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Heegaard splittings and virtually Haken Dehn filling

J. Masters, W. Menasco and X. Zhang

Abstract. We use Heegaard splittings to give some examples of virtually Haken 3-manifolds.

A compact connected 3-manifold is said to be virtually Haken if it has a finite sheeted covering space which is Haken. The virtual Haken conjecture states that every compact, connected, orientable, irreducible 3-manifold with infinite fundamental group is virtually Haken. Since virtually Haken 3-manifolds and Haken 3-manifolds possess similar properties, such as geometric decompositions and, in the closed case, topological rigidity, the resolution of this conjecture would provide solutions to several fundamental problems about compact 3-manifolds with infinite fundamental groups.

Some recent results in attacking the conjecture can be found in [CL] [BZ] [M] [DT]. A summary of earlier results can be found in [K, Problem 3.2]. For connections between the virtual Haken conjecture, Heegaard splittings, and the Property τ conjecture, see [L].

Motivated by the work of Casson and Gordon ([CG]), we shall show that lifted Heegaard surfaces can often be compressed to become essential. Our techniques can be used to produce many families of non-Haken but virtually Haken 3-manifolds, a few of which are given here to illustrate the method. A more general result will be proved in a forthcoming paper.

We proceed to give the examples. Let K_{2n+1} be the twist knot in S^3 as shown in Figure 1. Let M_n be the exterior of K_{2n+1}, with standard meridian-longitude framing on ∂M_n. Recall that a connected, compact, orientable 3-manifold whose boundary is a torus is called *small* if every closed, orientable, embedded, incompressible surface is parallel to the boundary, and called *large* otherwise.

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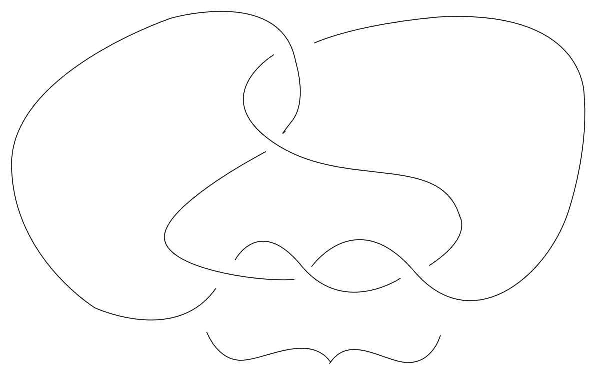

FIGURE 1. The twisted knot K_{2n+1}

Theorem 1. *The* 3*-fold cyclic cover of* M_n *is large for every* $n > 0$*. Every Dehn filling of* M_n *with slope* $3p/q$*,* $(3p, q) = 1$*,* $|p| > 1$*, yields a virtually Haken* 3*-manifold.*

Note that by [HT], M_n is hyperbolic, small, and has exactly three boundary slopes, for every $n > 0$. It follows (combining with [CGLS, Theorem 2.0.3]) that all but exactly three Dehn fillings of M_n give irreducible non-Haken 3-manifolds. Also note that each K_{2n+1}, $n > 0$, is a non-fibered knot with a genus one Seifert surface, and thus by [CL] it was known that every m-fold cyclic cover of M_n, $m \geq 4$, is large and every Dehn filling of M_n with slope p/q, $(p, q) = 1$, $|p| \geq 8$, is virtually Haken. It is also known that that all but finitely many Dehn fillings on M_n have virually positive Betti number [DT].

Proof. Let $\widetilde{M_n}$ be the 3-fold cyclic cover of M_n with induced meridian-longitude framing on $\partial\widetilde{M}_n$. We shall show that $\widetilde{M}_n$ contains a connected, essential (i.e., orientable, incompressible, non-boundary-parallel) genus two closed surface which has an essential simple closed curve isotopic to a longitude curve of the cover. It follows from [CGLS, Theorem 2.4.3] that the surface remains incompressible in every Dehn filling of $\widetilde{M}_n$ with slope p/q, $(p, q) = 1$, $|p| > 1$. As every Dehn filling of M_n with slope $3p/q$, $(3p, q) = 1$, $|p| > 1$, is free covered by Dehn filling of $\widetilde{M}_n$ with slope p/q, $(p, q) = 1$, $|p| > 1$, the second conclusion of the theorem will follow.

To make the illustration simple, we first prove the theorem with all details in case $n = 1$, i.e., for the 5_2 knot $K = K_3$. The knot K is tunnel number one, and Figure 2 shows an unknotting tunnel. Also pictured in Figure 2 is a longitude λ of K. Let N be a regular neighborhood of K in S^3, $M = M_1 = \overline{S^3 - N}$, B a regular neighborhood of the unknotting tunnel in M, and $H = \overline{M - B}$. Then H is a handlebody of genus two. Let D be a meridian disk of the 1-handle B whose boundary is shown in Figure 2. We deform the handlebody $H' = N \cup B$ by an isotopy in S^3 so that its exterior H can be recognized as a standard handlebody in S^3 and at the same time we trace the corresponding deformation of ∂D and λ under the isotopy. The process is shown through Figures 3–6.

