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Power weak mixing does not imply multiple recurrence in infinite measure and other counterexamples

Kate Gruher, Fred Hines, Deepam Patel, Cesar E. Silva, and Robert Waelder

Abstract. We show that for infinite measure-preserving transformations, power weak mixing does not imply multiple recurrence. We also show that the infinite measure-preserving "Chacon transformation" known to have infinite ergodic index is not power weakly mixing, and is 3-recurrent but not multiply recurrent. We also construct some doubly ergodic infinite measure-preserving transformations that are not of positive type but have conservative Cartesian square. Finally, we study the power double ergodicity property.

Contents

1. Preliminaries

In 1977, Furstenberg [F81] showed that if T is a finite measure-preserving (invertible) transformation on a measure space (X, λ), then for all integers $d > 0$ and all sets A of positive measure there exists an integer $n > 0$ such that $\lambda(T^{dn}(A) \cap T^{(d-1)n}(A) \cap \cdots \cap A) > 0$; this property is called d-recurrence, and if this property holds for all $d > 0$ then it is called multiple recurrence. This theorem is a far reaching generalization of Poincaré's Recurrence Theorem, which is probably the first result in measurable dynamics and asserts that if T is a finite measure-preserving transformation then for any set A of positive measure there

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exists an integer $n > 0$ such that $\lambda(T^n(A) \cap A) > 0$; this property of T is called recurrence or conservativity. Furstenberg used his Multiple Recurrence Theorem to give another proof of Szemerédi's Theorem on the existence of arithmetic progessions in sequences of integers of positive upper Banach density. For some time now there has been interest in studying the multiple recurrence property for transformations preserving an infinite measure and investigating whether it is possible to use similar results for infinite measure-preserving tranformations to prove combinatorial properties for certain sequences of integers. For a discussion of these questions we refer the reader to [AN00].

It is well-known that infinite measure-preserving transformations need not be recurrent; however if T is ergodic and invertible on a non-atomic space then it must be recurrent. But Eigen, Hajian and Halverson in [EHH98] constructed examples of ergodic, invertible, rank one infinite measure-preserving transformations that are not multiply recurrent. More recently, Aaronson and Nakada [AN00] have shown that if T is an infinite measure-preserving Markov shift, then T is d-recurrent if and only if the Cartesian product of d copies of T is recurrent. Markov shifts are of a different nature than rank one tranformations, and using techniques from [AFS97] one can observe that the transformations of Eigen, Hajian and Halverson have recurrent but non-ergodic Cartesian products. Thus it is of interest to investigate if there are some dynamical properties that force multiple recurrence or d-recurrence for general infinite measure-preserving transformations. A counterexample in this direction was obtained by Adams, Friedman and Silva [AFS01], who showed that there exists an infinite measure-preserving rank one T with all finite Cartesian products of T ergodic (and recurrent), but such that T is not 2-recurrent and hence not multiply recurrent; so infinite ergodic index (i.e., all finite Cartesian products ergodic), for infinite measure-preserving transformations, does not imply multiple recurrence. However, the example T in [AFS01] is such that $T \times T^2$ is not conservative, and thus it is of interest to ask whether conservativity of products of powers implies multiple recurrence. A transformation is said to be power weakly mixing if all finite Cartesian products of arbitrary non-zero powers of the transformation are ergodic. The transformation of [AFS01] is an infinite ergodic index transformation that is not power weakly mixing.

In Section 2 we show that the infinite measure-preserving transformation that was shown in [DGMS99] to be power weakly mixing is not multiply recurrent; we in fact show that it is 3-recurrent but not 16-recurrent, and thus power weak mixing does not imply multiple recurrence for infinite measure-preserving transformations. After this work was completed, T. Adams told the fourth-named author that is is possible to modify the construction of [AFS01] to obtain a power weakly mixing transformation that is not 2-recurrent (unpublished). More recently Danilenko and the fourth-named author have generalized these examples to actions of countable discrete Abelian groups. However, the constuctions of [AFS01] are more complex than the simple geometric construction which we show is not multiply recurrent, and the more recent constructions of Danilenko and Silva are algebraic in nature and with methods different from those in this paper.

Section 3 studies the infinite measure-preserving "Chacon transformation" that was shown in [AFS97] to be of infinite ergodic index. We show that this transformation is not power weakly mixing, but while arbitrary finite products of powers with absolute value is less than or equal to 4 of this transformation are recurrent, the transformation is not 3-recurrent.

The examples in Sections 2 and 3 are rank one transformations constructed by the process of cutting and stacking, where in particular each column in the inductive construction is cut into a fixed, constant, number of subcolumns. This means that the transformations have some partial rigidity (i.e., there is a sequence so that sets come back to themselves at a constant rate when iterated along this sequence), and by [AFS97] this implies that all their finite Cartesian products are conservative. In Sections 4 and 5 our examples are also cutting and stacking transformations but constructed in a different way, and are called tower staircases. Staircase constructions gained importance when Adams [A98] used them to construct the first explicit examples of finite measure-preserving rank one transformations that are mixing. In the construction of staircases, columns are cut into an increasing, non-constant, number of subcolumns, and *spacers* are added in a "staircase" fashion. They can be of finite or infinite measure, and in our case infinite measure is obtained by adding a small "tower". In [BFMS01], it was shown that one can have infinite measure-preserving tower staircases with strong dynamical properties such a double ergodicity but with non-conservative Cartesian square. Sections 4 and 5 build on these constructions and study tower staircases with some additional properties.

For finite measure-preserving transformations, Furstenberg [F81, Theorem 4.31] showed that the double ergodicity property is equivalent to weak mixing. Double ergodicity was studied in [BFMS01] for infinite measure-preserving transformations and shown to be strictly weaker than conservative Cartesian square. It is also shown in [BFMS01, Proposition 4.1] that in the infinite measure-preserving (and nonsingular) case double ergodicity implies weak mixing. A proof that the converse of this is not true is also given in [BFMS01]; however, we use this opportunity to note that, while [AFS97] does not mention the double ergodicity concept, the proof in [AFS97, Theorem 1.5] that shows that weak mixing does not imply ergodic Cartesian square already shows that weak mixing does not imply double ergodicity.

In [AN00], it was shown that the property of positive type implies that all finite Cartesian products of the transformation are conservative. In Section 4 we construct a transformation that is doubly ergodic, with conservative Cartesian square, but not of positive type. In Section 5 we introduce the property of power double ergodicity, which is stronger than double ergodicity, and study some of its properties. We then construct a class of tower staircases $\mathcal{T}$ that are power doubly ergodic, and observe that $\mathcal{T}$ contains transformations in a class of tower staircases that was shown in [BFMS01] not to have conservative Cartesian square. We also show that $\mathcal{T}$ contains transformations that are not of positive type.

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Let $(X, \mathcal{B}, \lambda)$ denote a measure space isomorphic to the positive reals with Lebesgue measure λ. A transformation $T : X \to X$ is said to be measurable if for all $A \in \mathcal{B}$, $T^{-1}(A) \in \mathcal{B}$; if in addition $\lambda(A) = \lambda(T^{-1}(A))$ we say that T is *measure-preserving*. All our transformations will be invertible. A transformation T is *ergodic* if for all $A \in \mathcal{B}$, $T^{-1}(A) = A \Rightarrow \lambda(A)\lambda(A^c) = 0$. T is *conservative* if for all sets A of positive measure, there exists an $n > 0$ such that $\lambda(T^n(A) \cap A) > 0$. Note that T is *conservative ergodic* if and only if for all sets A, B of positive measure, there exists an $n > 0$ such that $\lambda(T^n(A) \cap B) > 0$. In our context (invertible transformations in nonatomic spaces), ergodicity implies conservativity. T has *infinite ergodic index* if all finite products $T \times T \times \ldots T$ are ergodic. T is *power weakly mixing* if for all $k_1, \ldots, k_i \in \mathbb{Z} \setminus \{0\}$, $T^{k_i} \times \cdots \times T^{k_1}$ is ergodic.

We describe a geometric construction for rank one transformations using "cut and stack" procedures, see e.g. [F70]. A *column* C is an ordered set of $h > 0$ pairwise disjoint intervals in $\mathbb{R}$ of the same measure. We think of the intervals in a column as being stacked so that element $i+1$ is directly above element i, $0 \leq i \leq h-2$. The elements of C are called *levels* and h is the *height* of C. When clear from the context we also let C denote the union of the levels of the column. The column C partially defines a transformation $T = T_C$ on all levels in C except level $h-1$, by the (unique orientation preserving) translation that takes level i to level $i+1$. In other words, T maps a point in any level i, $0 \leq i < h-1$, to the point directly above it in level $i+1$. Thus if we let B be the bottom level in C, we can write the i^{th} level as $T^i(B)$, $i = 0, \ldots, h-1$. A *cut and stack* construction for a measure-preserving transformation $T : X \longrightarrow X$ consists of a sequence of columns $C_n = \{B_n, T(B_n), \ldots, T^{h_n-1}(B_n)\}$ of height h_n such that:

i) C_{n+1} is obtained by (vertically) cutting C_n into $r_n \geq 2$ equal-measure subcolumns (or *copies* of C_n), putting a number of *spacers* (new levels of the same measure as any of the levels in the r_n subcolumns) above each subcolumn, mapping the top level of each subcolumn to the spacer above it, and stacking left under right (i.e., the top level (or top spacer if it exists) of each subcolumn is mapped by translation to the bottom subinterval of the adjacent subcolumn to its right). In this way C_{n+1} consists of r_n copies of C_n, possibly separated by spacers. Given a level I in C_n we denote by $I^{[i]}$ the portion of I in subcolumn i of C_n, $0 \leq i \leq r_n - 1$.
ii) B_n is a union of elements from $\{B_{n+1}, T(B_{n+1}), \ldots, T^{h_{n+1}-1}(B_{n+1})\}$.
iii) $\bigcup_n C_n$ generates the Borel sets, i.e., for all subsets A in X with $\lambda(A) > 0$ and for all $\varepsilon > 0$, there exists B, a finite union of elements from C_n, for some n, such that $\lambda(A \Delta B) < \varepsilon$.

