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Improving tameness for metabelian groups

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Abstract. We show that any finitely generated metabelian group can be embedded in a metabelian group of type F_3. More generally, we prove that if n is a positive integer and Q is a finitely generated abelian group, then any finitely generated $\mathbb{Z}Q$-module can be embedded in a module that is n-tame. Combining with standard facts, the F_3 embedding theorem follows from this and a recent theorem of R. Bieri and J. Harlander.

Contents

1. Metabelian groups

This paper is about finiteness and geometric properties of metabelian groups. The story begins in the 1970s with a series of papers by G. Baumslag and V. R. Remeslennikov, who independently investigated finitely generated and finitely presented metabelian groups and showed that the theory of these groups is more complex than one might expect. For example [2, 9], there is a finitely presented metabelian group that contains a free abelian subgroup of infinite rank. Nevertheless, they proved [3, 9] that every finitely generated metabelian group can be embedded in a finitely presented one.

Finite generation and finite presentability are the first two in a hierarchy of increasingly strong finiteness properties of groups. A group G is of *type* F_n if there is a connected aspherical CW complex with fundamental group isomorphic to G (that is, an Eilenberg–Maclane complex of type $K(G,1)$) with finite n-skeleton. Type F_1 is equivalent to finite generation, type F_2 is equivalent to finite presentability, and type F_{n+1} implies type F_n. The main general result of this paper is the following:

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Theorem 1.1. *Every finitely generated metabelian group can be embedded in a metabelian group of type F_3.*

Our eventual aim is to improve this result to enable embeddings in metabelian groups of type F_n for arbitrarily large n and we make steps in that direction in this paper. This is the best one can hope for within the class of metabelian groups since a result of R. Bieri and J. R. J. Groves implies that if a metabelian group G admits a $K(G,1)$ with finitely many cells in each dimension (so that G is of type F_∞), then there is a uniform bound on the rank of the free abelian subgroups of G. That this embedding restriction also applies in the more general context of soluble groups follows from subsequent work of P. H. Kropholler [8].

In order to detect higher finiteness properties, R. Bieri and R. Strebel introduced the *Sigma theory* in [6]. The theory continues to evolve and there are many papers on the subject; see [1, 4, 5, 6, 7, 10] and the references therein. This paper deals only with the Sigma invariants of a module M over the group ring $\mathbb{Z}Q$ of a finitely generated abelian group Q. We summarize those portions of the theory that we need in §2, focusing on the concept of *n-tameness* for $\mathbb{Z}Q$-modules. The *F_n-Conjecture* asserts that an extension G of an abelian group Q by a module M is of type F_n if and only if M is n-tame. The conjecture is true for $n = 2$ [6], for metabelian groups G of finite Prüfer rank [1], for torsion modules M of Krull dimension one [7], and for split extensions $G = M \rtimes Q$ when $n = 3$ [5].

Our progress toward a general F_n-embedding theorem for finitely generated metabelian groups is best summarized as follows.

Theorem 1.2. *Given a positive integer n, any finitely generated metabelian group can be embedded in a split metabelian group of the form $M \rtimes Q$ where Q is a finitely generated abelian group and M is an n-tame $\mathbb{Z}Q$-module.*

One might paraphrase this to say that there are no restrictions, other than being metabelian, on the finitely generated subgroups of n-tame metabelian groups.

Embeddings are achieved using a *localization* procedure that played a central role in [3, 9]. We summarize this procedure and related facts about metabelian groups in §3. The heart of the paper is in §4 where we show that localization improves tameness. The concluding §5 introduces the idea of an *essential decomposition* for a module and uses this to complete the proofs of Theorems 1.1 and 1.2.

2. Tameness

For a finitely generated abelian group Q, $V(Q)$ denotes the set of real-valued homomorphisms (or characters) from Q to the additive group of real numbers: $V(Q) = \mathrm{Hom}(Q, \mathbb{R})$. This is a Euclidean space with dimension equal to the torsion-free rank of Q.