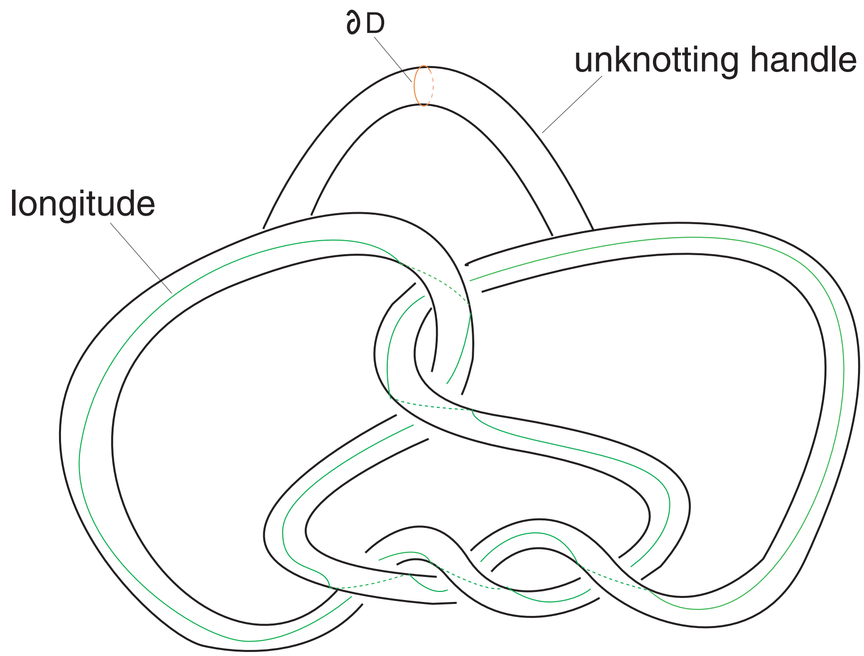

FIGURE 2. An unknotting tunnel, its co-core ∂D and a standard longitude of K

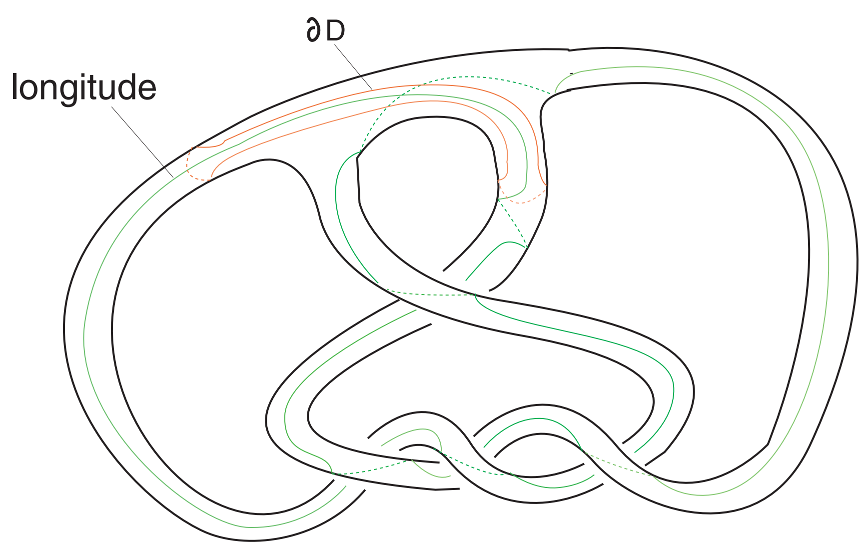

FIGURE 3. The deformation of H', ∂D and λ (part a)

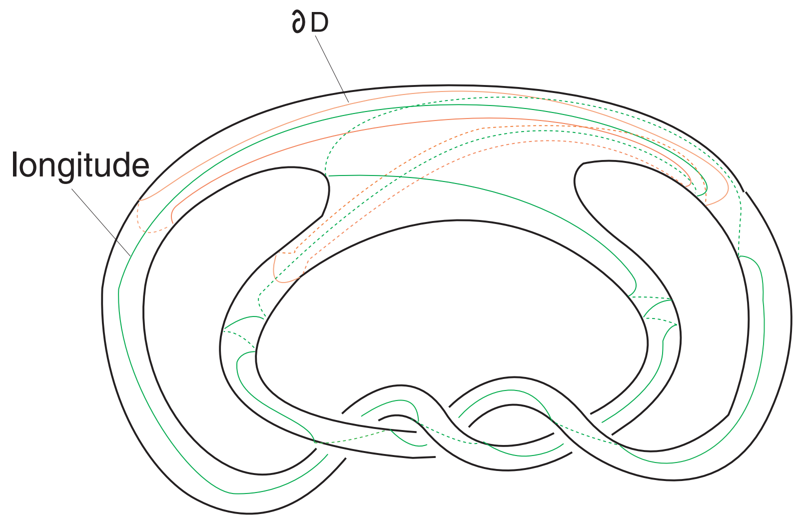

Figure 4. The deformation of H', ∂D and λ (part b)

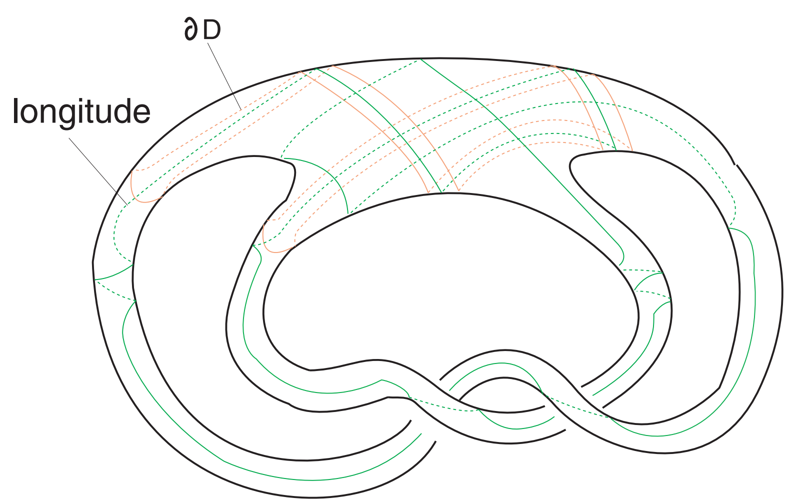

Figure 5. The deformation of H', ∂D and λ (part c)

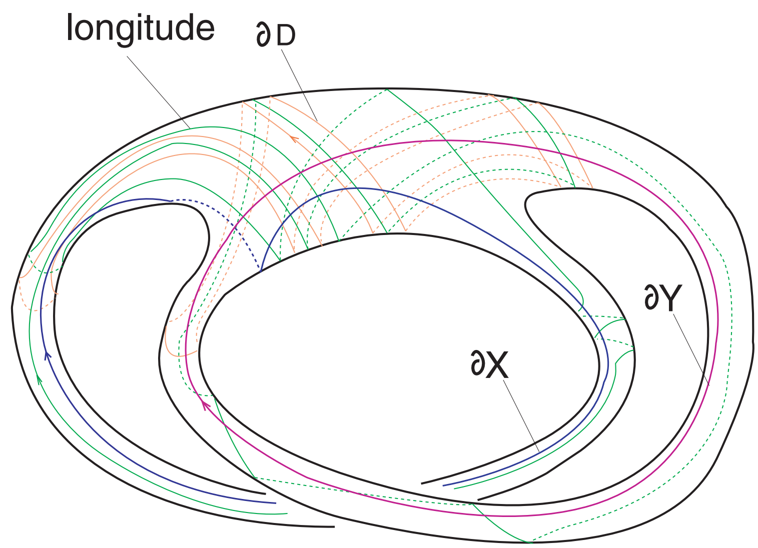

FIGURE 6. The deformation of H', ∂D and λ (part d)

A *meridian disk system* of a handlebody of genus g is a set of g properly embedded mutually disjoint disks in the handlebody such that cutting the handlebody along these disks results in a 3-ball. Let $\{X, Y\}$ be a meridian disk system of H whose boundary is shown in Figure 6. Following ∂D in the given orientation, we get a geometric presentation of the fundamental group $\pi_1(M)$ of M:

$$\pi_1(M) = \langle x, y; x^{-1}y^{-1}x^{-1}yxyxy^{-1}x^{-1}y^{-1}xyxy\rangle,$$

where x is chosen such that it has a representative curve which is a simple closed curve in ∂H which is disjoint from ∂Y and intersects ∂X exactly once and y is also chosen similarly. (We shall call such generators *dual to* the disk system.) Also we can read off the longitude in terms of these two generators:

$$\lambda = yxyx^{-1}y^{-1}x^{-1}y^{-2}x^{-1}y^{-1}x^{-1}yxyx^2.$$

Cutting H along X and Y, we get a 3-ball. Figure 7 shows the boundary 2-sphere of the 3-ball, which records X^+, X^-, Y^+, Y^- and ∂D. Figure 8 shows H in a standard position, and ∂D in ∂H.

The exterior of H in M is a compression body which we denote by C. Topologically, C is $\partial M \times [0, 1]$ with a 1-handle attached on $\partial M \times \{1\}$. It has two boundary components: one is $\partial M = \partial M \times \{0\}$ and the other is the genus two surface ∂H. We have that $H \cup_{\partial H} C$ is a Heegaard splitting of M.