Note that $I = T^k(B_\ell)$ is in C_ℓ, for $k = 0, \ldots, h_\ell - 1$. For any $n > \ell$, I is the union of some elements in $C_n = \{B_n, T(B_n), \ldots, T^{h_n-1}(B_n)\}$. We call the elements in this union *copies* of I. We denote by $I_{n,j}$, for $0 \leq j \leq r_\ell \cdot \cdots \cdot r_{n-1} - 1$, the j^{th} copy of I in C_n, numbered from the bottom up, and by $I_{n,j}^{[i]}$ the portion of $I_{n,j}$ which is in the subcolumn i, $0 \leq i \leq r_n - 1$.

Definition 1.1. Let $N > 0$ and $\ell \geq k > 0$. The *distance* between levels $T^\ell(B_N)$ and $T^k(B_N)$ in C_N is defined to be $\ell - k$. For $k \geq 0$ and I in C_N define $\sigma_{N,k}$ to be the finite ordered set consisting of the distances between the copies of I in C_{N+k+1}, starting from the bottom of C_{N+k+1}, so that the i^{th} element of $\sigma_{N,k}$ is the distance between copy i and copy $i+1$ of I in C_{N+k+1}. (Note that $\sigma_{N,k}$ is independent of the choice of I in C_N.) We denote by $\operatorname{ord}(\sigma_{N,k})$ the number of terms in the sequence $\sigma_{N,k}$ and by $\Sigma(\sigma_{N,k})$ the sum of the terms in the sequence $\sigma_{N,k}$. Note that $\mathrm{ord}(\sigma_{N,k}) < \infty$. In general, given a finite set A we denote by $\Sigma(A)$ the sum of the elements of A.

Given a level I in C_N, we have $\lambda(I \cap T^j(I)) > 0$ for an integer $j > 0$ if and only if j is equal to the sum of consecutive terms of $\sigma_{N,k}$ for some k, since the intersection must be of copies of I in C_{N+k+1} for some k. We call this sequence of terms of $\sigma_{N,k}$ summing to j a *j-subseries* of $\sigma_{N,k}$. We will have $\lambda(I \cap T^j(I) \cap \cdots \cap T^{dj}(I)) > 0$ if and only if, for some k, $\sigma_{N,k}$ contains d adjacent j-subseries. We call the sequence composed of d adjacent j-subseries a *d-sum sequence*, and we call j the *common sum* of this sequence. If $r_n = r$ is independent of n and $\sigma_{N,k}$ contains a d-sum sequence with common sum j, then for a level I in C_N, since the intersections are of full levels in C_{N+k+1}, we have

$$\lambda(I \cap T^j(I) \cap \cdots \cap T^{dj}(I)) \geq \left(\frac{1}{r}\right)^{k+1} \lambda(I).$$

2. Power weakly mixing but not multiply recurrent

An invertible transformation T is *d-recurrent* if for any set A of positive measure, there exists an integer $n > 0$ such that $\lambda(T^{dn}(A) \cap T^{(d-1)n}(A) \cap \cdots \cap A) > 0$. If T is d-recurrent for all integers $d > 0$, then T is called *multiply recurrent*.

In this section we describe a rank one, infinite measure-preserving transformation T that is known to be power weakly mixing and show that it is not multiply recurrent. We also show that for infinitely many positive integers d there exist power weakly mixing transformations that are d-recurrent but not $d+1$-recurrent. Finally, we describe a large class of transformations which are not multiply recurrent.

We now construct the rank one transformation T shown in [DGMS99] to be power weakly mixing. First we define inductively a sequence of columns. Let C_0 have base $B = [0, 1)$ and height $h_0 = 1$. Given a column C_n with base $B_{n,0} = \left[0, \frac{1}{4^n}\right)$ and height h_n, C_{n+1} is formed as follows: C_n is cut vertically three times so that $B_{n,0}$ is cut into the intervals $B_{n,0}^{[0]} = [0, \frac{1}{4^{n+1}})$, $B_{n,0}^{[1]} = \left[\frac{1}{4^{n+1}}, \frac{1}{2}(\frac{1}{4^n})\right)$, $B_{n,0}^{[2]} = \left[\frac{1}{2}(\frac{1}{4^n}), \frac{3}{4}(\frac{1}{4^n})\right)$, $B_{n,0}^{[3]} = \left[\frac{3}{4}(\frac{1}{4^n}), \frac{1}{4^n}\right)$. Next, a column of spacers h_n high is added to the subcolumn whose base is $B_{n,0}^{[1]}$, and a single spacer is added to the top of the subcolumn whose base is $B_{n,0}^{[3]}$. Finally, the four subcolumns are stacked left under right, i.e., the top level of a subcolumn is sent to the bottom level of the subcolumn to the right by the translation map. The resulting column C_{n+1} now has base $B_{n+1,0} = \left[0, \frac{1}{4^{n+1}}\right)$ and height $h_{n+1} = 5h_n + 1 = \frac{5^{n+2}-1}{4}$. Note that $\bigcup_{n\geq 0} C_n = [0, \infty)$. We see that $\sigma_{N,0} = h_N, 2h_N, h_N$ and $\Sigma(\sigma_{N,0}) = 4h_N$. Furthermore, for $k \geq 1$, we see that

$$\begin{aligned}\sigma_{N,k} =& \sigma_{N,k-1}, h_{N+k} - \Sigma(\sigma_{N,k-1}), \sigma_{N,k-1}, 2h_{N+k} - \Sigma(\sigma_{N,k-1}),\\ & \sigma_{N,k-1}, h_{N+k} - \Sigma(\sigma_{N,k-1}), \sigma_{N,k-1}.\end{aligned}$$

Then

$$\Sigma(\sigma_{N,k}) = 4h_{N+k} + \Sigma(\sigma_{N,k-1}) = 4\sum_{j=N}^{N+k} h_j = 5^{N+1}\left(\sum_{j=0}^{k} 5^j\right) - (k+1)$$

$$\Sigma(\sigma_{N,k}) = \frac{5^{N+1}(5^{k+1}-1)}{4} - k - 1$$

This yields $h_{N+k} - \Sigma(\sigma_{N,k-1}) = h_N + k$ and thus

$$\sigma_{N,k} = \sigma_{N,k-1}, h_N + k, \sigma_{N,k-1}, h_{N+k} + h_N + k, \sigma_{N,k-1}, h_N + k, \sigma_{N,k-1}.$$

Note that $\sigma_{N,k}$ is symmetric about its center term.

Lemma 2.1. *If $\sigma_{N,k}$ contains a d-sum sequence for $d \geq 4$, then the sequence does not contain the center term of $\sigma_{N,k}$ (the $h_{N+k} + h_N + k$ term).*

Proof. We will prove this by contradiction. Suppose $\sigma_{N,k}$ contains a d-sum sequence with common sum j that contains the center term of $\sigma_{N,k}$, $d \geq 4$. Then $j \geq h_{N+k} + h_N + k$. Also, either the terms right of the center term or the terms left of the center term must completely contain at least 2 j-subseries. By symmetry we may assume without loss of generality that the terms left of the center term contain at least 2 complete j-subseries. Thus the sum of the terms in $\sigma_{N,k}$ left of the center term is at least $2j$, so

$$2\Sigma(\sigma_{N,k-1}) + h_N + k \geq 2(h_{N+k} + h_N + k)$$

Then

$$\frac{5^{N+k+1} - 5^{N+1}}{2} - 2k + h_N + k \geq \frac{5^{N+k+1} - 1}{2} + 2h_N + 2k$$

which simplifies to

$$0 \geq 3h_N + 3k.$$

However, $h_N \geq 1$ and $k \geq 0$, so this is a contradiction. Thus if $\sigma_{N,k}$ contains a d-sum sequence for $d \geq 4$, then the sequence does not contain the center term of $\sigma_{N,k}$. $\square$

Given a real number $0 < \varepsilon < 1$ and a set $A \subset X$, $\lambda(A) > 0$, we say that a subset $I \subset X$ of finite positive measure is (at least) $(1-\varepsilon)$-*full* of A if

$$\lambda(I \cap A) > (1-\varepsilon)\lambda(I).$$

Theorem 2.1. *Let T be the rank one transformation defined above. Then T is a power weakly mixing transformation that is 3-recurrent but not multiply recurrent. Hence power weak mixing does not imply multiple recurrence.*

Proof. T was shown to be power weakly mixing in [DGMS99]. First we will show that T is 3-recurrent. Given a set $A \subset X$, $\lambda(A) > 0$, we may pick a level I in some C_N such that I is $(1 - \frac{1}{512})$-full of A. We define subsequences S_1, S_2, and S_3 of $\sigma_{N,2}$ where S_1 consists of terms 2 through 8, S_2 terms 9 through 19, and S_3 terms 20 through 26 of $\sigma_{N,2}$. Then

$$\begin{aligned}\Sigma(S_1) &= 2h_N + h_N + (h_N + 1) + h_N + 2h_N + h_N + (h_{N+1} + h_N + 1) \\ &= 14h_N + 3,\end{aligned}$$

and similarly,

$$\Sigma(S_2) = \Sigma(S_3) = 14h_n + 3.$$

Thus $\sigma_{N,2}$ contains a 3-sum sequence with $j = 14h_N+3$, so $\lambda(I \cap T^j I \cap T^{2j} I \cap T^{3j} I) \geq (\frac{1}{64})\lambda(I)$. Also, since I is $\left(1 - \frac{1}{512}\right)$-full of A, each copy of I in C_{N+3} is $\frac{7}{8}$-full of A. Hence $\lambda(I \cap A \cap T^j(I \cap A) \cap T^{2j}(I \cap A) \cap T^{3j}(I \cap A)) > \frac{1}{128}\lambda(I)$. Therefore $\lambda(A \cap T^j(A) \cap T^{2j}(A) \cap T^{3j}(A)) > \frac{1}{128}\lambda(I) > 0$.

