The Picard Group of the Moduli of Higher Spin Curves

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ABSTRACT. This article treats the Picard group of the moduli (stack) of r-spin curves and its compactification. Generalized spin curves, or r-spin curves are a natural generalization of 2-spin curves (algebraic curves with a theta-characteristic), and have been of interest lately because they are the subject of a remarkable conjecture of E. Witten, and because of the similarities between the intersection theory of these moduli spaces and that of the moduli of stable maps.

We generalize results of Cornalba, describing and giving relations between many of the elements of the Picard group of the stacks. These relations are important in the proof of the genus-zero case of Witten's conjecture given in [14]. We use these relations to show that when 2 or 3 divides r, the Picard group has non-zero torsion. And finally, we work out some specific examples.

CONTENTS

1. Introduction

In this article we study the Picard group of the moduli (or rather the stack) $\overline{\mathfrak{S}}_g^{1/r}$ of higher spin curves, or r-spin curves, over the Deligne-Mumford stack of stable curves $\overline{\mathfrak{M}}_g$. Smooth r-spin curves consist of a smooth algebraic curve X, a line bundle (invertible coherent sheaf) $\mathcal{L}$, and an isomorphism from the rth tensor power $\mathcal{L}^{\otimes r}$ to the cotangent bundle ω_X. The compactification of the stack of smooth spin curves uses *stable spin curves*. These consist of a stable curve X, and for any divisor d of r, a rank-one, torsion-free sheaf $\mathcal{E}_d$ which is almost a dth root of the canonical (relative dualizing) sheaf. This is made precise in Definition 2.5.

The stack of spin curves provides a finite cover of the stack of stable curves. Although this stack has many similarities to the stack of stable maps, including the existence of classes analogous to Gromov-Witten classes, and an associated cohomological field theory [14], it is not the stack of stable maps into any variety [14, §5.1].

These moduli spaces are especially interesting because of a conjecture of E. Witten relating the intersection theory on the moduli space of r-spin curves and KdV (Gelfand-Dikii) hierarchies of order r [21, 22]. This conjecture is a generalization of an earlier conjecture of his, which was proved by Kontsevich (see [15, 16] and [17]). As in the case of Gromov-Witten theory, one can construct a cohomology class $c^{1/r}$ and a potential function from the intersection numbers of $c^{1/r}$ and the tautological ψ classes associated to the universal curve. Witten conjectures that the potential of the theory corresponds to the tau-function of the order-r Gelfand-Dikii (KdV_r) hierarchy. In genus zero, the Witten conjecture is true [14], and the relations of Theorem 3.8 play a role in the proof.

Construction of the stack $\mathfrak{S}_g^{1/r}$, and its compactification $\overline{\mathfrak{S}}_g^{1/r}$, was done in [4] for $r = 2$ and in [13] for all $r \geq 2$. This article focuses on giving a description of the Picard group of $\mathfrak{S}_g^{1/r}$ and $\overline{\mathfrak{S}}_g^{1/r}$.

1.1. **Overview and outline of the paper.** In Section 2.1 and 2.2 we give definitions of smooth and stable spin curves and some examples. The examples of Section 2.2.3 are typical of the spin curves that arise over the boundary divisors of Pic $\overline{\mathfrak{M}}_g$. In Section 2.3 we recall the basic properties of the moduli spaces from [13]. In Section 3 we treat the Picard group of these spaces.

The Picard group that we will work with in this paper is the Picard group of the stack Pic $\overline{\mathfrak{S}}_g^{1/r}$ (also called Pic $_{fun}$), as defined in [18] or [11]. The exact definition of the Picard group is given in Section 3.1, and the definitions of the boundary

divisors and the tautological elements of $\operatorname{Pic} \overline{\mathfrak{S}}_g^{1/r}$ are given in Section 3.2. We show in Section 3.2.4 that the boundary divisors and the Hodge class are independent in $\operatorname{Pic} \overline{\mathfrak{S}}_g^{1/r}$. This turns out to be a useful step toward showing that torsion exists in $\operatorname{Pic} \mathfrak{S}_g^{1/r}$.

In Section 3.3 we compute the main relations between elements of the Picard group, and we show some of the consequences of those relations. The general results are given in Theorems 3.8 and 3.8.bis, and some special cases of those results are given in Corollary 3.10. In the special case that $r = 2$, these results reduce to those of [4] and [5].

In Section 3.4 we discuss the existence of torsion in $\operatorname{Pic} \mathfrak{S}_g^{1/r}$ when 2 or 3 divides r. In particular, Proposition 3.14 shows that when 2 divides r, $\operatorname{Pic} \mathfrak{S}_g^{1/r}$ has 4-torsion, and when 3 divides r, $\operatorname{Pic} \mathfrak{S}_g^{1/r}$ has 3-torsion. This is in stark contrast to the case of $\operatorname{Pic} \mathfrak{M}_g$, and $\operatorname{Pic} \overline{\mathfrak{M}_g}$, which are known to be free for $g \geq 3$. [2].