Let $\widetilde{M} = \widetilde{M}_1$ be the 3-fold cyclic cover of $M = M_1$. Note that each of x and y is a generator of $H_1(M; \mathbb{Z}) = \mathbb{Z}$. Let $\widetilde{M}$ have the induced Heegaard splitting from that of M. We can easily give the Heegaard diagram of $\widetilde{M}$, as shown in Figure 9.

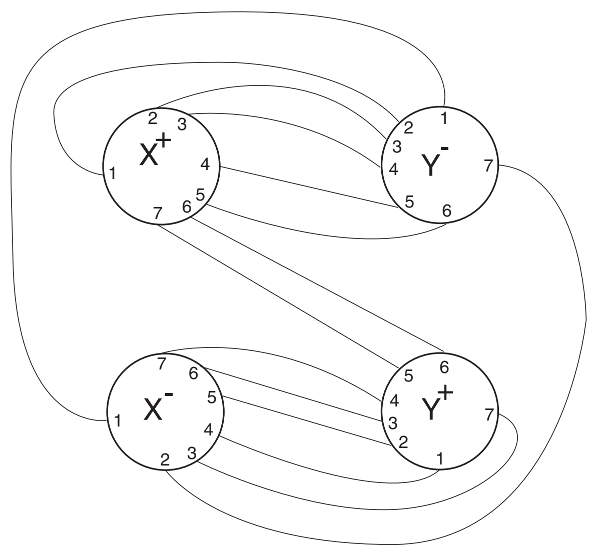

FIGURE 7. ∂D on the sphere $\partial(\overline{H - \{X \times I \cup Y \times I\}})$

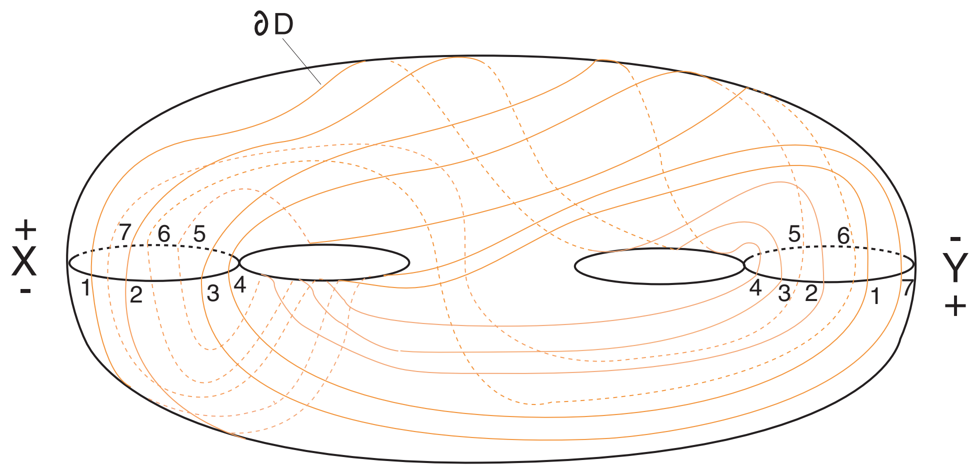

FIGURE 8. H and ∂D in standard position

The genus four handlebody $\widetilde{H}$ in Figure 9 is the corresponding cover of H. The corresponding cover $\widetilde{C}$ of C is a compression body obtained by attaching three 1-handles to $\partial\widetilde{M} \times [0,1]$ on the side $\partial\widetilde{M} \times \{1\}$. The disk X lifts to three disks X_1, X_2, X_3; and the disk Y lifts to three disks Y_1, Y_2, Y_3, as shown in Figure 9. Pick the meridian disk X_4 of $\widetilde{H}$ as shown in Figure 9. Then $\{X_1,\ X_2,\ X_3,\ X_4\}$ forms a disk system of $\widetilde{H}$. The disk D lifts to three disks $\{W_1, W_2,\ W_3\}$ whose boundary

$\{\partial W_1, \partial W_2, \partial W_3\}$ is shown in Figure 9. Figure 9 also shows the longitude $\widetilde{\lambda}$ of $\widetilde{M}$, which is a lift of λ.

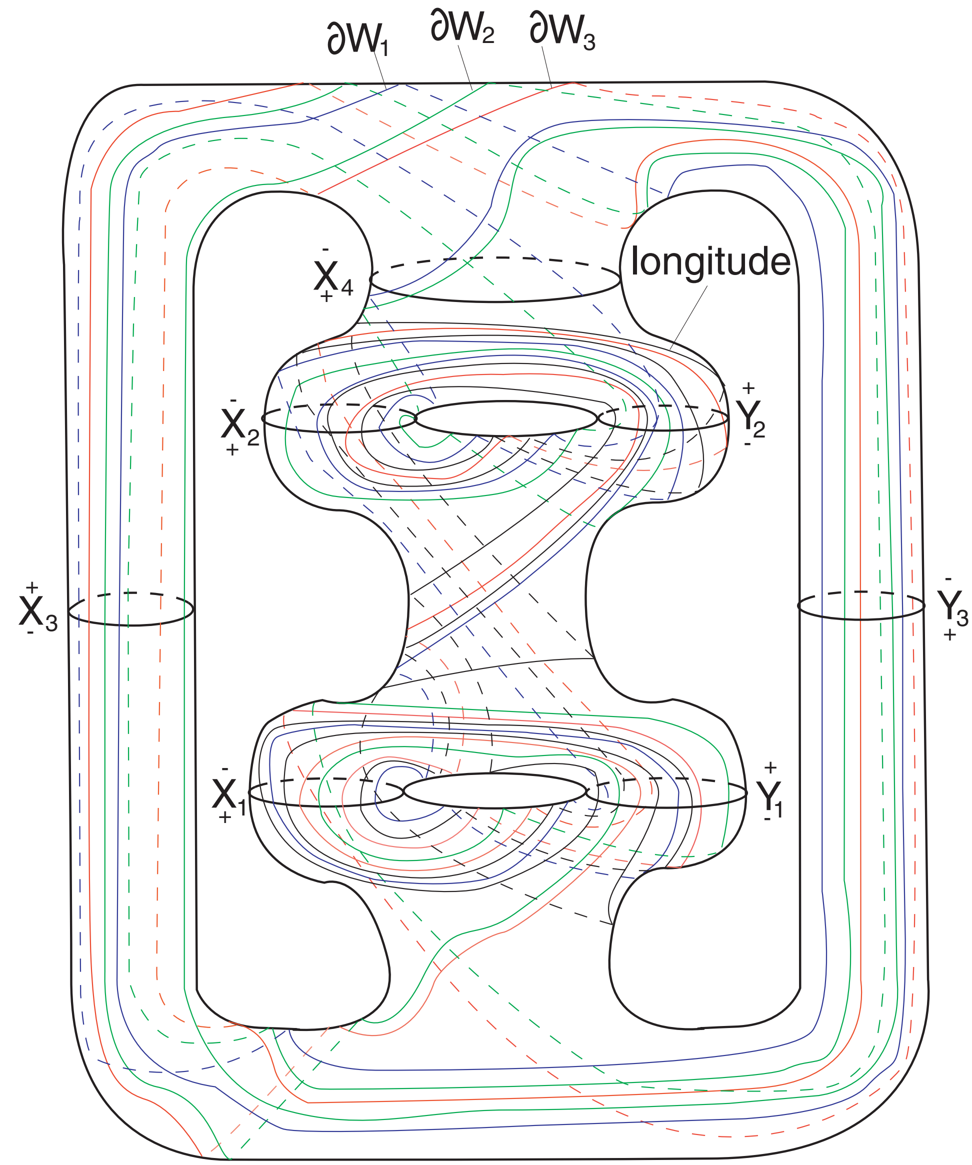

FIGURE 9. The Heegaard diagram of the 3-fold cyclic cover $\widetilde{M}$ and the longitude $\widetilde{\lambda}$