Now we will prove that T is not 16-recurrent by contradiction; in fact, we will show that for any integers $N, k \geq 0$, $\sigma_{N,k}$ does not contain a 16-sum sequence. Suppose T is 16-recurrent. Choose an integer $N \geq 0$ and a level I in C_N. Then there exist integers $j \geq 1$ and $k \geq 2$ such that $\sigma_{N,k}$ contains a 16-sum sequence with common sum j and $k = \min\{h | \sigma_{N,h}$ contains a 16-sum sequence with common sum $j\}$. By Lemma 2.1 the 16-sum sequence may not contain the center term of $\sigma_{N,k}$. Since

$$\sigma_{N,k} = \sigma_{N,k-1}, h_N + k, \sigma_{N,k-1}, h_{N+k} + h_N + k, \sigma_{N,k-1}, h_N + k, \sigma_{N,k-1}$$

the 16-sum sequence must be entirely contained in

$$S_{N,k} = \sigma_{N,k-1}, h_N + k, \sigma_{N,k-1}$$

and by the definition of k it must contain the center term thereof (otherwise the 16-sum sequence is entirely contained in $\sigma_{N,k-1}$, contradicting the definition of k). Thus we see that one of the $\sigma_{N,k-1}$ sequences must contain at least 8 complete, adjacent j-subseries; without loss of generality let it be the further right one and define ℓ as the sum of the terms in the right-side $\sigma_{N,k-1}$ that are in the same j-subseries as the $h_N + k$ term. Note that $0 \leq \ell < j$. Now let $k' = \min\{k | \sigma_{N,k}$ contains an 8-sum sequence with common sum $j\}$, so $k' < k$. If we start from the left end of $\sigma_{N,k'}$ and take the first 8 consecutive j-subseries, we may define ℓ' as the sum of the terms in $\sigma_{N,k'}$ to the left of all 8 j-subseries. Then $\ell' \leq \ell$ since the first $\mathrm{ord}(\sigma_{N,k'})$ terms of $\sigma_{N,k-1}$ are the same as $\sigma_{N,k'}$. As before all 8 j-subseries must be contained in $S_{N,k'} = \sigma_{N,k'-1}, h_N + k', \sigma_{N,k'-1}$ and one of these j-subseries must contain the center term thereof. Since we took the first 8 j-subseries in $\sigma_{N,k'}$ starting from the left, the left-side $\sigma_{N,k'-1}$ in $S_{N,k'}$ must contain at least 4 complete, adjacent j-subseries. Then by Lemma 2.1 none of these j-subseries may contain the center term of $\sigma_{N,k'-1}$, and since the j-subseries contained in $\sigma_{N,k'-1}$ must be adjacent to an additional j-subseries containing the $h_N + k'$ term of $S_{N,k'}$, all of the j-subseries in the left-side $\sigma_{N,k'-1}$ must be to the right of the center term of $\sigma_{N,k'-1}$. Thus all of the j-subseries in $\sigma_{N,k'}$ must be to the right of the center term of the left-most $\sigma_{N,k'-1}$ and thus $\ell' > \frac{1}{2}\Sigma(\sigma_{n,k'-1}) > 4j$. Then $\ell \geq \ell' > j$ which is a contradiction. Thus there is no k such that $\sigma_{N,k}$ contains a 16-sum sequence, and thus T is not 16-recurrent. $\square$

Remark 2.1. By this method it is possible to find infinitely many positive integers d such that there exists a power weakly mixing transformation T that is d-recurrent but not $d+1$-recurrent. We consider a class of transformations T_r defined by columns C_n, where we cut C_n into $r \geq 3$ subcolumns, place a column of spacers h_n high over the second subcolumn from the left and a single spacer over the subcolumn furthest to the right, and then stack left under right to form C_{n+1}. (Note that the transformation previously considered is T_4.) Then each T_r is d_r-recurrent but not d_r+1-recurrent; $2 \leq d_3 < 13$ and for $r \geq 4$, $r - 3 \leq d_r < r^3 - r^2 - r$.

Proposition 2.1. *Let T be a rank one transformation constructed by cutting C_n into r_n subcolumns and placing $s_{n,i}$ spacers over subcolumn i, $0 \leq i \leq r_n - 1$. If $\{r_n\}$ is bounded, then T is not multiply recurrent if*

$$\liminf_{n\to\infty} \frac{a_n}{h_n} > 0, \quad \textit{where} \quad a_n = \max_{0 \leq i \leq r_n - 1} s_{n,i}.$$

Proof. Choose an integer $R > 0$ such that $\forall n > 0,\ r_n \leq R$ and choose integers $N, q > 0$ such that for $n \geq N$, $\frac{a_n}{h_n} > \frac{1}{q}$. Define b_n to be the smallest integer greater than $\frac{R(h_n+a_n)}{a_n}$ and define $p_n = \max_{N \leq k \leq n} b_k$. Now $\sigma_{N,n-N}$ contains a term larger than a_n, so a p_n-sum sequence contained in $\sigma_{N,n-N}$ cannot contain this term since $p_n a_n > R(h_n + a_n) > \Sigma(\sigma_{N,n-N})$. Now suppose for some $k > N$, $\sigma_{N,k-N}$ contains an $(R^2 p_k + R^2 - 1)$-sum sequence with common sum j. Removing the $R-1$ possible terms that are not completely contained in some $\sigma_{N,k-N-1}$, we see that at least one $\sigma_{N,k-N-1}$ contains an m-sum sequence, $m \geq (p_k R + R - 1)$, no further than j from one of its ends (by this we mean that the sum of consecutive terms starting from the end of $\sigma_{N,k-N-1}$ which are not included in the m-sum sequence is less than j). Then considering the lowest ℓ such that $\sigma_{N,\ell}$ contains an m-sum sequence with common sum j, we see that at least one of the $\sigma_{N,\ell-1}$ sequences must contain a p_k-sum sequence and thus the m-sum sequence in $\sigma_{N,\ell}$ cannot contain the larger than $a_{N+\ell}$ term in any $\sigma_{N,\ell-1}$. However, this implies that the m-sum sequence is not within j of the end of $\sigma_{N,\ell}$, which is a contradiction. Thus for no $n \geq N$ does $\sigma_{N,n-N}$ contain an $(R^2 p_n + R^2 - 1)$-sum sequence. However, $(q+3)R^3 > (R^2 p_n + R^2 - 1)$ for $n \geq N$, so T is not $(q+3)R^3$-recurrent. Thus T is not multiply recurrent. $\square$

The following proposition shows that the converse of Proposition 2.1 is not true.

Proposition 2.2. *There exists an infinite measure-preserving transformation T with $\{r_n\}$ bounded such that*

$$\limsup_{n\to\infty} \frac{a_n}{h_n} > 0$$

and

$$\liminf_{n\to\infty} \frac{a_n}{h_n} = 0$$

but T is not multiply recurrent.

Proof. Let T be a cut and stack transformation with $r_n = 2$ for all n. Let $s_{n,i} = 2h_n$ if $i = 0$ and n is odd, and let $s_{n,i} = 0$ otherwise. Then $\limsup_{n\to\infty} \frac{a_n}{h_n} = 2$ and $\liminf_{n\to\infty} \frac{a_n}{h_n} = 0$. We will prove that T is not 8-recurrent. We see that for n odd,

$$\sigma_{N,n-N} = \sigma_{N,n-N-1}, 3h_n - \Sigma(\sigma_{N,n-N-1}), \sigma_{N,n-N-1}$$

and for n even

$$\begin{aligned}\sigma_{N,n-N} &= \sigma_{N,n-N-1}, h_n - \Sigma(\sigma_{N,n-N-1}), \sigma_{N,n-N-1} \\ &= \sigma_{N,n-N-2}, 3h_{n-1} - \Sigma(\sigma_{N,n-N-2}), \sigma_{N,n-N-2}, \\ &\quad h_n - \Sigma(\sigma_{N,n-N-1}), \sigma_{N,n-N-2}, 3h_{n-1} - \Sigma(\sigma_{N,n-N-2}), \sigma_{N,n-N-2}.\end{aligned}$$

Suppose $\sigma_{N,n-N}$ contains a d-sum sequence for $d \geq 4$. Then for n odd the sequence cannot contain the center term of $\sigma_{N,n-N}$, and for n even it cannot contain the $3h_{n-1} - \Sigma(\sigma_{N,n-N-2})$ term.