Let M be a $\mathbb{Z}Q$-module. Given a character $\chi \in V(Q)$, there is the submonoid Q_χ of Q consisting of those $q \in Q$ for which $\chi(q) \geq 0$, and there is the subring $\mathbb{Z}Q_\chi$ of $\mathbb{Z}Q$. The *Sigma set* of the $\mathbb{Z}Q$-module M is

$$\Sigma(M,Q) = \{\chi \in V(Q) : M \text{ is finitely generated as a } \mathbb{Z}Q_\chi\text{-module}\}$$

and the *Sigma complement* is

$$\Sigma(M,Q)^c = V(Q) - \Sigma(M,Q).$$

A great deal is known about the Sigma set and the geometry of its complement (see e.g., [5]), but we will require only the following fact:

Lemma 2.1 ([6, Proposition 2.1])**.** *Let Q be a finitely generated abelian group and let M be a finitely generated $\mathbb{Z}Q$-module. A nonzero character $0 \neq \chi \in V(Q)$ lies in $\Sigma(M,Q)$ if and only if there is a centralizing element $c \in \operatorname{Cent}_{\mathbb{Z}Q}(M)$ such that $\chi(q) > 0$ for all q in the support of c.*

Tameness is formulated in terms of the geometry of the Sigma complement. The complement of every hyperplane in the Euclidean space $V(Q)$ consists of two convex open subspaces, called *open half-spaces*. For a positive integer n, the $\mathbb{Z}Q$-module is said to be *n-tame* if each n-element subset of the Sigma complement $\Sigma(M,Q)^c$ is contained in some open half-space of $V(Q)$. To be explicit, the module M is n-tame if whenever $\chi_1, \dots, \chi_n \in \Sigma(M,Q)^c$, then the only nonnegative solution to $\sum_{i=1}^{n} t_i\chi_i = 0$ is $t_1 = \cdots = t_n = 0$.

Thus, M is 1-tame if and only if $0 \in \Sigma(M,Q)$, which amounts to saying that M is finitely generated as a $\mathbb{Z}Q$-module. Higher tameness properties are unaffected if we view the Sigma complement in the sphere $S(Q)$ consisting of positive rays of nonzero characters in $V(Q)$ and formulate the property in terms of open hemispheres. For example, 2-tameness amounts to saying that $\Sigma(M,Q)^c$ contains no antipodal points in the sphere $S(Q)$. Note that $(n+1)$-tame implies n-tame.

3. Localization

In order to embed a finitely generated metabelian group in one with better finiteness properties, one can first reduce to the split case.

Lemma 3.1 ([3, Lemma 3])**.** *Every finitely generated metabelian group G can be embedded in a finitely generated split metabelian group of the form $M \rtimes Q$ where $Q = G^{\mathrm{ab}}$ is a finitely generated abelian group and M is a finitely generated $\mathbb{Z}Q$-module.*

This reduces the problem to one involving embeddings of modules.

In showing how to embed finitely generated split metabelian groups in finitely presented ones, Baumslag [3] made essential use of a localization construction from commutative ring theory. We briefly recall the details. Let Q be a finitely generated abelian group and let M be a $\mathbb{Z}Q$-module. Consider a subset S of the group ring $\mathbb{Z}Q$ with the following properties:

- S is unital, in that $1 \in S$;
- S is multiplicatively closed, in that $S \cdot S \subseteq S$;
- S acts freely on M, in that if $s \in S$ and $m \in M$ are such that $sm = 0$, then $m = 0$;

Suppose that $\mathbf{y}$ generates S as a multiplicative submonoid of $\mathbb{Z}Q$, and let $\overline{Q}$ be the direct product of Q with the free abelian group with basis consisting of elements z_y, $y \in \mathbf{y}$. Localization produces a $\mathbb{Z}\overline{Q}$-module $\overline{M}$ that arises as the set of equivalence classes of the relation on $S \times M$ given by

$$(s,m) \sim (s',m') \Leftrightarrow sm' = s'm.$$

(The fact that $sm = 0$ only if $m = 0$ is required to verify that this relation is transitive.) The algebraic structure of $\overline{M}$ is determined as follows:

$$\begin{aligned} (s,m) + (s',m') &= (ss', sm' + s'm) \\ \lambda(s,m) &= (s, \lambda m) \quad (\lambda \in \mathbb{Z}Q) \\ z_y(s,m) &= (s, ym) \\ z_y^{-1}(s,m) &= (sy, m) \end{aligned}$$