Finally, in Section 4 we work out examples for genus 1 and general r, and for $r = 2$ and general genus.

1.2. **Previous work.** The Picard group of the moduli space of curves is now fairly well understood, thanks primarily to the work of Arbarello-Cornalba [2] and Harer [9]. Most of the progress toward understanding the Picard group of the moduli of 2-spin curves is also due to Cornalba [4, 5] and Harer [10]. Section 3.3 on relations between classes in the Picard group is strongly motivated by Cornalba's work in [4] and [5].

In [13] and [12] several different compactifications of the moduli of r-spin curves are constructed. The best-behaved of these compactifications, and the one we will use here, is the stack of what are called *stable r-spin curves* or just *r-spin curves* in [13]. In the case of prime r, these are the same as the *pure spin curves* of [12].

1.3. **Conventions and notation.** By a *curve* we mean a reduced, complete, connected, one-dimensional scheme over an algebraically closed field k. A *semi-stable curve of genus g* is a curve with only ordinary double points such that $H^1(X, \mathcal{O}_X)$ has dimension g. And an *n-pointed stable curve* is a semi-stable curve X together with an ordered n-tuple of non-singular points $(p_1, \ldots, p_n)$, such that at least three marked points or double points of X lie on every smooth irreducible component of genus 0, and at least one marked point or double point of X lies on every smooth component of genus one. A family of stable (or semi-stable) curves is a flat, proper morphism $\mathcal{X} \to T$ whose geometric fibres $\mathcal{X}_t$ are (semi) stable curves. Irreducible components of a semi-stable curve which have genus 0 (i.e., are birational to $\mathbb{P}^1$) but which meet the curve in only two points will be called *exceptional* curves.

By *line bundle* we mean an invertible (locally free of rank one) coherent sheaf. By canonical sheaf we will mean the relative dualizing sheaf of a family of curves $f : \mathcal{X} \to T$, and this sheaf will be denoted $\omega_{\mathcal{X}/T}$ or ω_f. Note that for a semi-stable curve, the canonical sheaf is a line bundle. When T is the spectrum of an algebraically closed field, we will often write X for $\mathcal{X}$, and ω_X for $\omega_{X/T}$.

2. Spin curves

2.1. **Smooth spin curves.** For any $g \geq 0$, fix a positive integer r with the property that r divides $2g - 2$. We define a smooth *r-spin curve* to be a triple $(X, \mathcal{L}, b)$ of

a smooth curve X of genus g, a line bundle $\mathcal{L}$, and an isomorphism b, of the rth tensor power of $\mathcal{L}$ to the canonical bundle ω_X of X; that is, $b : \mathcal{L}^{\otimes r} \xrightarrow{\sim} \omega_X$. For a given X, any $(\mathcal{L}, b)$ making $(X, \mathcal{L}, b)$ into a spin curve will be called an *r-spin structure*. Families of smooth r-spin curves are triples $(\mathcal{X}/T, \mathcal{L}, b)$ of a family $\mathcal{X}/T$ of smooth curves, a line bundle $\mathcal{L}$, and an isomorphism b, which induces an r-spin structure on each geometric fibre of $\mathcal{X}/T$.

Example ($\mathbf{r = 2}$). A 2-spin curve is what has classically been called a spin curve (a curve with a theta-characteristic $\mathcal{L}$) *with an explicit isomorphism $b : \mathcal{L}^{\otimes 2} \xrightarrow{\sim} \omega$.*

For any choice of $\mathbf{m} = (m_1, m_2, m_3, \ldots, m_n)$, such that r divides $2g - 2 - \sum m_i$, we may also define an n-pointed r-spin curve of type $\mathbf{m}$ as a triple $((X, (p_1, p_2, \ldots, p_n)), \mathcal{L}, b)$ such that $(X, (p_1, p_2, \ldots, p_n))$ is a smooth, n-pointed curve, and b is an isomorphism from $\mathcal{L}^{\otimes r}$ to $\omega_X(-\sum m_i p_i)$. Families of n-pointed r-spin curves are defined analogously.

Two spin structures $(\mathcal{L}, b)$ and $(\mathcal{L}', b')$ are isomorphic if there exists an isomorphism between the bundles $\mathcal{L}$ and $\mathcal{L}'$ which respects the homomorphism b and b'. Over a fixed curve X, any two r-spin structures $(\mathcal{L}, b)$ and $(\mathcal{L}, b')$ which differ only by their isomorphism b or b' must be isomorphic, provided the base is algebraically closed. The set $\mathfrak{S}_g^{1/r}[X]$ of isomorphism classes of r-spin structures on X is a principal homogeneous space over the r-torsion $\mathrm{Jac}\,_r X$ of the Jacobian of X; thus it has r^{2g} elements in it.