Now choose a level I in C_N and suppose T is 8-recurrent. Then there exists a least integer $n > N$ such that $\sigma_{N,n-N}$ contains an 8-sum sequence. Let j denote the common sum of this sequence and note that n must be even since this sequence must contain the center term of $\sigma_{N,n-N}$. Now the sequence cannot contain the $3h_{n-1} - \Sigma(\sigma_{N,n-N-2})$ term in $\sigma_{N,n-N}$ so one of the two center $\sigma_{N,n-N-1}$ sequences must contain a 4-sum sequence with common sum j within j of its end. Consider the lowest integer ℓ such that $\sigma_{N,\ell-N}$ contains a 4-sum sequence with common sum

j (note that ℓ must be even). This sequence must contain the center term of $\sigma_{N,\ell-N}$, but it cannot contain the $3h_{\ell-1} - \Sigma(\sigma_{N,\ell-N-2})$ term. Thus the 4-sum sequence is farther than j from the end of $\sigma_{N,\ell-N}$ and any 4-sum sequence in $\sigma_{N,n-N-1}$ is more than j from the end of $\sigma_{N,n-N-1}$, which is a contradiction. Therefore T is not 8-recurrent. $\square$

Remark 2.2. There is a multiply recurrent infinite measure-preserving transformation T with $\{r_n\}$ bounded such that

$$\limsup_{n\to\infty} \frac{a_n}{h_n} > 0$$

but

$$\liminf_{n\to\infty} \frac{a_n}{h_n} = 0.$$

In fact, let T be a cut and stack transformation with $r_n = 2$ for all n. Let $s_{n,i} = 0$ unless $n = 2^k$ for some integer $k \geq 1$, in which case $s_{n,0} = 2h_n$ and $s_{n,1} = 0$. The proof is left to the reader.

If $\{r_n\}$ is bounded, then T is not multiply recurrent if

$$\liminf_{n\to\infty} \frac{s_{n,r_{n-1}}}{h_{n+2}} > 0.$$

3. 4-power conservative but not 3-recurrent

A nonsingular, invertible transformation is called *d-power conservative* if $T^{k_1} \times \cdots \times T^{k_r}$ is conservative for any sequence of nonzero integers $\{k_1 \dots k_r\}$ such that $|k_i| \leq d$. There exist examples of infinite ergodic index transformations that are not 2-power conservative [AFS01]; this example is also not 2-recurrent. In this section we show that the infinite measure-preserving "Chacon transformation" T of [AFS97] that has infinite ergodic index, is 4-power conservative but is not 3-recurrent and is not 7-power conservative, and so not power weakly mixing. We use the notion of partial rigidity to show conservativity of products. A measure-preserving transformation T is *partially rigid* if there exists a fixed α, $0 < \alpha \leq 1$, and a strictly increasing sequence $\{a_n\}$ such that for any measurable set A of finite measure,

$$\liminf_{n\to\infty} \lambda(T^{a_n}(A) \cap A) \geq \alpha\lambda(A).$$

It is clear that if T is partially rigid then it must be conservative. It was shown in [AFS97] that if T and S are partially rigid along the same sequence $\{a_n\}$ then $T \times S$ is partially rigid along $\{a_n\}$, and so if T is partially rigid all its finite Cartesian products are conservative.

The transformation T is a cut and stack transformation constructed by cutting column C_n into three pieces, so $r_n = 3$ for all $n \geq 0$, adding no spacers over subcolumn 0, a single spacer over subcolumn 1 and a tower of $3h_n + 1$ spacers over subcolumn 2, and then stacking left under right. More precisely, let the first column C_0 have base $B = [0, 1)$ and height $h_0 = 1$ and, given a column C_n with base $B_{n,0} = \left[0, \frac{1}{3^n}\right)$ and height h_n, C_{n+1} is formed as follows: C_n is cut vertically twice so that B_n is cut into the intervals $B^{[1]}_{n,0} = \left[0, \frac{1}{3^{n+1}}\right)$, $B^{[2]}_{n,0} = \left[\frac{1}{3^{n+1}}, \frac{2}{3^{n+1}}\right)$ and $B^{[3]}_{n,0} = \left[\frac{2}{3^{n+1}}, \frac{1}{3^n}\right)$. Then add one spacer to the top of the subcolumn whose base is $B_{n,0}^{[2]}$ and $3h_n+1$ spacers to the top of the subcolumn whose base is $B_{n,0}^{[3]}$, and stack left to right. The height of C_n is given by:

$$h_n = 6h_{n-1} + 2 = \left(\frac{7}{5}\right) 6^n - \frac{2}{5}.$$

We see that $\sigma_{1,0} = 8, 9$ and

$$\sigma_{1,n} = \sigma_{1,n-1}, h_{n+1} - \Sigma(\sigma_{1,n-1}), \sigma_{1,n-1}, h_{n+1} - \Sigma(\sigma_{1,n-1}) + 1, \sigma_{1,n-1},$$

$$\Sigma(\sigma_{1,n-1}) = 2\left(\sum_{k=1}^{n} h_k\right) + n = \frac{14}{25}6^{n+1} + \frac{n}{5} - \frac{84}{25}.$$

Then $h_n - \Sigma(\sigma_{1,n-2}) = \left(\frac{21}{25}\right)6^n - \frac{n}{5} + \frac{79}{25}$, so

$$\sigma_{1,n-1} = \sigma_{1,n-2}, \left(\frac{21}{25}\right)6^n - \frac{n}{5} + \frac{79}{25}, \sigma_{1,n-2}, \left(\frac{21}{25}\right)6^n - \frac{n}{5} + \frac{104}{25}, \sigma_{1,n-2}.$$

Lemma 3.1. *The transformation T defined above is not 3-recurrent.*

Proof. We will prove this by contradiction. Choose a level I in C_1. Suppose we have $\lambda(I \cap T^j(I) \cap T^{2j}(I) \cap T^{3j}(I)) > 0$ for some integer $j > 0$. Then there exists a lowest integer $k \geq 2$ such that $\sigma_{1,k-1}$ contains a 3-sum sequence with common sum j, and one of the j-subseries must contain either the $(\frac{21}{25})6^k - \frac{k}{5} + \frac{79}{25}$ or the $(\frac{21}{25})6^k - \frac{k}{5} + \frac{104}{25}$ term of $\sigma_{1,k-1}$. Then $j \geq (\frac{21}{25})6^k - \frac{k}{5} + \frac{79}{25} > \Sigma(\sigma_{1,k-2})$. Since $\sigma_{1,k-1}$ contains 3 j-subseries, at least one of the j-subseries must be completely contained in one of the $\sigma_{1,k-2}$ sequences, which requires $j \leq \Sigma(\sigma_{1,k-2})$, a contradiction. □

Lemma 3.2. *T is 4-power conservative.*

Proof. As shown in [AFS97], if each of $T^{k_1}, \dots, T^{k_r}$ is partially rigid along the same sequence $\{a_n\}$, then $T^{k_1} \times \cdots \times T^{k_r}$ is partially rigid along $\{a_n\}$ and so is conservative. Also, it is sufficient to check the partial rigidity property on levels [AFS97]. Given a level I in C_N, we see that for $N \leq n$, $0 \leq j \leq 3^{(n-N)}$:

$$\lambda(T^{2h_n+1}(I_{n,j}^{[1]}) \cap I_{n,j}^{[3]}) = \left(\frac{1}{3}\right)\lambda(I_{n,j}),$$

so since I consists of 3^{n-N} levels in C_n, $\lambda(T^{2h_n+1}(I) \cap I) \geq (\frac{1}{3})\lambda(I)$. Thus T is partially rigid along $\{2h_n+1\}_{n \geq N}$. Furthermore,

$$\lambda(T^{4h_n+2}(I_{n,j}^{[3]}) \cap I_{n,j}^{[1]}) \geq \left(\frac{1}{9}\right)\lambda(I_{n,j}).$$

Therefore $\lambda(T^{4h_n+2}(I) \cap I) \geq (\frac{1}{9})\lambda(I)$, so T^2 is partially rigid along $\{2h_n+1\}_{n \geq N}$. Also,

$$\lambda(T^{6h_n+3}(I_{n,j}^{[2]}) \cap I_{n,j}^{[2]}) \geq \left(\frac{1}{9}\right)\lambda(I_{n,j})$$

so that T^3 is partially rigid along $\{2h_n+1\}_{n \geq N}$. Finally,

$$\lambda(T^{8h_n+4}(I_{n,j}^{[1]}) \cap I_{n,j}^{[3]}) \geq \left(\frac{1}{9}\right)\lambda(I_{n,j})$$

so T^4 is also partially rigid along $\{2h_n+1\}_{n \geq N}$. Note that for any $\alpha > 0$ and any integer $m > 0$ such that $\lambda(T^m(I) \cap I) > \alpha\lambda(I)$, we also have $\lambda(I \cap T^{-m}(I)) > \alpha\lambda(I)$ since T is measure-preserving. Thus $T^{k_1} \times \cdots \times T^{k_r}$ is partially rigid for any $|k_i| \leq 4$, which completes the proof. □

Theorem 3.1. *Let T be the rank one transformation where C_{n+1} is created by cutting C_n into 3 subcolumns and placing 1 spacer over the second subcolumn and $3h_n + 1$ spacers over the third (defined above). Then T has infinite ergodic index [AFS97], is not 3-recurrent, and is 4-power conservative but is not 7-power conservative and therefore is not power weakly mixing.*

Proof. That T has infinite ergodic index was shown in [AFS97]. 3-recurrence and 4-power conservatity were shown for T in Lemma 3.1 and Lemma 3.2, respectively. We will prove that T is not 7-power conservative by contradiction. Fix an integer $N \geq 3$. Choose a level I in C_N and let $A = \Pi_{i=1}^{7} I$. Suppose there exists an integer $m > 0$ such that

$$\lambda((T \times T^2 \times \cdots \times T^7)^m (A) \cap A) > 0.$$

Then $\exists\, n > N$ such that $h_{n-1} < m \leq h_n$.

