(\delta^{-1}x).$$
Also define
$$\phi_k : \bigcup_{K\in\mathcal{F}_k(W,S)} Q_{1K} \to \bigcup_{K\in\mathcal{F}_k(W,S)} Q_{2K}.$$
Then ϕ_k is a Δ-homeomorphism.

After completing the construction up to $\mathcal{F}_0(W,S)$, we get a Δ-paneled homeomorphism
$$\phi : Q_1 \to Q_2.$$
Since the Δ-action is compatible with α, it induces, by Proposition 2.5, a Γ-homeomorphism $\mathcal{U}(W,\phi) : \mathcal{U}(W,Q_1) \to \mathcal{U}(W,Q_2)$, which agrees with f in the complement of the Γ-orbit of a compact subset. That completes the proof of the Main Theorem. □

Remark 5.7. 1. There is no rigidity assumption on the Coxeter group W. The only requirement is that W acts on an appropriate space by reflections and that all the Coxeter generating sets are conjugate. The Coxeter group serves as a "blueprint" used for gluing the fixed point subspaces.

2. The condition that W is of type PM can be weakened. The action of W by reflections on M_i, $i = 1, 2$, determines two sets of Coxeter generators S_i, $i = 1, 2$, for W ([13]). Lemma 3.5 shows that an element of W acts as a reflection on M_1 if and only if it acts as a reflection on M_2. Thus the two Coxeter presentations (W, S_i), $i = 1, 2$ have identical sets of reflections. So the condition that W is of type PM can be weakened to that W is *reflection rigid*, i.e., for any two set of Coxeter generators S_i, $i = 1, 2$, that determine the same set of reflections in W, there is an automorphism ω of W such that $\omega(S_1) = S_2$. If ω is an inner automorphism, W is called *strongly reflection rigid*. Coxeter groups of type PM are strongly reflection rigid. The automorphism ω induces an isomorphism of the Coxeter graphs. So the conditions required for the Coxeter group are:
   a) W is reflection rigid.
   b) There is a Γ-manifold with boundary X on which W acts by reflections and it is properly Γ-homotopy equivalent to M_i, $i = 1, 2$.

   Classes of reflection rigid (or simply rigid) Coxeter groups are given in [1], [5], [20], [24]. Notice though that Condition (b) forces the maximal finite parabolic subgroups of W to have the same rank and thus making W very close to being a group of type PM.
3. The rigidity assumptions are not needed for all the subgroups of Δ but rather for the subgroups Δ_J where $J \in \mathcal{F}(W, S)$.
4. There are no dimension assumptions in the rigidity theorem. The rigidity assumption on the subgroups of Δ is much stronger, in general.
5. Usually, the subgroups of Δ satisfy the rigidity assumption in higher dimensions (bigger than or equal to 5). In this case, Theorem 5.6 is true if we assume that f is already an equivariant homeomorphism on the fixed point sets of lower dimensions. Thus, in this case, a relative version of Theorem 5.6 holds.

6. Special cases

6.1. Trivial actions. As a special case of Theorem 5.6 and Remark 5.7 (2), when the action α is trivial, i.e., when $\Gamma = W \times \Delta$. Then Theorem 5.6 is true when the rigidity assumption for Δ holds:

Theorem 6.1. *Let M_i, $i = 1, 2$, be two Γ-manifolds of type $\mathcal{E}\Gamma$. Let*

$$f : (M_1, \partial M_1) \to (M_2, \partial M_2)$$

be a Γ-homotopy equivalence that is a homeomorphism outside the Γ-orbit of a compact subset of M_1. Assume:

1. *W is of type* PM.
2. *There is a W-manifold with boundary X of type $\mathcal{E}\Gamma$ properly Γ-homotopic to M_i, with a homotopy equivalence that restricts to a homeomorphism on the boundary, on which W acts by reflections.*
3. *Δ satisfies Assumption* (R).

Then f is Γ-homotopic to a Γ-homeomorphism.