This Heegaard splitting of $\widetilde{M}$ is weakly reducible: ∂X_4 is disjoint from ∂W_3. We now show that the closed, genus 2 surface S obtained by compressing the Heegaard surface $\partial\widetilde{H}$ using the disks W_3 and X_4 is essential in $\widetilde{M}$. It is enough to show that the surface S is incompressible in $\widetilde{M}(2)$, which is the manifold obtained by Dehn filling $\widetilde{M}$ with the slope 2. $\widetilde{M}(2)$ has the induced Heegaard splitting $\widetilde{H}\cup\widetilde{C}(2)$. Note that $\widetilde{M}(2)$ is the free 3-fold cyclic cover of $M(6)$, extending the cover $\widetilde{M}\to M$, and that $\widetilde{C}(2)$ is a handlebody of genus four covering the handlebody $C(6)$ of genus two, extending the cover $\widetilde{C}\to C$. Let $\widetilde{V}$ be the filling solid torus in $\widetilde{M}(2)$ and let W_4 be a meridian disk of $\widetilde{V}$. Then $\{W_1, W_2, W_3, W_4\}$ is a disk system of the handlebody $\widetilde{C}(2)$.

Cutting $\widetilde{H}$ along X_4, we get a handlebody $H_\#$ of genus three, and $\{X_1, X_2, X_3\}$ is a disk system of $H_\#$. Using the Whitehead algorithm [S], we see that $\partial H_\# - \partial W_3$ is incompressible in $H_\#$. In fact, from Figure 9, we can read off the Whitehead graph of ∂W_3 with respect to the disk system $\{X_1, X_2, X_3\}$ of $H_\#$, which is given as Figure 10. The graph is connected with no cut vertex, which means, by the Whitehead algorithm, that ∂W_3 must intersect every essential disk of $H_\#$. Now by the Handle Addition Lemma due to Przytycki [P] and Jaco [J], the manifold $H_\# \cup W_3 \times I$, obtained by attaching the 2-handle $W_3 \times I$ to $H_\#$, has incompressible boundary.

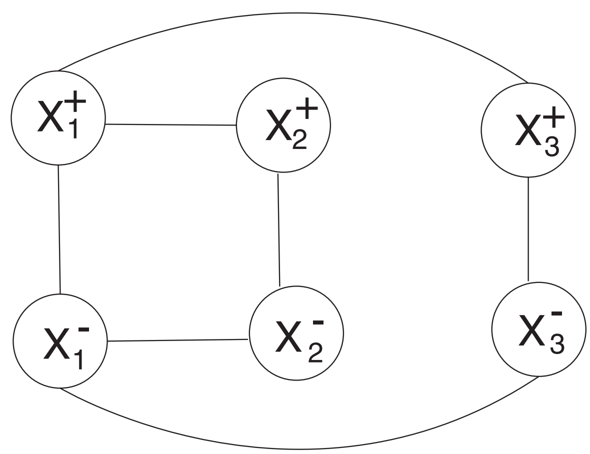

Figure 10. The Whitehead graph of ∂W_3 with respect to the disk system $\{X_1, X_2, X_3\}$ of the handlebody $H_\#$

On the other hand, cutting the handlebody $\widetilde{C}(2)$ along the disk W_3, we get a handlebody H_*, which is homeomorphic to $\widetilde{V}$ with the two 1-handles $W_1 \times I$ and $W_2 \times I$ attached on $\partial\widetilde{V}$. The genus of H_* is three, and $\{W_1, W_2, W_4\}$ gives a disk system. Let $\alpha \subset \partial M$ be an essential simple closed curve of slope 6. We can easily see that with respect to the generators x, y of $\pi_1(M)$,

$$\alpha = \lambda x^6 = yxyx^{-1}y^{-1}x^{-1}y^{-2}x^{-1}y^{-1}x^{-1}yxyx^8.$$

Let $\widetilde{\alpha} \subset \partial\widetilde{M}$ be a lift of α. Then $\widetilde{\alpha}$ has slope 2 in $\partial\widetilde{M}$ which can be considered as the boundary of the disk W_4. Figure 11 shows $\widetilde{\alpha} = \partial W_4$, ∂W_1 and ∂W_2 in $\partial\widetilde{H}$.

Again using the Whitehead algorithm, we see that $\partial H_* - \partial X_4$ is incompressible in $\widetilde{H}_*$. In fact, from Figure 11, we can read off the Whitehead graph of ∂X_4