First we claim that $m \neq h_n$. Every term of $\sigma_{3,n-4}$ is at least 302, so any two copies of I are a distance $\ell \geq 302$ apart. Consider $T^{2h_n+1}(J)$, where J is a copy of I in C_n. $T^{2h_n+1}(J^{[1]}) = J^{[3]}$, so $T^{2h_n}(J^{[1]}) \cap I = \emptyset$. Furthermore, $T^{2h_n}(J^{[2]})$ and $T^{2h_n}(J^{[3]})$ are both in the spacers above the third subcolumn. Thus $T^{2h_n}(I) \cap I = \emptyset$ and so $m \neq h_n$.

Since the tower of spacers that is added to C_{n-1} to build C_n consists of half of the height of C_n, all the copies of I in C_n are in the bottom $\frac{h_n}{2}$ levels. Then the distance (i.e., number of levels measured in C_{n+1}) from the lowest copy of I in the first subcolumn of C_n (seen as a level in C_{n+1}) to the highest copy of I in the third subcolumn is no more than $\frac{5}{2}h_n + 1$, and the distance from the highest copy of I in the third subcolumn to the top of C_{n+1} is at least $\frac{7}{2}h_n + 1$. Therefore $T^{km}(I) \cap I = \emptyset$ if $\frac{5}{2}h_n + 1 < km \leq \frac{7}{2}h_n + 1$. We will prove inductively that $km \leq \frac{5}{2}h_n + 1$ for all k such that $1 \leq k \leq 7$. For $k = 1$, $m \leq h_n < \frac{5}{2}h_n + 1$. Next suppose that $km \leq \frac{5}{2}h_n + 1$ for some $k, 1 \leq k < 7$. Then since $m < h_n$,

$$(k+1)m \leq \frac{5}{2}h_n + 1 + m < \frac{7}{2}h_n + 1.$$

But $T^{(k+1)m}(I) \cap I = \emptyset$ if $\frac{5}{2}h_n + 1 < (k+1)m \leq \frac{7}{2}h_n + 1$, so $(k+1)m \leq \frac{5}{2}h_n + 1$. Therefore $km \leq \frac{5}{2}h_n + 1$ for $1 \leq k \leq 7$.

Let $k = 7$. Then

$$h_{n-1} < m < \frac{5}{14}h_n + 1 < 3h_{n-1},$$

since $h_{n-1} \geq 302$. Furthermore, we have seen that it cannot be the case that $\frac{5}{2}h_{n-1} + 1 < km \leq \frac{7}{2}h_{n-1} + 1$, so we cannot have $\frac{5}{2}h_{n-1} + 1 < m \leq \frac{7}{2}h_{n-1} + 1$, $\frac{5}{2}h_{n-1} + 1 < 2m \leq \frac{7}{2}h_{n-1} + 1$, or $\frac{5}{2}h_{n-1} + 1 < 3m \leq \frac{7}{2}h_{n-1} + 1$. Thus either

$$\frac{7}{6}h_{n-1} + 1 < m \leq \frac{5}{4}h_{n-1}$$

or

$$\frac{7}{4}h_{n-1} + 1 < m \leq \frac{5}{2}h_{n-1}.$$

Clearly the first case occurs only if $n \geq N + 2$; the only possibilities for the second case with $n = N + 1$ are $m = 2h_N$ and $m = 2h_N + 1$. However, $T^{4h_N}(I) \cap I = \emptyset$

and $T^{10h_N+5}(I) \cap I = \emptyset$, so we need only consider the case when $n \geq N+2$. We will consider first the case when $\frac{7}{6}h_{n-1} < m \leq \frac{5}{4}h_{n-1}$, which is equivalent to

$$42h_{n-2} + 14 < 6m \leq 45h_{n-2} + 15. \tag{1}$$

Given a copy of I in C_{n-2}, we see that in order to have $\lambda(T^p(I) \cap I) > 0$ while sending I through the tower of spacers above the third subcolumn of C_{n-1} no more than once, we must have $p \leq 38h_{n-2}$. However, to send I through the tower of C_{n-1} twice, we require $p > \frac{91}{2}h_{n-2}$. Thus for $1 \leq k \leq 7$, we cannot have

$$38h_{n-2} < km \leq \frac{91}{2}h_{n-2}.$$

However, this contradicts (1), since $h_{n-2} \geq 50$.

We now consider the other case:

$$\frac{7}{4}h_{n-1} < m \leq \frac{5}{2}h_{n-1},$$
$$\frac{21}{2}h_{n-2} + \frac{7}{2} < m \leq 15h_{n-2} + 5.$$

Then $4m \geq 42h_{n-2} + 14 > 38h_{n-2}$. Then $4m > \frac{91}{2}h_{n-2}$, so we have

$$\frac{91}{8}h_{n-2} < m \leq 15h_{n-2} + 5,$$
$$79h_{n-2} < 7m \leq 105h_{n-2} + 35.$$

Now, the distance between copies of I along a path not passing through the tower in C_n can be no greater than $\frac{149}{2}h_{n-2}$, and the distance between copies of I along a path that does pass through the C_n tower must be at least $108h_{n-2}$. So $7m \geq 108h_{n-2}$, which is a contradiction. □

4. Conservative product, doubly ergodic but not positive type

Examples of conservative ergodic, infinite measure-preserving transformations T such that $T \times T$ is not conservative have been know for some time. The first examples were constructed by Kakutani and Parry [KP63] and were infinite Markov shifts. Markov shifts were also used in [ALW79] to construct an infinite measure-preserving transformation T that is weakly mixing but such that $T \times T$ is not conservative, hence not ergodic. An infinite (or finite) measure-preserving transformations T is said to be *weakly mixing* if for all ergodic finite measure-preserving transformations S, the product $T \times S$ is ergodic. Rank one versions of the first example, and of the second example but with $T \times T$ conservative but not ergodic, were constructed in [AFS97]. As remarked in Section 3, one technique to show conservativity of products is partial rigidity. It is easy to see that, for the rank one transformations of the previous sections, if the number of cuts $\{r_n\}$ is bounded then the transformation is partially rigid (along the sequence of heights $\{h_n\}$). A property that is weaker than partial rigidity, originally introduced by Hajian and Kakutani, is positive type. A transformation T is said to be of *positive type* if for all measurable sets A such that $\lambda(A) > 0$:

$$\limsup_{n\to\infty} \lambda(T^n(A) \cap A) > 0.$$

It was recently shown by Aaronson and Nakada [AN00] that positive type implies all finite Cartesian products conservative. Here we construct an infinite measure-preserving transformation T such that $T \times T$ is conservative but T is not of positive type, and such that in addition T satisfies an interesting dynamical property called double ergodicity.

While in the finite measure-preserving case weak mixing and power weak mixing are equivalent, it is now well-known that in the case of infinite measure-preserving transformations there is an increasing hiearchy of properties between weak mixing and power weak mixing. The first counterexample in this direction is due to Kakutani and Parry [KP63] where, in particular, they construct an infinite measure-preserving transformations T such that $T \times T$ is ergodic but $T \times T \times T$ is not. A transformation T is *doubly ergodic* if for all A, B with $\lambda(A)\lambda(B) > 0$, there exists an $n > 0$ such that $\lambda(T^{-n}(A) \cap A) > 0$ and $\lambda(T^{-n}(A) \cap B) > 0$. Furstenberg [F81] showed that for finite measure-preserving transformations double ergodicity is equivalent to weak mixing. However, in [BFMS01] it is shown that double ergodicty of T does not imply that $T \times T$ is ergodic. This is done by showing that all tower staircases (defined below) are doubly ergodic and then showing that there are tower staircases with $T \times T$ not conservative, hence not ergodic. It is also shown in [BFMS01] that double ergodicity implies weak mixing; as mentioned earlier, a proof that the converse is not true is also given in [BFMS01], but the proof in [AFS97, Theorem 1.5] that shows weak mixing does not imply ergodic Cartesian square already shows that weak mixing does not imply double ergodicity.

In this section we construct a class of tower staircases, which by [BFMS01] are doubly ergodic, and show that they are not of positive type but have conservative Cartesian square. A cut and stack transformation is called a *tower staircase* if $s_{n,i} = i$ for $0 \leq i \leq r_n - 2$ and $r_n \to \infty$. They may be finite or infinite measure-preserving depending of the choice of r_n and of the "tower" s_{n,r_n-1}. If in addition $s_{n,r_n-1} = r_n - 1$ they are called *pure staircases*. Pure staircases were used by Adams [A98] to construct explicit examples of finite measure-preserving rank one transformations that are mixing. In our case here (as in [BFMS01]) we use the staircase part ($s_{n,i} = i$, $0 \leq i \leq r_n - 2$) to obtain double ergodicity, and the tower part (s_{n,r_n-1}) to obtain not positive type, while still obtaining conservative Cartesian square. One property of tower staircases is that the sequence of cuts $\{r_n\}$ increases to ∞ and so the earlier argument of partial rigidity cannot be used to show conservativity of products; in fact, by [BFMS01] there exist tower staircases with non-conservative Cartesian square. The proof that our special class of tower staircases has conservative Cartesian square follows the technique in [BFMS01] that uses Siegel's Lemma from Diophantive equations. In what follows of this section T will be a tower staircase transformation with $r_n = n$ and the number of spacers on the tower in column C_n given by

$$s_{n,r_n-1} = (nh_n + (n-1)(n-2)/2)^2.$$

We first show that $T \times T$ is conservative. In [BFMS01] it is shown that if T is a tower staircase with $\{r_n\} \uparrow \infty$ such that

$$(r_n)^{1/n} < M, \text{ for some } M, \text{ and } \prod_{n=\ell}^{\infty} \left(1 - \frac{1}{r_n}\right) > 0,$$

then $T \times T$ is conservative. Our contribution here is to note that essentially the same proof in [BFMS01] works for the case when $r_n = n$; we include the details for completeness. Before proceeding further, we introduce some notation. Let I be a level in C_n and let $j \geq 0$. We describe a way to index copies of I in C_{n+j}. In C_{n+j} there are $n+j-1$ copies of C_{n+j-1}, separated by spacers. Each copy can be indexed from bottom to top with an integer from 0 to $n+j-2$ and we denote the k_1-copy, for $0 \leq k_1 < n+j-1$, by $C_{n,j}[k_1]$. In turn, there are $n+j-2$ copies of C_{n+j-2} in C_{n+j-1} and when viewed in C_{n+j} each $C_{n,j}[k_1]$ has copies of C_{n+j-2}, and the k_2 such copy, $0 \leq k_2 < n+j-2$, in $C_{n,j}[k_1]$, is denoted by $C_{n,j}[k_1, k_2]$. We continue in this way until we see copies of C_n. For example, for $j = 3$, the copies of C_n in C_{n+3} are indexed by $C_{n,j}[k_1, k_2, k_3]$ with $0 \leq k_i < n-(3-i)$, $i = 1, 2, 3$. In this way each copy of I in C_{n+j} is in a unique $C_{n,j}[k_1, \ldots, k_j]$ for some $k_1, \ldots, k_j$, and so it can be indexed by a j-tuple of numbers $[a_1, a_2, \ldots, a_j]$, $0 \leq a_i < r_{n+j-i}$ where $a_i = k_i$. The following proposition follows by induction and its proof can be found in [BFMS01].