It is straightforward to verify the following standard properties of the localized module $\overline{M}$:

- M embeds in $\overline{M}$ as a $\mathbb{Z}Q$-submodule.
- Any $\mathbb{Z}Q$-generating set for M determines a $\mathbb{Z}\overline{Q}$-generating set for $\overline{M}$.
- The annihilator $\operatorname{Ann}_{\mathbb{Z}\overline{Q}}(\overline{M})$ is the smallest ideal in $\mathbb{Z}\overline{Q}$ containing $\operatorname{Ann}_{\mathbb{Z}Q}(M)$ and all elements of the form $z_y - y$, $y \in \mathbf{y}$.

The localized module is usually denoted by $\overline{M} = S^{-1}M$ in the literature, and is viewed as a module over a localized version $S^{-1}\mathbb{Z}Q$ of the group ring $\mathbb{Z}Q$. It is more convenient for our purposes to focus on the status of $\overline{M}$ as a module over the expanded group ring $\mathbb{Z}\overline{Q}$.

A nonconstant polynomial with integer coefficients is called *special* if its leading and constant coefficients are both 1. When q is an element of infinite order in a finitely generated abelian group Q, a special polynomial $p(X)$ uniquely determines an element $p(q) \in \mathbb{Z}Q$ that is neither a unit nor a zero divisor in the group ring. Baumslag used the fact that the group ring $\mathbb{Z}Q$ is noetherian to show that for any element q of infinite order in Q and any finitely generated $\mathbb{Z}Q$-module M, there is a special polynomial $p(X)$ such that $p(q)$ acts freely on M:

Lemma 3.2 ([3, Lemma 7]). *Suppose that we are given an element q of infinite order in a finitely generated abelian group Q. If M is a finitely generated $\mathbb{Z}Q$-module, then there is a special polynomial $p(X)$ such that if $m \in M$ and $p(q)m = 0$, then $m = 0$.*

Given $q \in Q$, M, and $p(X)$ as above, the multiplicative submonoid S of $\mathbb{Z}Q$ generated by $p(q)$ acts freely on M so we can embed M in the module $S^{-1}M = \overline{M}$ over $\mathbb{Z}\overline{Q}$ where $\overline{Q} = Q \times \langle z \rangle$. The action of $p(q)$ on M is then invertible in the larger module $\overline{M}$. For future reference, we shall say that the module $\overline{M}$ is obtained from M by a *special localization in the q direction*. More generally, we will consider simultaneous special localizations in directions q_i taken from linearly independent subsets of Q.

4. Improving tameness

We now show that localization can be used to improve tameness. We first use Lemma 2.1 to investigate how the Sigma complement behaves under passage to subgroups.

Lemma 4.1. *Let A be a subgroup of a finitely generated abelian group Q. Let M be a finitely generated $\mathbb{Z}Q$-module and let M_A be a finitely generated $\mathbb{Z}A$-module such that* $\operatorname{Ann}_{\mathbb{Z}A}(M_A) \subseteq \operatorname{Ann}_{\mathbb{Z}Q}(M)$. *If $\chi \in \Sigma(M,Q)^c$, then either $\chi|_A = 0$ or $\chi|_A \in \Sigma(M_A, A)^c$.*

Proof. Assume that $\operatorname{Ann}_{\mathbb{Z}A}(M_A) \subseteq \operatorname{Ann}_{\mathbb{Z}Q}(M)$ and $0 \neq \chi|_A \in \Sigma(M_A, A)$. It suffices to show that $\chi \in \Sigma(M,Q)$. By Lemma 2.1, there is a centralizing element $c \in \operatorname{Cent}_{\mathbb{Z}A}(M_A)$ such that $\chi(a) > 0$ for all a in the support of c. But $c \in \operatorname{Cent}_{\mathbb{Z}A}(M_A) = 1 + \operatorname{Ann}_{\mathbb{Z}A}(M_A) \subseteq 1 + \operatorname{Ann}_{\mathbb{Z}Q}(M) = \operatorname{Cent}_{\mathbb{Z}Q}(M)$ and so $\chi \in \Sigma(M,Q)$ by Lemma 2.1. $\square$