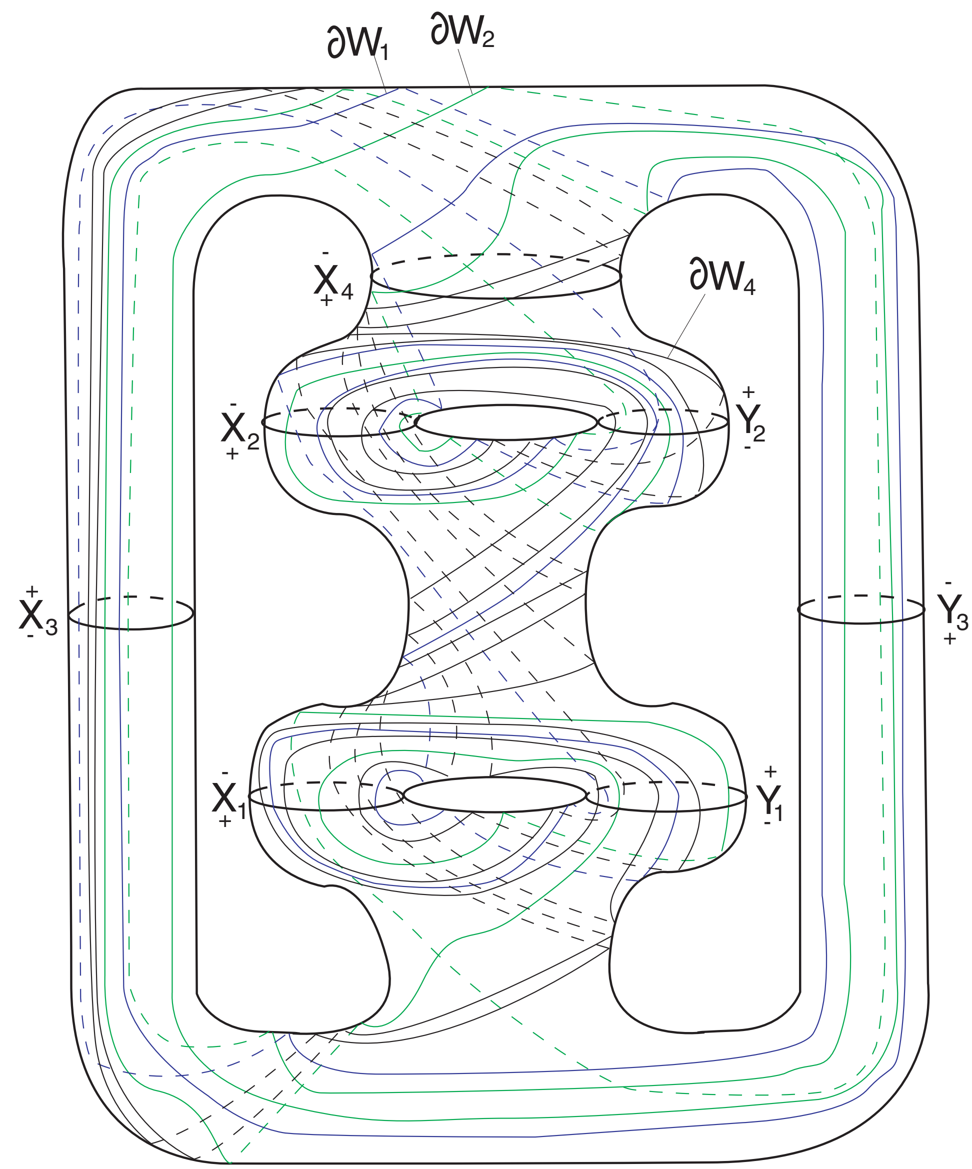

FIGURE 11. $\partial W_4 = \widetilde{\alpha}$, ∂W_1 and ∂W_2 on the Heegaard surface $\partial \widetilde{H}$

with respect to the disk system $\{W_1, W_2, W_4\}$, which is given as Figure 12. The graph is connected with no cut vertex, which means, by the Whitehead algorithm, that $\partial H_* - \partial X_4$ is incompressible in H_*. Again by the Handle Addition Lemma, the manifold $H_* \cup X_4 \times I$ has incompressible boundary of genus two. Note that $\partial(H_* \cup X_4 \times I) = \partial(\widetilde{H}_\# \cup Y_3 \times I) = S$ (up to a small isotopy), and thus S is

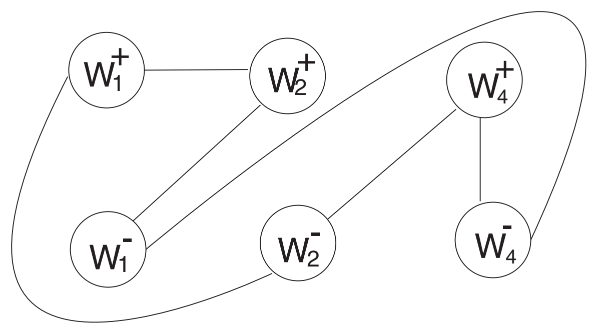

Figure 12. The Whitehead graph of ∂X_4 with respect to the disk system $\{W_1, W_2, W_4\}$ of the handlebody H_*

incompressible in $\widetilde{M}(2)$. But the surface S is contained $\widetilde{M}$, and thus it is an essential surface in $\widetilde{M}$.

In Figure 9, we see that the longitude $\widetilde{\lambda}$ is disjoint from the boundaries of X_4 and W_3, thus it is isotopic to an essential simple closed curve in the surface S. The proof of Theorem 1 is complete for $n = 1$.

The proof for general K_{2n+1}, $n > 0$, is similar. The knot K_{2n+1} is tunnel number one, with an unknotting tunnel shown in Figure 2 (replacing the bottom three crossings by $2n + 1$ crossings). Let M_n be the exterior of K_{2n+1}, H' the handlebody which is a regular neighborhood of the knot and its unknotting tunnel, $H = \overline{M_n - H'}$, and D a meridian disk of the unknotting tunnel. There is a corresponding Heegaard splitting $M_n = H \cup_{\partial H} C$, where C is a compression body. We let λ be a standard longitude. Again we deform the handlebody H' by isotopy in S^3 so that its exterior H can be recognized as a standard handlebody in S^3, while tracing the corresponding deformations of ∂D and λ under the isotopy. In fact, the Heegaard diagram together with the longitude diagram of M_n, for $n > 1$, can be simply obtained by $(n - 1)$ full Dehn twists the diagram Figure 3. Pick two essential disks X and Y for H in a similar way as in $n = 1$ case. From ∂D, we get a geometric presentation of the fundamental group $\pi_1(M_n)$ of M_n with respect to the disk system $\{X, Y\}$:

$$\pi_1(M_n) = \langle x, y; (x^{-1}y^{-1})^n x^{-1}(yx)^{n+1}y^{-1}(x^{-1}y^{-1})^n(xy)^{n+1}\rangle.$$

Also we get

$$\lambda = y(xy)^n(x^{-1}y^{-1})^n x^{-1}y^{-2}(x^{-1}y^{-1})^n x^{-1}(yx)^{n+1}x.$$

Let $\widetilde{M_n}$ be the 3-fold cyclic cover of M_n and let $\widetilde{M_n} = \widetilde{H} \cup_{\partial \widetilde{H}} \widetilde{C}$ have the induced Heegaard splitting from that of M_n, where $\widetilde{H}$ is a genus four handlebody which is the corresponding 3-fold cyclic cover of H and $\widetilde{C}$ a compression body which covers C. Again the disk X lifts to three disks X_1, X_2, X_3; and the disk Y lifts to three disks Y_1, Y_2, Y_3, as shown in Figure 9 (ignore the ∂W_i and $\widetilde{\lambda}$ part), and we pick the meridian disk X_4 of $\widetilde{H}$ as shown in Figure 9. Then $\{X_1,\ X_2,\ X_3,\ X_4\}$ forms a disk system of $\widetilde{H}$. The disk D lifts to three disks $\{W_1, W_2,\ W_3\}$ which form a disk system of $\widetilde{C}$. Again exactly one of the disks $\{W_1,\ W_2,\ W_3\}$, say W_3, is disjoint from X_4, which shows that the Heegaard splitting of $\widetilde{M}_n$ is weakly reducible. Again one can show that the surface S obtained by compressing the Heegaard surface $\partial\widetilde{H}$ using the disks W_3 and X_4 is an essential closed genus two surface in $\widetilde{M}_n$. In fact, cutting $\widetilde{H}$ along X_4, we get a handlebody $H_\#$ of genus three and $\{X_1, X_2, X_3\}$ is a disk system of $H_\#$. The Whitehead graph of ∂W_3 with respect to the disk system $\{X_1, X_2, X_3\}$ of $H_\#$ is given as Figure 13. The graph is connected with no cut vertex, which means that $\partial H_\# - \partial W_3$ is incompressible. Thus by the handle addition lemma, the manifold $H_\# \cup W_3 \times I$, obtained by attaching the 2-handle $W_3 \times I$ to $H_\#$, has incompressible boundary.