Proposition 4.1. *The distance $d(a, b)$ between levels $a = [a_1, a_2, \ldots, a_j]$ and $b = [b_1, b_2, \ldots, b_j]$ is given by*

$$d(a, b) = \sum_{k=1}^{j} \left((b_k - a_k) h_{n+j-k} + \frac{(b_k - a_k)(b_k + a_k - 1)}{2} \right).$$

This result is negative if a is above b; the usual positive distance is given by $|d(a, b)|$.

Given a, b copies of I in C_{n+j} we can specify a j-tuple c by letting $c_k = b_k - a_k$. For convenience, we refer to this j-tuple c as $b - a$. We use a corollary from Siegel's Lemma below. The proof of the corollary is as in [BFMS01] but we include it here for completeness and to correct some typos in [BFMS01]; a proof of Siegel's Lemma may be found in [HS00].

Siegel's Lemma. *Suppose we have a system of equations with integer coefficients, with $n > m$:*

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

Suppose also that $|a_{ij}| \leq A$ where A is a positive integer. Then there is a nontrivial solution in the integers with

$$|x_i| < 1 + (nA)^{m/(n-m)} \quad (i = 1, \ldots, n).$$

Corollary 4.1. *Given $\{r_k\}_{k=1}^{\infty}$ increasing with $r_k^{\frac{1}{k}} < M$, for some $M > 0$, and given $n \geq 1$, there exists j such that for any $\{d_k\}_{k=1}^{j}$ with $|d_k| \leq r_{n+k}$ there exists a nontrivial solution in integers $\{x_k\}_{k=1}^{j}$ to $\sum_{k=1}^{j} d_k x_k = 0$ with $|x_k| < M + 2$.*

Proof. Using Siegel's Lemma for $m = 1$, for any j there always exists a solution with $|x_k| < 1 + (jr_{n+j})^{\frac{1}{j-1}} = 1 + (j^{\frac{1}{j-1}})(r_{n+j}{}^{\frac{1}{n+j}})^{\frac{n+j}{j-1}}$, which tends to $1 + (1)(M)$ as $j \to \infty$. Thus for each j sufficiently large there is a solution with $|x_k| < 2 + M$. $\square$

Proposition 4.2. *$T \times T$ is conservative.*

Proof. We first show a special type of conservativity on sets $I \times J$ where I and J are levels in column C_n. Let $C_{n+j}(I) = \{\text{copies of } I \text{ in } C_{n+j}\}$ and similarly for $C_{n+j}(J)$. The squares are indexed by pairs $(a, b), a \in C_{n+j}(I), b \in C_{n+j}(J)$. We say that a pair (a, b) is a unique distance pair if there are no other pairs (a', b') such that $d(a, b) = d(a', b')$.

Suppose we have a pair of copies in C_{n+j}, $a \in C_{n+j}(I)$ and $b \in C_{n+j}(J)$. Suppose also that $4 \leq a_k, b_k \leq n + j - k - 4$. Call the fraction of copies that satisfy this condition α^2. We claim that $\alpha^2 \to 1$ as $n \to \infty$. We show this as follows: There are $(n+j-8)(n+j-9)\ldots(n-7)$ choices for a and b independently. Thus, there are $[(n+j-8)\ldots(n-7)]^2$ choices for (a, b). Since there are a total of $[(n + j - 1)(n + j - 2)\ldots(n)]^2$ pairs, the fraction α^2 is thus given by

$$\alpha^2 = \left(\frac{(n+j-8)\ldots(n-7)}{(n+j-1)(n+j-2)\ldots(n)}\right)^2 .$$

Thus, $\alpha^2 \to 1$ as $n \to \infty$.

Let $c = b-a$. By Corollary 4.1 let j be large enough so that $\sum_{k=1}^{j} c_k x_k = 0$ has a solution with $|x_k| < 4$ for all k. Let $x = [x_1, x_2, \ldots, x_j]$ and set $a' = a + x$ and $b' = b + x$. By our choice of a and b, $a' \in C_{n+j}(I)$ and $b' \in C_{n+j}(J)$. We claim that $d(a', b') = d(a, b)$. Using Proposition 4.1:

$$\begin{aligned} d(a', b') &= \sum_{k=1}^{j}(b'_k - a'_k)\left(h_{n+j-k} + \frac{b'_k + a'_k - 1}{2}\right) \\ &= \sum_{k=1}^{j}(b_k - a_k)\left(h_{n+j-k} + \frac{b_k + x_k + a_k + x_k - 1}{2}\right) \\ &= \sum_{k=1}^{j}(b_k - a_k)\left(h_{n+j-k} + \frac{b_k + a_k - 1}{2}\right) + \sum_{k=1}^{j} c_k x_k \\ &= \sum_{k=1}^{j}(b_k - a_k)\left(h_{n+j-k} + \frac{b_k + a_k - 1}{2}\right) = d(a, b). \end{aligned}$$

It follows that if $T^k(a) = a'$, for some $k \neq 0$, then $T^k(b) = b'$, and so a square that corresponds to a non-unique distance pair lies in the set

$$E = \bigcup_{k \neq 0} ((I \times J) \cap (T \times T)^k (I \times J)),$$

so $\lambda \times \lambda(E) \geq \alpha^2 \lambda \times \lambda(I \times J)$.

This proves our preliminary version of conservativity.

Now let $A \subset X \times X$ be of positive measure. We can find levels I and J in C_n, where n can be as large as needed, such that $I \times J$ is at least $\frac{7}{8}$-full of A. Let j and (a, b) be as above. So a fraction α^2 of these (a, b)'s are not unique distance pairs. Suppose $(T \times T)^k(a \times b) = a' \times b'$ for some $k \neq 0$, and assume $|k|$ is the smallest for which this holds. If $k > 0$ we call (a, b) a Type 1 square, and if $k < 0$ a Type 2 square. There is a natural one-to-one correspondence between Type 1 and Type 2 squares, i.e., a Type 1 square corresponds to the Type 2 square it is sent to. Thus they have the same total measure, and so the total measure of the Type 1 squares, as well as of the Type 2 squares, is at least $\frac{1}{2}\alpha^2 \lambda \times \lambda(I \times J)$. Since $I \times J$ is $\frac{7}{8}$-full of A then more than $\frac{1}{2}$ of the Type 1 squares must be $\frac{1}{2}$-full of A and more than $\frac{1}{2}$ if the Type 2 squares must be $\frac{1}{2}$-full of A. By the pigeonhole principle, there must exist a pair related by the correspondance C and $(T \times T)^k(C)$, some $k \neq 0$, each $\frac{1}{2}$-full of A. So $\lambda \times \lambda((T \times T)^k(A \cap C) \cap (A \cap C)) > 0$, which shows $T \times T$ is conservative. $\square$

We need some notation before our next proposition. Let I be a level in column C_n. For $r \geq 0$ define an $I_{n,r}$-*cluster* to be a set of $n \dots (n+r)$ consecutive copies of I, I_i (indexed from bottom to top), in which the distance between I_i and I_{i+1} is given by the i^{th} element of $\sigma_{n,r}$. Define an $I_{n,-1}$ cluster to be a set consisting of single copy of I. Note that for $j > r+1$ if I is a level in C_n, then in C_{n+j} there are $(n+r+1) \dots (n+j-1)$ $I_{n,r}$ clusters. Otherwise, for $j = r+1$, there is one $I_{n,r}$ cluster in C_{n+j}. We let $I_{n,r}^{[i]}$ be the i^{th} such cluster, labeled from the bottom of the column to the top. Note that the number of spacers added to the tower at stage n is equal to the square of the number of remaining levels. So the height h_{n+1} of column C_{n+1} is given by

$$h_{n+1} = nh_n + (n-1)(n-2)/2 + (nh_n + (n-1)(n-2)/2)^2,$$

and the distance between adjacent $I_{n,j}$-clusters is greater than the square of their length. Also $\Sigma(\sigma_{n,k}) > (n+k-1)h_{n+k} + \Sigma(\sigma_{n,k-1})$. Hence $\Sigma(\sigma_{n,k}) > (n+k-1)!^2$.