It is worth noting that if M_A is the $\mathbb{Z}A$-submodule of M spanned by a $\mathbb{Z}Q$-generating set of M, then $\operatorname{Ann}_{\mathbb{Z}A}(M_A) \subseteq \operatorname{Ann}_{\mathbb{Z}Q}(M)$.

Our next objective is to see how special localization affects the Sigma complement.

Lemma 4.2. *Suppose that q and z are elements of a finitely generated abelian group Q and that M is a finitely generated $\mathbb{Z}Q$-module. Let $p(X)$ be a special polynomial of degree $d \geq 1$ such that $z - p(q) \in \operatorname{Ann}_{\mathbb{Z}Q}(M)$. If χ is an element of the Sigma complement $\Sigma(M,Q)^c$ of M, then exactly one of the following conclusions applies:*

(1) *If $\chi(q) = 0$, then $\chi(z) \geq 0$.*
(2) *If $\chi(q) > 0$, then $\chi(z) = 0$.*
(3) *If $\chi(q) < 0$, then $\chi(z) = d\chi(q)$.*

Proof. We have annihilating elements $z - p(q)$, $-z^{-1}(z - p(q))$, and $q^{-d}(z - p(q))$ in $\operatorname{Ann}_{\mathbb{Z}Q}(M)$. These determine centralizing elements c_1, c_2, and $c_3 \in \operatorname{Cent}_{\mathbb{Z}Q}(M)$ with supports contained in the following lists.

$$\begin{array}{ll} c_1: & z, q^i, i = 1, \ldots, d \\ c_2: & z^{-1}q^i, i = 0, \ldots, d \\ c_3: & q^{-d}z, q^{-i}, i = 1, \ldots, d. \end{array}$$

Since the polynomial $p(X)$ is special, the elements z^{-1} and $z^{-1}q^d$ are in the support of c_2. Lemma 2.1 implies that each c_k has a support element with nonpositive value under the character χ.

(1) Here $\chi(q) = 0$. The centralizing element c_2 implies that $\chi(z^{-1}q^i) \leq 0$ for some i, so that $\chi(z) \geq 0$.

(2) Here $\chi(q) > 0$. The centralizing element c_1 implies that $\chi(z) \leq 0$ and the centralizing element c_2 implies that $\chi(z^{-1}q^i) \leq 0$ for some $i = 0, \ldots, d$, that is, $\chi(z) \geq \min\{i\chi(q) : i = 0, \ldots, d\} = 0$. Thus $\chi(z) = 0$ in this case.

(3) Here $\chi(q) < 0$. The centralizing element c_3 implies that $\chi(q^{-d}z) \leq 0$, or rather, $\chi(z) \leq d\chi(q)$. The centralizing element c_2 implies that $\chi(z^{-1}q^i) \leq 0$ for some $i = 0, \ldots, d$, that is $\chi(z) \geq \min\{i\chi(q) : i = 0, \ldots, d\} = d\chi(q)$. We conclude that $\chi(z) = d\chi(q)$ in this case. $\square$

In the setting of Lemma 4.2, the character value $\chi(z)$ is precisely specified except when $\chi(q) = 0$. We say that an element q of infinite order in Q is an *essential direction* for M if $\chi(q) \neq 0$ for all $\chi \in \Sigma(M,Q)^c$. Our main innovation is to note that localization in an essential direction improves tameness.