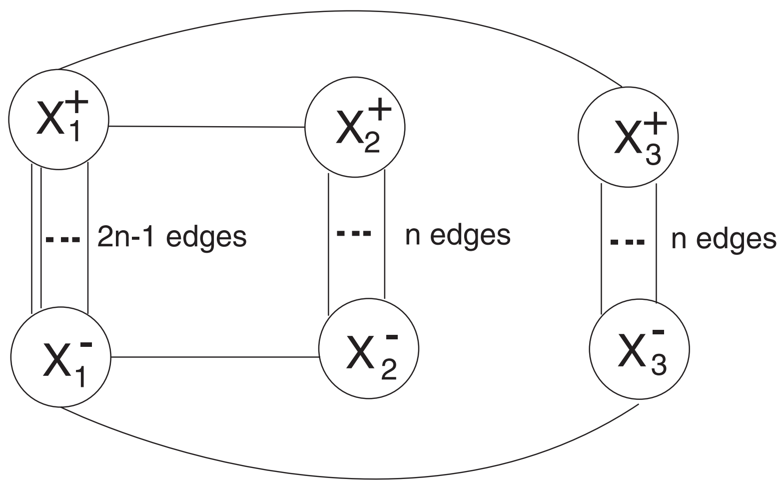

FIGURE 13. The Whitehead graph of ∂W_3 with respect to the disk system $\{X_1, X_2, X_3\}$ of the handlebody $H_\#$

On the other hand, letting $\widetilde{C}(2)$ be the handlebody obtained by Dehn filling $\widetilde{C}$ with slope 2 and letting W_4 be a meridian disk of the filling solid torus, then $\{W_1, W_2, W_3, W_4\}$ forms a disk system of $\widetilde{C}(2)$. Cutting $\widetilde{C}(2)$ along the disk W_3, we get a handlebody H_* with disk system $\{W_1, W_2, W_4\}$. Let $\alpha \subset \partial M$ be an essential simple closed curve of slope 6. Then with respect to the generators x, y of $\pi_1(M)$,

$$\alpha = \lambda x^6 = y(xy)^n(x^{-1}y^{-1})^n x^{-1}y^{-2}(x^{-1}y^{-1})^n x^{-1}(yx)^{n+1}x^7.$$

We may consider ∂W_4 as a lift of α. From the word α, we can draw ∂W_4 on $\partial\widetilde{H}$. Consequently we can read off the Whitehead graph of ∂X_4 with respect to the disk system $\{W_1, W_2, W_4\}$ and see that the graph is the same as that shown in Figure 12, showing that $\partial H_* - \partial X_4$ is incompressible in H_*. Thus the manifold $H_* \cup X_4 \times I$ has incompressible boundary of genus two. We thus have justified the incompressibility of the surface S in $\widetilde{M}_n(2)$ and thus in $\widetilde{M}_n$.

Finally the longitude $\widetilde{\lambda}$ in $\partial\widetilde{M}$ is isotopic to an essential simple closed curve in the surface S, which is obvious. The proof for the general case is complete. □

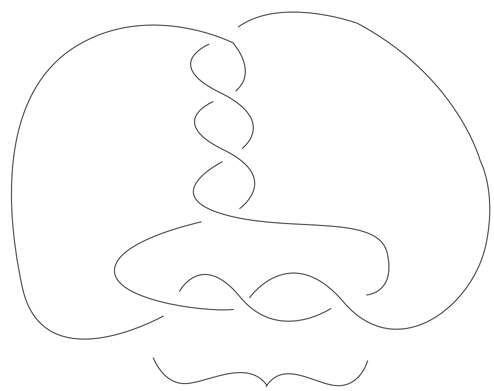

Figure 14. The knot J_{2n+1}

Let J_{2n+1}, $n > 0$, be the family of two bridge knots shown in Figure 14. Note that these knots are hyperbolic, small and non-fibered with genus two Seifert surfaces.

Theorem 2. *The* 5*-fold cyclic cover of the exterior of* J_{2n+1} *is large and every Dehn filling of the exterior of* J_{2n+1} *with slope* $5p/q$*,* $(5p, q) = 1$*,* $|p| > 1$*, yields a virtually Haken* 3*-manifold, for every* $n > 0$*.*

This theorem gives another family of non-Haken, virtually Haken 3-manifolds to which the results of [CL] do not apply (e.g., the fillings of the exterior of J_{2n+1} with slopes $5/q$, $(5, q) = 1$). As the proof of Theorem 2 is very similar to that of Theorem 1, we omit the details and indicate only the steps. In fact the exterior of J_{2n+1} is tunnel number one and a genus two Heegard splitting of it can be explicitly given as in the case for the exterior of the twist knot K_{2n+1}. In the 5-fold cyclic cover of the exterior of J_{2n+1}, the lifted Heegaard surface is of genus 6 and can be compressed along two reducing disks, one on each side of the Heegaard surface, to a closed incompressible surface of genus 4. Also a lift of the longitude can be isotoped into the resulting incompressible surface.

We now go back to the twist knots K_{2n+1} and prove the following Theorem 3. Although the result of the theorem is covered by [CL], we have included it primarily because its proof illustrates two complications which arise in more general settings. First, we have to deal with multi 2-handle additions, which requires the multi 2-handle addition theorem of Lei [L]. Also, one of the Whitehead graphs contains a cut vertex, and must be simplified using Whitehead moves.

Theorem 3. *The* 5*-fold cyclic cover of the exterior* M_n *of* K_{2n+1} *is large for every* $n > 0$*. Every Dehn filling of* M_n *with slope* $5p/q$*,* $(5p, q) = 1$*,* $|p| > 1$*, yields a virtually Haken* 3*-manifold.*