Proposition 4.3. *The transformation T described above is not of positive type.*

Proof. We show that T is not of positive type on levels. Fix $N > 1$ and let I be a level in C_N. Let $r > 0$ and choose $k = k_r$ such that $N + k - 1 = \Sigma(\sigma_{N,r})$; clearly $r < k$. From the construction of T it follows that

$$\Sigma(\sigma_{N,k}) = (N+k-1)h_{N+k} + (N+k-1)(N+k-2)/2 + \Sigma(\sigma_{N,k-1}).$$

Now, for each $r > 0$, we study $\lambda(T^j(I) \cap I)$ for $\Sigma(\sigma_{N,k-1}) < j \leq \Sigma(\sigma_{N,k})$. For this we consider the copies of I in column C_{N+k}. We first note that if $j < h_{N+k} - \Sigma(\sigma_{N,k-1})$, then clearly $\lambda(T^j(I) \cap I) = 0$. Next consider values of j such that

$$\begin{aligned} h_{N+k} - \Sigma(\sigma_{N,k-1}) &\leq j \\ &\leq (N+k-1)h_{N+k} \\ &\quad + (N+k-1)(N+k-2)/2 + \Sigma(\sigma_{N,k-1}). \end{aligned}$$

Since each copy of I in C_{N+k} is cut into $n+k$ subintervals, the image of each copy under T^j consists of a sequence of no more than $N+k-1$ subintervals in column C_{N+k}, plus at least one subinterval which gets lost in the tower of C_{N+k}. The distance between the top-most and bottom-most such subintervals in C_{N+k} (not counting those lost in the tower) is at most $(N+k-1)(N+k)/2$, which is clearly less than the spacing between the $I_{N,r+1}$-clusters in an $I_{N,r+2}$-cluster. Thus, each copy in C_{N+k} can intersect at most $N(N+1)\dots(N+r+1)$ copies under the action of T^j. Since there are $N(N+1)\dots(N+k-1)$ copies of I in C_{N+k}, and the measure of each subinterval is $\lambda(I)/N(N+1)\dots(N+k)$ then

$$\begin{aligned} \lambda(T^j(I) \cap I) &\leq \frac{N(N+1)\dots(N+r+1)}{N+k}\lambda(I) \\ &= \frac{N(N+1)\dots(N+r+1)}{\Sigma(\sigma_{N,r})+1}\lambda(I). \end{aligned}$$

Since

$$\lim_{r\to\infty} \frac{n\ldots(n+r+1)}{\Sigma(\sigma_{n,r})+1}\lambda(I) = 0,$$

it follows that $\lim_{j\to\infty} \lambda(T^n(I)\cap I) = 0$. Therefore T is not positive type. $\square$

We thus arrive at the following result:

Theorem 4.1. *There exists an infinite measure-preserving transformation T such that $T\times T$ is conservative and T is doubly ergodic, but T is not of positive type.*

5. Power double ergodicity

A transformation T is said to be *power doubly ergodic* if for all sets A, B of positive measure and all integers $k_1, k_2 > 0$ there is an integer $n > 0$ such that

$$\lambda(T^{-nk_1}(A)\cap A)\lambda(T^{-nk_2}(A)\cap B) > 0.$$

We note that power double ergodicity is a strictly stronger property than double ergodicity. It was shown in [AFS01] that there exists a transformation T such that T has infinite ergodic index, but $T\times T^2$ is not conservative on product sets. Thus, there exists a set $A\times B$ such that $\lambda((T\times T^2)^n(A\times B)\cap(A\times B)) = 0$ for all $n \neq 0$. It follows that $\lambda(T^n(A)\cap A)\lambda(T^{2n}(B)\cap B) = 0$ for all $n\neq 0$, which implies that T is not power doubly ergodic (since it fails for $k_1 = 1$ and $k_2 = 2$), but since $T\times T$ is ergodic (and conservative), T is doubly ergodic. In this section we first prove some basic properties of power double ergodicity and then show that a large class of tower staircases $\mathcal{T}$ are power doubly ergodic. We then observe that $\mathcal{T}$ contains transformations in a class of tower staircases that was shown in [BFMS01] not to have conservative Cartesian square. Finally we show that there are transformations in $\mathcal{T}$ that are not of positive type; it is not clear if there are transformations here with nonconservative Cartesian square.

To state and prove the basic properties of power double ergodicity we do not need that T is measure-preserving, we only require T to be *nonsingular*, i.e., $\lambda(A) = 0$ if and only if $\lambda(T^{-1}(A)) = 0$. The proofs of these properties are very similar to the proofs of the corrensponding porperties for double ergodicity in [BFMS01], and so we include the details only in the case of Theorem 5.1.

Theorem 5.1. *Let T be a nonsingular power doubly ergodic transformation. Let A, B, C and D be sets of positive measure. Then for all nonzero integers k_1, k_2 there exists an integer $n > 0$ such that $\lambda(T^{-nk_1}(A)\cap B) > 0$ and $\lambda(T^{-nk_2}(C)\cap D) > 0$.*

Proof. We first show that for any sets A, B, C of positive measure and for all $k_1, k_2 > 0$, there exists $\ell > 0$ such that $\lambda(T^{-\ell k_1}(A)\cap A)\lambda(T^{-\ell k_2}(C)\cap B) > 0$. Now, there exists a $j > 0$ such that $\lambda(T^{-j}(C)\cap A) > 0$ and $\lambda(T^{-j}(B)) > 0$. Also, there exists $\ell > 0$ such that

$$\lambda(T^{-\ell k_1}(T^{-j}(C)\cap A)\cap(T^{-j}(C)\cap A)) > 0 \text{ and}$$
$$\lambda(T^{-\ell k_2}(T^{-j}(C)\cap A)\cap T^{-j}(B)) > 0.$$

Thus,

$$\lambda(T^{-\ell k_1}(A)\cap A) > 0 \text{ and } \lambda(T^{-\ell k_2 - j}(C)\cap T^{-j}(B)) > 0.$$

Then, by nonsingularity, $\lambda(T^{-\ell k_2}(C)\cap B) > 0$, which proves this first part.

Now let k_1 and k_2 be given and let A, B, C, D be sets of positive measure. As before, there exists $m > 0$ and $j > 0$ such that:

$$\lambda(T^{-m}(A) \cap B)\lambda(T^{-m}(B)) > 0, \text{ and}$$
$$\lambda(T^{-j}(C) \cap D) > 0\lambda(T^{-j}(D)) > 0.$$

By the first part, there exists $\ell > 0$ such that

$$\lambda(T^{-\ell k_1}(T^{-m}(A) \cap B) \cap T^{-m}(B)) > 0 \text{ and}$$
$$\lambda(T^{-\ell k_2}(T^{-j}(C) \cap D) \cap T^{-j}(C) \cap D) > 0.$$

Also, there exists $p > 0$ such that

$$\lambda(T^{-pk_1}(T^{-\ell k_1}(T^{-m}(A) \cap B) \cap T^{-m}(B)) \cap T^{-\ell k_1}(T^{-m}(A) \cap B) \cap T^{-m}(B)) > 0,$$
$$\text{and } \lambda(T^{-pk_2}(T^{-\ell k_2}(T^{-j}(C) \cap D) \cap T^{-j}(C) \cap D) \cap T^{-j}(D)) > 0.$$

Thus,

$$\lambda(T^{-pk_1-\ell k_1-m}(A) \cap T^{-m}(B)) > 0 \text{ and}$$
$$\lambda(T^{-pk_2-\ell k_2-j}(C) \cap T^{-j}(D)) > 0.$$

Finally, letting $n = p + \ell$, we obtain

$$\lambda(T^{-nk_1}(A) \cap B)\lambda(T^{-nk_1}(C) \cap D) > 0.$$

□

The proof of the following theorem follows similar ideas and is left to the reader.

Theorem 5.2.

a) *Suppose T is a power doubly ergodic nonsingular endomorphism. Then for all integers $k_1, k_2 > 0$, and $p_1, \ldots, p_n$ where $p_i = k_1$ or k_2 and $A_1, \ldots, A_n$, $B_1, \ldots, B_n$ sets of positive measure, there exists $m > 0$ such that:*

$$\lambda(T^{-mp_1}(A_1) \cap B_1) > 0,$$
$$\vdots$$
$$\lambda(T^{-mp_n}(A_n) \cap B_n) > 0.$$

b) *Suppose T is a nonsingular automorphism on X. Then T is power doubly ergodic if and only if T^{-1} is power doubly ergodic.*
c) *If T power doubly ergodic then T^k power doubly ergodic for all $k > 0$.*

The remainder of this section is devoted to showing specific properties of classes of cutting and stacking transformations. Our first goal is to show that a large class of tower staircases is power doubly ergodic. We require the following lemma from [BFMS01, Lemma 5.4]. The proof simply uses the idea that if a sum of terms is positive then at least one of the terms is positive.

Lemma 5.1. *Let $A, B \subset X$ be sets of positive measure, and let levels $I, J \subset C_m$ be such that $\lambda(I \cap A) + \lambda(J \cap B) > \delta\lambda(I)$, with I a distance $\ell \geq 0$ above J. If we cut I and J into r_m equal pieces $I_0, \ldots, I_{r_m-1}$ and $J_0, \ldots, J_{r_m-1}$, respectively (numbered from left to right), then there is some k such that*

$$\lambda(I_k \cap A) + \lambda(J_k \cap B) > \delta\lambda(I_k)$$

and I_k is ℓ above J_k in C_{m+1}.