Lemma 4.3. *Let Q be a finitely generated abelian group and let M be a finitely generated $\mathbb{Z}Q$-module with essential direction $q \in Q$. Let $p(X)$ be a special polynomial of degree $d \geq 1$. Let $\overline{Q} = Q \times \langle z \rangle$ be the direct product of Q with the infinite cyclic group generated by z and suppose that $\overline{M}$ is a finitely generated $\mathbb{Z}\overline{Q}$-module such that $z - p(q) \in \operatorname{Ann}_{\mathbb{Z}\overline{Q}}(\overline{M})$ and $\operatorname{Ann}_{\mathbb{Z}Q}(M) \subseteq \operatorname{Ann}_{\mathbb{Z}\overline{Q}}(\overline{M})$. We conclude the following:*

(1) *If M is k-tame, then $\overline{M}$ is $(k+1)$-tame.*
(2) *$\overline{M}$ has an essential direction $\bar{q} \in \overline{Q}$.*

Proof. Assume that M is k-tame. Let $\overline{\chi}_1, \ldots, \overline{\chi}_{k+1} \in \Sigma(\overline{M}, \overline{Q})^c$ and suppose that we are given nonnegative real scalars $t_1, \ldots, t_{k+1}$ such that $\sum_i t_i \overline{\chi}_i = 0$. We must show that $t_i = 0$ for all $i = 1, \ldots, k+1$. For each i, let $\chi_i = \overline{\chi}_i|_Q$.

Consider the case when $\chi_i \neq 0$ for all i. Then $\chi_i \in \Sigma(M, Q)^c$ by Lemma 4.1 so $\chi_i(q) \neq 0$ for all i since q is essential for M. Let $q^* = qz^{-2} \in \overline{Q}$. For any given i, if $\chi_i(q) > 0$, then $\overline{\chi}_i(q^*) = \chi_i(q)$, while if $\chi_i(q) < 0$, then $\overline{\chi}_i(q^*) = (1-2d)\chi_i(q)$ by Lemma 4.2. From this we conclude that $\overline{\chi}_i(q^*) > 0$ for all i. Then the fact that $0 = \sum_i t_i \overline{\chi}_i(q^*)$ implies that $t_i = 0$ for all i.

Now suppose that $\ell \leq k$ and that $\chi_i \neq 0$ if and only if $i \leq \ell$. Then $0 = \sum_{i=1}^{\ell} t_i \chi_i$ where $\chi_i \in \Sigma(M, Q)^c$ for $i = 1, \ldots, \ell$ by Lemma 4.1. The fact that M is k-tame thus implies that $t_i = 0$ for those i. We thus conclude that $\sum_{i=\ell+1}^{k+1} t_i \overline{\chi}_i = 0$. Given $\ell+1 \leq i \leq k+1$, Lemma 4.2 shows that $\overline{\chi}_i(z) \geq 0$. But since $\overline{M}$ is finitely generated, that is, 1-tame as a $\mathbb{Z}\overline{Q}$-module, and $\overline{Q} = Q \times \langle z \rangle$, we must have $\overline{\chi}_i(z) \neq 0$. Thus $\overline{\chi}_i(z) > 0$ for $i = \ell+1 \ldots, k+1$. Then the fact that $0 = \sum_{\ell+1}^{k+1} t_i \overline{\chi}_i(z)$ implies that $t_i = 0$ for all i. Thus, $\overline{M}$ is $(k+1)$-tame.

To see that $\overline{M}$ has an essential direction, set $\bar{q} = qz$ and let $\overline{\chi} \in \Sigma(\overline{M}, \overline{Q})^c$. If $\overline{\chi}(q) = 0$, then Lemma 4.1 implies that $\overline{\chi}|_Q = 0$ since q is essential for M. The fact that $\overline{M}$ is finitely generated implies that $\overline{\chi} \neq 0$, so we must have $\overline{\chi}(\bar{q}) = \overline{\chi}(z) \neq 0$ since $\overline{Q}$ is generated by Q and z. Next, suppose that $\overline{\chi}(q) > 0$. Then Lemma 4.2 shows that $\overline{\chi}(\bar{q}) = \overline{\chi}(q) \neq 0$. Finally, if $\overline{\chi}(q) < 0$ then Lemma 4.2 shows that $\overline{\chi}(\bar{q}) = (1+d)\overline{\chi}(q) \neq 0$. Thus we find that $\bar{q} \in \overline{Q}$ is essential for $\overline{M}$. $\square$