6.2. Virtual Poincaré duality groups. Let $\Gamma = W \rtimes \Delta$, as always, and assume that Γ is virtually torsion-free and there is a cocompact manifold, without boundary, of type $\mathcal{E}\Gamma$. Then Γ is a virtual Poincaré Duality group. Suppose also that M_1 is a cocompact manifold, without boundary, of type $\mathcal{E}\Gamma$. Proposition 4.4 implies that Δ and W are also virtual Poincaré Duality Groups. Then [9] implies that W splits as a product $W_1 \times W_2$ with W_1 a manifold-reflection group and W_2 finite. Proposition 4.4, Part (iv), implies that W_2 acts trivially on M_i, $i = 1, 2$. In this case, we do not need the assumption that W is of type PM (Remark 3.2, Part (iv)). Also Corollary 4.5 implies that the space X in (i.1) exits. The case $\Delta = \{e\}$ is treated in [23] under an assumption on low dimensional fixed point sets that can be removed if the 3-dimensional Poincaré Conjecture is true ([22]).

Theorem 6.2. *Let Γ be a virtual Poincaré Duality Group and*

$$f : (M_1, \partial M_1) \to (M_2, \partial M_2)$$

a Γ-homotopy equivalence between cocompact Γ-manifolds of type $\mathcal{E}\Gamma$, which restricts to a Γ-homeomorphism on the boundaries. Assume that Δ_J satisfies Assumption (R) *for each $J \in \mathcal{F}(W, S)$. Then f is Γ-homotopic to a Γ-homeomorphism. If Δ is the trivial group, then f is Γ-homotopic to a Γ-homeomorphism provided the* 3*-dimensional Poincaré Conjecture is true.*

6.3. Δ is trivial. The question is to what extent the trivial group satisfies Assumption (R). If the manifolds have dimension 1 or 2 the result is trivial. If the dimension is larger than 3 the result follows from the surgery exact sequence and the Poincaré Conjecture. In dimension 3, Theorem 1 in [6] implies that the rigidity holds, provided both manifolds are P^2-irreducible 3-manifolds (i.e., they contain no two-sided embedded projective planes and every embedded 2-sphere bounds a 3-cell).

Theorem 6.3. *Let W be a Coxeter group of type* PM. *Let*

$$f : (M_1, \partial M_1) \to (M_2, \partial M_2)$$

is a Γ-homotopy equivalence between cocompact manifolds of type $\mathcal{E}\Gamma$, which restricts to a Γ-homeomorphism on the complement of the W-orbit of a compact subset. Assume that:

1. *There is a manifold with boundary of type $\mathcal{E}W$ X properly W-homotopy equivalent to M_i, with a homotopy that restricts to a homeomorphism on the boundary such that W acts by reflections on X.*
2. *Any* 3*-dimensional fixed point sets are P^2-irreducible manifolds.*

Then f is Γ-homotopic to a Γ-homeomorphism.

6.4. W is the infinite dihedral group. Let $\Gamma = D_\infty \rtimes \Delta$. There are two cases:

1. Δ acts on D_∞ trivially. Then, as in 6.1 we only need the rigidity assumption for Δ.
2. Δ acts on D_∞ nontrivially. Let Δ_0 be the kernel of the action. In this case, we need only the rigidity assumption for Δ_0 and Δ.

Theorem 6.4. *Let $\Gamma = D_\infty \rtimes \Delta$. Let M_i, $i = 1, 2$, be two Γ-manifolds of type $\mathcal{E}\Gamma$. Let*

$$f : (M_1, \partial M_1) \to (M_2, \partial M_2)$$

be a Γ*-homotopy equivalence that is a homeomorphism outside the* Γ*-orbit of a compact subset of* M_1*. Assume:*

1. *There is a* W*-manifold with boundary* X *of type* $\mathcal{E}\Gamma$ *properly* Γ*-homotopic to* M_i*, with a homotopy that restricts to a homeomorphism on the boundary and on which* W *acts by reflections.*
2. *If the action of* Δ *on* D_∞ *is trivial assume that* Δ *satisfies Assumption* (R)*. If the action is nontrivial assume that* Δ *and* Δ_0 *satisfy Assumption* (R)*.*

Then f *is* Γ*-homotopic to a* Γ*-homeomorphism.*

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