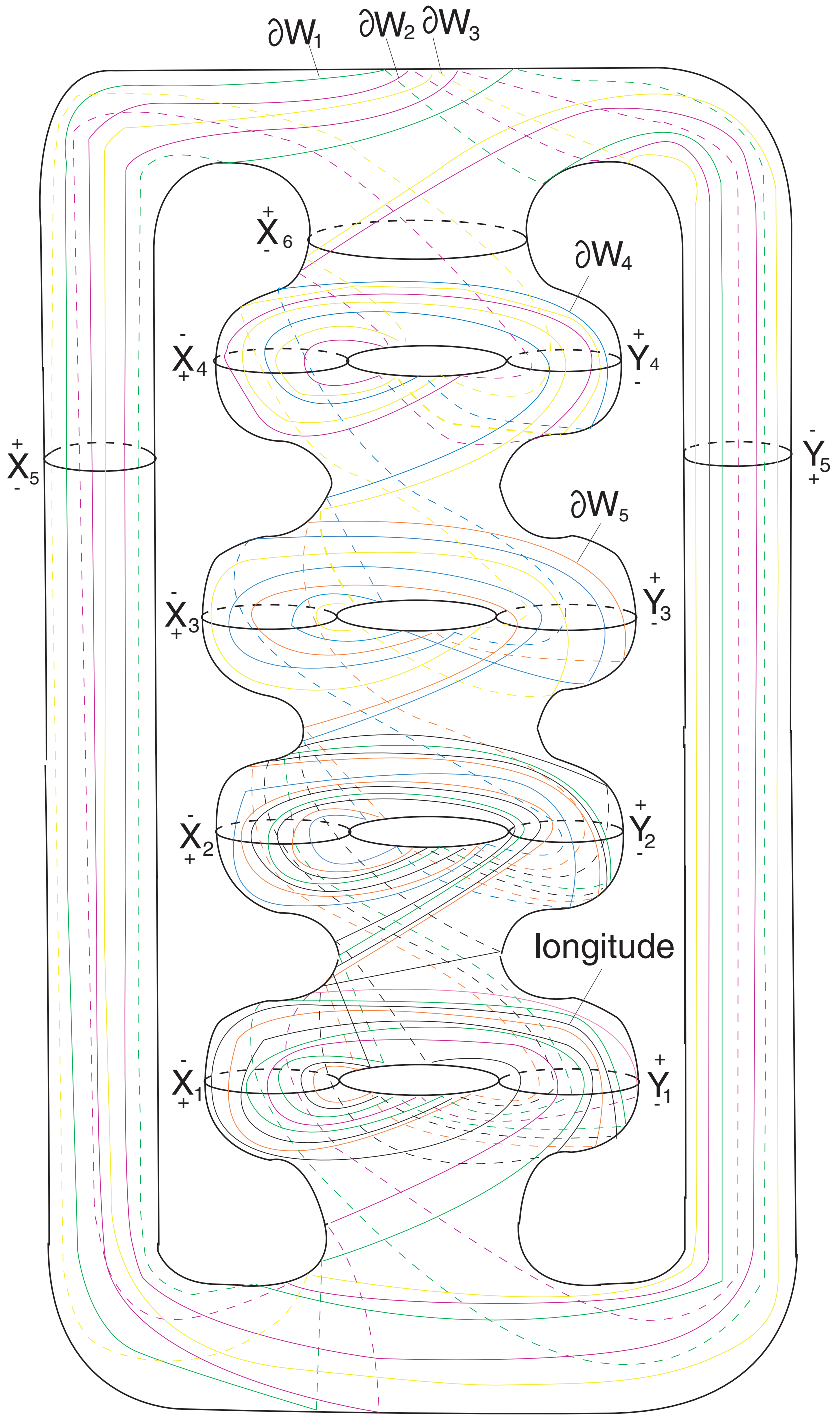

FIGURE 15. The Heegaard splitting of the 5-fold cover of M

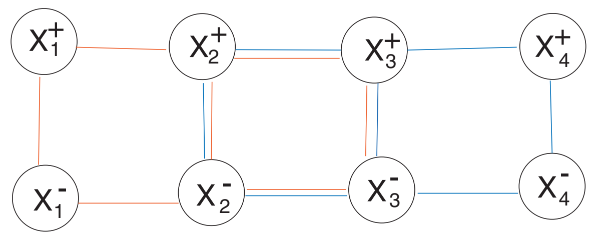

FIGURE 16. The Whitehead graph of $\{\partial W_4, \partial W_5\}$ with respect to the disk system $\{X_1, X_2, X_3, X_4\}$ of the handlebody $H_\#$

Proof. Again we give details only for the $n = 1$ case. We continue to use the Heegaard splitting of $M = M_1 = H \cup C$ as given in the proof of Theorem 1. Let $\widetilde{M}$ be the 5-fold cyclic cover of M with the induced Heegaard splitting from that of M. The Heegaard diagram of $\widetilde{M}$ is shown in Figure 15. The genus six handlebody of Figure 14 is $\widetilde{H}$ which covers H. The disks X and Y of H lift to disks $X_1, ..., X_5$ and $Y_1, ..., Y_5$, as shown in Figure 15. Pick the meridian disk X_6 of $\widetilde{H}$ as shown in Figure 15. Then $\{X_1, X_2, X_3, X_4, Y_5, X_6\}$ forms a disk system of $\widetilde{H}$. The disk D lifts to five disks $\{W_1, W_2, W_3, W_4, W_5\}$ whose boundaries are shown in Figure 15. Figure 15 also shows a longitude $\widetilde{\lambda}$ of $\widetilde{M}$, which is a lift of the longitude λ of M.

This Heegaard splitting of $\widetilde{M}$ is weakly reducible: $\{\partial Y_5, \partial X_6\}$ is disjoint from $\{\partial W_4, \partial W_5\}$. We now show that the surface S obtained by compressing the Heegaard surface $\partial\widetilde{H}$ using these four disks is an essential closed genus two surface in $\widetilde{M}$. It is enough to show that the surface S is incompressible in $\widetilde{M}(2)$, which is the free 5-fold cyclic cover of $M(10) = H \cup C(10)$, and has the induced Heegaard splitting $\widetilde{H} \cup \widetilde{C}(2)$. Let $\widetilde{V}$ be the filling solid torus in $\widetilde{M}(2)$ and let W_6 be a meridian disk of $\widetilde{V}$. Then $\{W_1, ..., W_5, W_6\}$ is a disk system of the handlebody $\widetilde{C}(2)$.

Cutting $\widetilde{H}$ along Y_5, X_6, we get a handlebody $H_\#$ of genus four and $\{X_1, X_2, X_3, X_4\}$ is a disk system of $H_\#$. The Whitehead graph of $\{\partial W_4, \partial W_5\}$ with respect to the disk system $\{X_1, ..., X_4\}$ of $H_\#$ is given in Figure 16. The graph is connected with no cut vertex, which means that the surface $\partial H_\# - \{\partial W_4, \partial W_5\}$ is incompressible in $H_\#$. Moreover as ∂W_4 is disjoint from the disk X_1, and ∂W_5 is disjoint from the disk X_4, each of the surfaces $\partial H_\# - \partial W_4$ and $\partial H_\# - \partial W_5$ is compressible in $H_\#$. Therefore all the conditions of the multi-handle addition theorem of [L] are satisfied, and thus the manifold $H_\# \cup W_4 \times I \cup W_5 \times I$ has incompressible boundary.

On the other hand, cutting the handlebody $\widetilde{C}(2)$ along the disks W_4 and W_5, we get a handlebody H_*, with disk system $\{W_1, W_2, W_3, W_6\}$. Let $\alpha \subset \partial M$ be an essential simple closed curve of slope 10. Then

$$\alpha = \lambda x^{10} = yxyx^{-1}y^{-1}x^{-1}y^{-2}x^{-1}y^{-1}x^{-1}yxyx^{12}.$$

Let $\widetilde{\alpha} \subset \partial\widetilde{M}$ be a lift α. Then $\widetilde{\alpha}$, which can be considered as the boundary of the disk W_6, has slope 2 in $\partial\widetilde{M}$. Figure 17 shows $\widetilde{\alpha} = \partial W_6$, ∂W_1,∂W_2, ∂W_3 in $\partial\widetilde{H}$.