5. Essential decompositions

Lemma 4.3 shows that if M is a finitely generated $\mathbb{Z}Q$-module with an essential direction, then special localization in that direction produces a new module with an essential direction and with improved tameness. Unfortunately, not every module has an essential direction. For example, the free cyclic module over the free abelian group of rank two has no essential directions. This follows from the fact that the group ring $\mathbb{Z}\mathbb{Z}^2$ has no zero divisors, so that $\Sigma(\mathbb{Z}\mathbb{Z}^2, \mathbb{Z}^2)^c = V(\mathbb{Z}^2) - \{0\}$ by Lemma 2.1.

We circumvent this difficulty with the following more general concept. A *k-essential decomposition* for a finitely generated $\mathbb{Z}Q$-module M consists of a direct product decomposition $Q = Q_1 \times \cdots \times Q_r$ of Q, together with, for each $i = 1, \ldots, r$, a $\mathbb{Z}Q_i$-submodule M_i of M such that:

- $\mathrm{Ann}_{\mathbb{Z}Q_i}(M_i) \subseteq \mathrm{Ann}_{\mathbb{Z}Q}(M)$.
- M_i is k-tame.
- M_i has an essential direction $q_i \in Q_i$.

Lemma 5.1. *Let Q be a finitely generated abelian group. If a finitely generated $\mathbb{Z}Q$-module M possesses a k-essential decomposition, then M is k-tame.*

Proof. Suppose that we are given characters $\chi_1, \ldots, \chi_k \in \Sigma(M, Q)^c$ and nonnegative real scalars $t_1, \ldots, t_k$ such that $\sum_j t_j \chi_j = 0$. By Lemma 4.1, those χ_j that do not vanish on Q_i restrict to characters $\chi_j|_{Q_i} \in \Sigma(M_i, Q_i)^c$, and so the fact that

M_i is k-tame implies that

$$\chi_j|_{Q_i} \neq 0 \Rightarrow t_j = 0.$$

On the other hand, since M is finitely generated, that is, 1-tame as a $\mathbb{Z}Q$-module, we know that $\chi_j \neq 0$ for all j. This means that for each $j = 1, \ldots, k$, there exists $1 \leq i \leq r$ such that $\chi_j|_{Q_i} \neq 0$, and hence $t_j = 0$. □

Lemma 5.2. *If Q is a finitely generated abelian group, then every finitely generated $\mathbb{Z}Q$-module M admits a 1-essential decomposition.*

Proof. Given Q and M, we can decompose Q as a direct product $Q = Q_1 \times \cdots \times Q_r$ where each Q_i has torsion-free rank one. Choose a finite generating set $\mathbf{x}$ for the $\mathbb{Z}Q$-module M and for each i, let M_i be the $\mathbb{Z}Q_i$-submodule of M generated by $\mathbf{x}$: $M_i = \mathbb{Z}Q_i \cdot \mathbf{x}$. Then $\operatorname{Ann}_{\mathbb{Z}Q_i}(M_i) \subseteq \operatorname{Ann}_{\mathbb{Z}Q}(M)$ and M_i is 1-tame. In addition, any element q_i of infinite order in Q_i is essential for M_i since $0 \notin \Sigma(M_i, Q_i)^c$ and no nonzero character on the virtually infinite cyclic group Q_i can vanish on q_i. □

Lemma 5.3. *Let Q be a finitely generated abelian group and let M be a finitely generated $\mathbb{Z}Q$-module that admits a k-essential decomposition. There is a finitely generated abelian overgroup $\overline{Q}$ of Q and a $\mathbb{Z}\overline{Q}$-module $\overline{M}$ that contains M as a $\mathbb{Z}Q$-submodule and which admits a $(k+1)$-essential decomposition.*