Theorem 5.3. *Let T be a tower staircase transformation with sequence of cuts $\{r_n\}$. If for all $K > 0$, there exists N such that for all integers $n > N$, $r_n > Kh_n$, then T is power doubly ergodic.*

Proof. Let $k_1 > 0$ and $k_2 > 0$ be given and let $k = \max\{k_1, k_2\}$. Let $\varepsilon = \frac{1}{16k}$. As the levels approximate measurable sets, for any sets A, B of positive measure there exist levels I' and J' in some column C_m, with I' ℓ levels above J', for some $\ell > 0$, and such that I' and J' are $(1 - \frac{\varepsilon}{2})$-full of A and B, respectively. Let N be such that for all $n > N$, $r_n > 16(k+1)h_n$. Now

$$\lambda(I' \cap A) + \lambda(J' \cap B) > (1 - \varepsilon/2)(\lambda(I') + \lambda(J')) = (2 - \varepsilon)\lambda(I').$$

Let $n > N$. Appling Lemma 5.1 $n - m + 1$ times, we get levels I and J in C_n, such that $\lambda(I \cap A) + \lambda(J \cap B) > (2 - \varepsilon)\lambda(I)$, where I is ℓ above J. The inequality above implies that I and J are $(1 - \varepsilon)$-full of A and B, respectively. Now let $I_0, \ldots, I_{r_n - 1}$ and $J_0, \ldots, J_{r_n-1}$ be the r_n sub-intervals of I and J in C_n, numbered from left to right. From the staircase construction it follows that for all j, $0 \le j \le r_n - (k-1)h_n - k\ell - 1$,

$$\begin{aligned}
T^{k_1(h_n+j)}(I_{(k_1-1)h_n+k_1j}) &= T^{h_n+(k_1-1)h_n+k_1j}(I_{(k_1-1)h_n+k_1j}) \\
&= I_{(k_1-1)h_n+k_1j+1}; \\
T^{(k_2(h_n+j)}(I_{(k_2-1)h_n+k_2j+\ell}) &= T^{h_n+(k_2-1)h_n+k_2j)}(I_{(k_2-1)h_n+k_2j+\ell}) \\
&= T^{-\ell}(I_{(k_2-1)h_n+k_2j+\ell+1}) \\
&= J_{(k_2-1)h_n+k_2j+\ell+1}.
\end{aligned}$$

Now we introduce some notation. Let

$$P = \{0, 1, \ldots, r_n - [(k-1)h_n + k\ell + 1]\},$$

and set $G = \{I_{(k_1-1)h_n+k_1p} : p \in P\}$, $G' = \{I_{(k_1-1)h_n+k_1p+1} : p \in P\}$, $H = \{I_{(k_2-1)h_n+k_2p+\ell} : p \in P\}$, and $H' = \{J_{(k_2-1)h_n+k_2p+\ell+1} : p \in P\}$. Let G_p denote $I_{(k_1-1)h_n+k_1p}$, G'_p denote $I_{(k_1-1)h_n+k_1p+1}$, H_p denote $I_{(k_2-1)h_n+k_2p+\ell}$, and H'_p denote $J_{(k_2-1)h_n+k_2p+\ell+1}$.

Now, G, G', H, H', and K all have the same number of elements. By our choice of N,

$$\begin{aligned}
|G| &= r_n - (k-1)h_n - k\ell - 1 \\
&\ge r_n - (k+1)h_n \\
&> r_n - \frac{r_n}{16} = \frac{15}{16} r_n,
\end{aligned}$$

so

$$\lambda\left(\bigcup_p G_p\right) > \frac{15}{16} r_n \frac{\lambda(I)}{r_n} = \frac{15}{16}\lambda(I).$$

Using that I is $(1-\varepsilon)$-full of A,

$$\begin{aligned}\lambda\left(\left(\bigcup G_p\right)\cap A\right) &= \lambda(I\cap A)-\lambda\left(\left(I\backslash\bigcup G_p\right)\cap A\right)\\ &\geq (1-\varepsilon)\lambda(I)-\lambda\left(I\backslash\bigcup G_p\right)\\ &\geq (1-\varepsilon)\lambda(I)-\lambda(I)+\lambda\left(\bigcup G_p\right)\\ &> \left(\frac{15}{16}-\varepsilon\right)\lambda(I)=\frac{7}{8}\lambda(I).\end{aligned}$$

Similarly, $\bigcup_p G'_p$ and $\bigcup_p H_p$ are both $\frac{7}{8}$-full of A and $\bigcup_p H'_p$ is $\frac{7}{8}$ full of B. Thus more than $\frac{3}{4}$ of the intervals in G, G' and H are at least $\frac{1}{2}$-full of A, and more than $\frac{3}{4}$ of the intervals in H' are $\frac{1}{2}$-full of B. Now let

$$\begin{aligned}K_G &= \left\{k\in K : G_k \text{ and } G'_k \text{ are both at least } \tfrac{1}{2}\text{-full of } A\right\},\\ K_H &= \left\{k\in K : H_k \text{ and } H'_k \text{ are at least } \tfrac{1}{2}\text{-full of } A \text{ and } B\text{, respectively}\right\}.\end{aligned}$$

Note that if $k\in K_G$ then $T^{k_1(h_n+k)}(G_k)=G'_k$, and if $k\in K_H$ then $T^{k_2(h_n+k)}(H_k)=H'_k$. Also, $|H_G|>\frac{1}{2}|K|$, and $|H_G|>\frac{1}{2}|K|$. Therefore $K_G\cap K_H\neq\emptyset$. So for $k\in K_G\cap K_H$ it follows that $\lambda(T^{k_1(h_n+k)}(A)\cap A)\lambda(T^{k_2(h_n+k)}(A)\cap B)>0$. □

We end with two counterexamples. For the first one we use a construction in [BFMS01] where a class of tower staircases are constructed for which $T\times T$ is not conservative. The conditions on these staircases are that r_n must grow very fast, namely $r_n > n^2(2r_{n-1}h_{n-1}+(r_{n-1}-1)(r_{n-1}-2))$, and that the tower s_{n,r_n-1} must be very big. (For the precise condition on s_{n,r_n-1} we refer to [BFMS01, p.1014] as it is not needed here.) By [BFMS01, Theorem 3], a tower staircase with these properties satisfies that $T\times T$ is not conservative. By Theorem 5.3, a tower staircase transformation with $\{r_n\}$ such that for all $K>0$, there exists N, such that for all integers $n>N$, $r_n>Kh_n$, is power doubly ergodic. As this theorem puts no condition on s_{n,r_n-1} and clearly one can find $\{r_n\}$ to satisfy the conditions of the two theorems just mentioned, it follows that there exist power double ergodic staircases for which $T\times T$ is not conservative, hence not ergodic. So power double ergodicity does not imply conservative Cartesian square. For our last example we show that power double ergodicity does not imply positive type.

Proposition 5.1. *Let T be a tower staircase with $r_n=\lceil h_n(1+\cdots+\frac{1}{n})\rceil$ and the number of spacers on the last subcolumn given by $\left(r_nh_n+\frac{(r_n-1)(r_n-2)}{2}\right)^2$. Then T is not of positive type.*

We require some notation before the proof. Let w_n denote the measure of a level in column C_n. Define a *diagonal* D in C_n to be a set of intervals contained in C_n of measure $\frac{w_n}{r_n}$ such that:

(1) For all $I\in D$, I is in a subcolumn of C_n.
(2) No two members of D are in the same subcolumn.
(3) There exists k such that for every interval $I\in D$, if $J\in D$ and J is in the subcolumn to the right of the subcolumn containing I then J is k levels below I.

We define a diagonal D to be a *full-diagonal* if it is not contained in any other diagonal.

Proof. We show that T is not of positive type for a level $I \subset C_n$ for a fixed $n > 1$. Choose $\Sigma(\sigma_{n+k-1}) < j \leq \Sigma(\sigma_{n+k})$ and consider the intersection of the image of a copy of I in C_{n+k} under T^j with I. Note that in C_{n+k} there are $r_{n+k-1} \dots r_n$ copies of I. For convenience, denote this number by $q = q_{n,k}$. Note that the image of a copy of I in C_{n+k} consists of a sequence of full-diagonals in C_{n+k}. Since each copy of I in C_{n+k} is cut into $r_{n+k} = \lceil h_{n+k}(1 + \cdots + \frac{1}{n+k}) \rceil$ pieces, the image of a copy of I under T^j consists of no more than $2(n+k)$ full-diagonals. For each of these full-diagonals, there can be no more than q intersections with I. Since there are q copies of I in C_{n+k}, we get:

$$\begin{aligned} \lambda(T^j(I) \cap I) &< \frac{2(n+k)q^2}{r_{n+k} \dots r_n} \lambda(I) \\ &= \frac{2(n+k)q}{r_{n+k}} \lambda(I). \end{aligned}$$

Since

$$\begin{aligned} r_{n+k} &> h_{n+k} \\ &= r_{n+k-1}h_{n+k-1} + \frac{(r_{n+k-1}-1)(r_{n+k-1}-2)}{2} \\ &\quad + \left(r_{n+k-1}h_{n+k-1} + \frac{(r_{n+k-1}-1)(r_{n+k-1}-2)}{2} \right)^2, \end{aligned}$$

it follows that $r_{n+k} > (r_{n+k-1}!)^2$. Thus,

$$\begin{aligned} \lambda(T^j(I) \cap I) &< \frac{2(n+k)q}{q^2} \lambda(I) \\ &= \frac{2(n+k)}{q} \lambda(I) \\ &< \frac{2(n+k)}{(n+k-1)!} \lambda(I). \end{aligned}$$

Thus $\lim_{n\to\infty} \lambda(T^n(I) \cap I) = 0$. □

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University of Chicago, Chicago, IL 60637, USA
kagruher@uchicago.edu

Williams College, Williamstown, MA 01267, USA
fhines@wso.williams.edu

Brandeis University, Waltham, MA 02454, USA
dns97@hotmail.com

Department of Mathematics, Williams College, Williamstown, MA 01267, USA
csilva@williams.edu

University of California, Berkeley, CA 94720, USA
rwaelder@hotmail.com

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