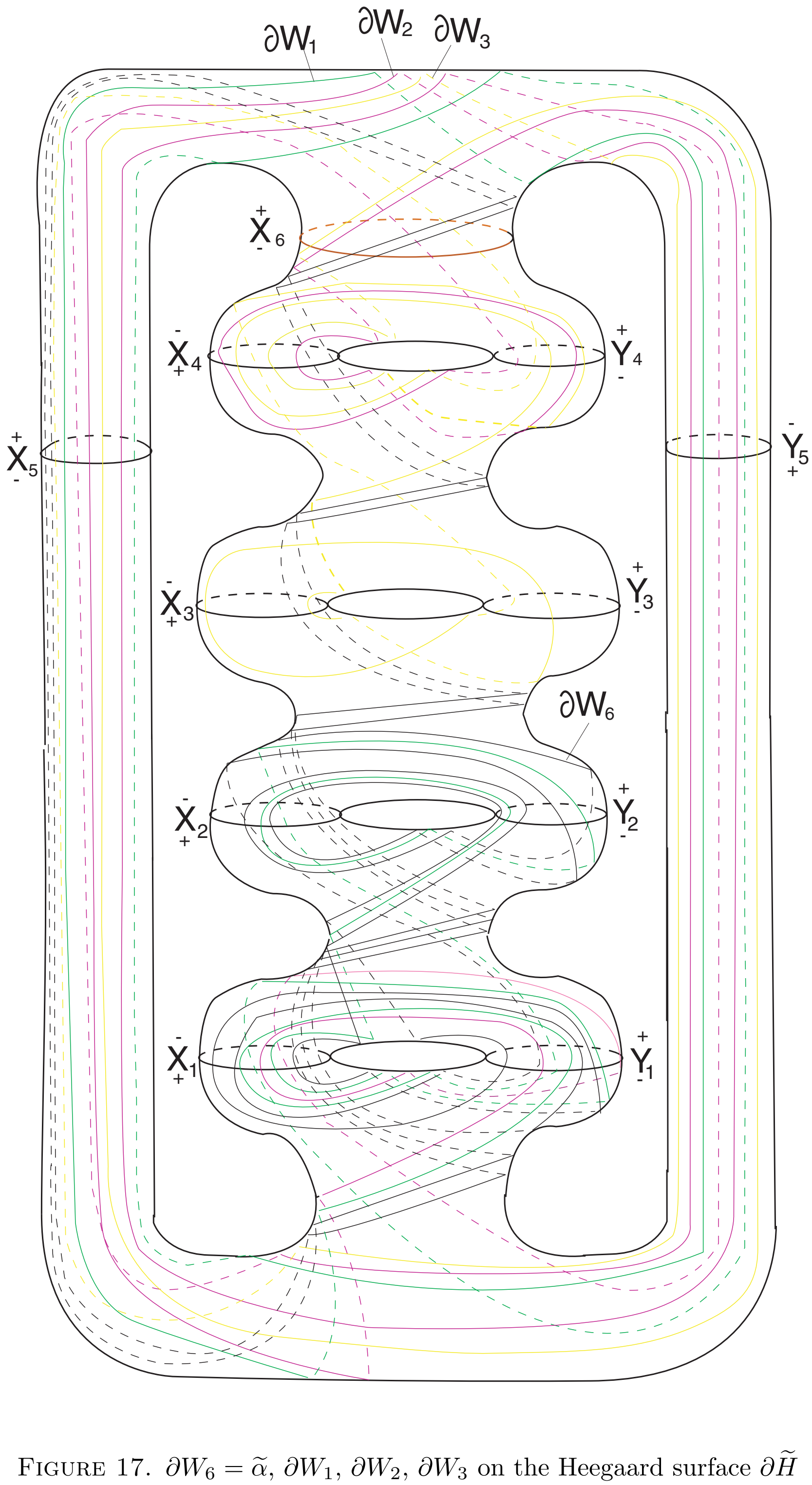

FIGURE 17. $\partial W_6 = \widetilde{\alpha}$, ∂W_1, ∂W_2, ∂W_3 on the Heegaard surface $\partial \widetilde{H}$

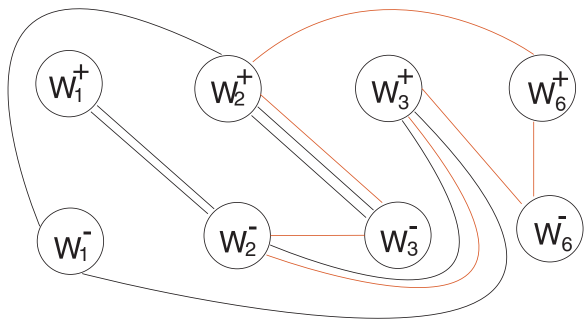

FIGURE 18. The Whitehead graph of $\{\partial Y_5, \partial X_6\}$ with respect to the disk system $\{W_1, W_2, W_3, W_6\}$ of the handlebody H_*

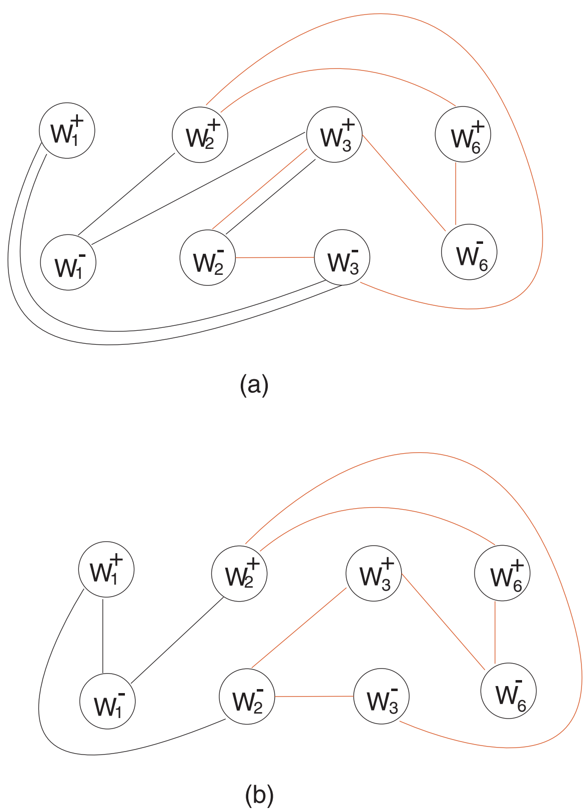

FIGURE 19. (a) The resulting graph after the Whitehead move with respect to the cut vertex W_2^- of Figure 18. (b) The resulting graph after the Whitehead move with respect to the cut vertex W_3^- of part (a).

From Figure 17, we can read off the Whitehead graph of $\{\partial Y_5, \partial X_6\}$ with respect to the disk system $\{W_1, W_2, W_3, W_6\}$ of H_*, which is given as Figure 18. The graph is connected but has a cut vertex (the vertex W_2^-). Applying Whitehead moves to the graph twice with results shown in Figure 19, we end up with a graph (shown in Figure 19 (b)) which is connected with no cut vertex. This means that the surface $\partial H_* - \{\partial Y_5 \cup \partial X_6\}$ is incompressible in H_*. From Figure 16, we also see that ∂Y_5 is disjoint from ∂W_6 and ∂X_6 is disjoint from ∂W_1. Thus each of the surfaces $\partial H_* - \partial Y_5$ and $\partial H_* - \partial X_6$ is compressible in H_*. Again the multi-handle addition theorem of [L] implies that the manifold $H_* \cup X_6 \times I \cup Y_5 \times I$ has incompressible boundary. Therefore the genus two surface $S = \partial(H_* \cup X_6 \times I \cup Y_5 \times I) = \partial(H_\# \cup W_4 \times I \cup W_5 \times I)$ is incompressible in $\widetilde{M}(2)$ and thus is essential in $\widetilde{M}$.

Obviously $\widetilde{\lambda}$ can be isotoped into S. The proof of Theorem 3 is complete in case $n = 1$. The proof for the general case is similar (cf. the proof of Theorem 1 in general case). We leave the details to the reader to verify. □

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