Proof. Given the essential data $Q = Q_1 \times \cdots \times Q_r$, M, M_i, and $q_i \in Q_i$, Lemma 3.2 allows us to select special polynomials $p_1(X), \ldots, p_r(X)$ such that for $i = 1, \ldots, r$, $p_i(q_i)$ acts freely on M. We form the direct product $\overline{Q} = Q \times \langle z_1 \rangle \times \cdots \times \langle z_r \rangle$ and the unital multiplicative submonoid S of $\mathbb{Z}Q$ generated by the $p_i(q_i)$. Since S acts freely on M, the special localization $\overline{M} = S^{-1}M$ is a $\mathbb{Z}\overline{Q}$-module that contains M as a $\mathbb{Z}Q$-submodule and is $\mathbb{Z}\overline{Q}$-generated by any given finite $\mathbb{Z}Q$-generating set $\mathbf{x}$ for M. In addition, $\operatorname{Ann}_{\mathbb{Z}Q}(M) \subseteq \operatorname{Ann}_{\mathbb{Z}\overline{Q}}(\overline{M})$ and $z_i - p_i(q_i) \in \operatorname{Ann}_{\mathbb{Z}\overline{Q}}(\overline{M})$ for all i. For $i = 1, \ldots, r$, we set $\overline{Q}_i = Q_i \times \langle z_i \rangle$ and let $\overline{M}_i = \mathbb{Z}\overline{Q}_i \cdot \mathbf{x}$ be the $\mathbb{Z}\overline{Q}_i$-submodule of $\overline{M}$ generated by $\mathbf{x}$. We have that $z_i - p_i(q_i) \in \operatorname{Ann}_{\mathbb{Z}\overline{Q}_i}(\overline{M}_i)$ for all i. Since $\operatorname{Ann}_{\mathbb{Z}Q_i}(M_i) \subseteq \operatorname{Ann}_{\mathbb{Z}Q}(M)$, we know that $\operatorname{Ann}_{\mathbb{Z}Q_i}(M_i)$ annihilates $\mathbf{x}$, and hence $\operatorname{Ann}_{\mathbb{Z}Q_i}(M_i) \subseteq \operatorname{Ann}_{\mathbb{Z}\overline{Q}_i}(\overline{M}_i) \subseteq \operatorname{Ann}_{\mathbb{Z}\overline{Q}}(\overline{M})$ for all i. By Lemma 4.3, each $\overline{M}_i$ is $(k+1)$-tame and has an essential direction $\bar{q}_i \in \overline{Q}_i$. □

We can now indicate the proofs of the main results. Let a positive integer n be given as in Theorem 1.2. Any given finitely generated metabelian group G can be embedded in a split one $G_1 = M_1 \rtimes Q_1$ by Lemma 3.1. By Lemmas 5.2 and 5.3, G_1 can be embedded in $G_n = M_n \rtimes Q_n$ where the $\mathbb{Z}Q_n$-module M_n admits an n-essential decomposition, and hence is n-tame by Lemma 5.1. Theorem 1.1 then follows since the F_3-Conjecture is true for split extensions [5]. Indeed, we see that G would embed in a metabelian group of type F_n if the F_n-Conjecture were known to be true for split extensions.

To illustrate the foregoing analysis and to suggest what remains to be done, suppose that Q is a finitely generated abelian group of torsion-free rank one and that M is a finitely generated $\mathbb{Z}Q$-module. We may view the Sigma complement in the sphere $S(Q)$, which consists of exactly two points (corresponding to rational characters). Localizing in an essential direction, we obtain a finitely generated abelian overgroup $\overline{Q}$ of Q with torsion-free rank two and a finitely generated 2-tame

$\mathbb{Z}\overline{Q}$-module $\overline{M}$ with an essential direction. Viewed in the sphere $S(\overline{Q})$, Lemma 4.2 shows that $\Sigma(\overline{M},\overline{Q})^c$ consists of at most three (rational) points. Continuing this process, after $n-1$ successive special localizations, we end up with an n-tame module over a finitely generated abelian group of torsion-free rank n whose Sigma complement, when viewed in the $(n-1)$-sphere, is contained in a set of $n+1$ rational points.

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