Similarly, two spin curves $(X, \mathcal{L}, b)$ and $(X', \mathcal{L}', b')$ are isomorphic if there is an isomorphism $\tau : X \xrightarrow{\sim} X'$ and an isomorphism of spin structures $i : \mathcal{L} \xrightarrow{\sim} \tau^* \mathcal{L}$.

Example ($\mathbf{g = 1}$, $\mathbf{r \geq 2}$, $\mathbf{m = 0}$). If $n \geq 1$ and $\mathbf{m} = \mathbf{0}$, then, up to isomorphism, an n-pointed r-spin curve of genus 1 is just an n-pointed curve of genus 1 with a point of order r on the curve. However, the automorphisms of the underlying curve identify some of these r-spin structures. In particular, when $n = 1$ and r is odd, the elliptic involution acts freely on all the non-trivial r-spin structures, thus there are only $1 + (r^2 - 1)/2$ isomorphism classes of r-spin curves over a generic 1-pointed curve of genus 1.

Let $\mathfrak{S}_g^{1/r}$ denote the stack of smooth r-spin curves of genus g. And let $\mathfrak{S}_{g,n}^{1/r, \mathbf{m}}$ denote the stack of n-pointed spin curves (for a given $\mathbf{m}$). When $g = 1$, and $n = 1$, we will also write $\mathfrak{S}_1^{1/r}$ to denote the stack $\mathfrak{S}_{1,1}^{1/r, \mathbf{0}}$. If X has no automorphisms, the sets $\mathfrak{S}_g^{1/r}[X]$ and $\mathfrak{S}_{g,n}^{1/r, \mathbf{m}}[X]$, as defined above, are just the fibres of $\mathfrak{S}_g^{1/r}$, or $\mathfrak{S}_{g,n}^{1/r, \mathbf{m}}$ over the point corresponding to X in $\mathfrak{M}_g$, or in $\mathfrak{M}_{g,n}$, respectively.

2.2. **Stable spin curves.** To compactify the moduli of spin curves, it is necessary to define a spin structure for stable curves. To do this we need not just line bundles, but also rank-one torsion-free sheaves over stable curves. Some additional structure, as given in Definitions 2.3 and 2.4, is also necessary to ensure that the compactified moduli space $\overline{\mathfrak{S}}_{g,n}^{1/r, \mathbf{m}}$ is separated and smooth.

2.2.1. DEFINITIONS. To begin we need the definition of torsion-free sheaves.

Definition 2.1. A *relatively torsion-free sheaf* (or just torsion-free sheaf) on a family of stable or semi-stable curves $f : \mathcal{X} \to T$ is a coherent $\mathcal{O}_\mathcal{X}$-module $\mathcal{E}$ that is flat over T, such that on each fibre $\mathcal{X}_t = \mathcal{X} \times_T \mathrm{Spec}\, k(t)$ the induced $\mathcal{E}_t$ has no associated primes of height one.

We will only be concerned with rank-one torsion-free sheaves. Such sheaves are called *admissible* by Alexeev [1] and *sheaves of pure dimension* 1 by Simpson [20]. Of course, on the open set where f is smooth, a torsion-free sheaf is locally free.

Note 2.2. It is well-known and easy to check that if a rank-one, torsion-free sheaf $\mathcal{E}$ is not locally free (also called *singular*) at a node p of X, then the completion $\hat{\mathcal{O}}_{X,p}$ of the local ring of X near p is isomorphic to $A = k[[x, y]]/xy$, and $\mathcal{E}$ corresponds to an A-module $E \cong xk[[x]] \oplus yk[[y]] = \langle \zeta_1, \zeta_2 \mid y\zeta_1 = x\zeta_2 = 0 \rangle$ [19, Prop. 11.3].

Definition 2.3. Given an n-pointed, semi-stable curve $(X, p_1, \ldots, p_n)$, and a rank-one, torsion-free sheaf $\mathcal{K}$ on X, and given an n-tuple $\mathbf{m} = (m_1, \ldots, m_n)$ of integers, we denote by $\mathcal{K}(\mathbf{m})$ the sheaf $\mathcal{K} \otimes \mathcal{O}(-\sum m_i p_i)$.

A *dth root of $\mathcal{K}$ of type* $\mathbf{m}$ is a pair $(\mathcal{E}, b)$ of a rank-one, torsion-free sheaf $\mathcal{E}$, and an $\mathcal{O}_X$-module homomorphism $b : \mathcal{E}^{\otimes d} \to \mathcal{K}(\mathbf{m})$ with the following properties:

1. $d \cdot \deg \mathcal{E} = \deg \mathcal{K} - \sum m_i$.
2. b is an isomorphism on the locus of X where $\mathcal{E}$ is locally free.
3. For every point $p \in X$ where $\mathcal{E}$ is not free, the length of the cokernel of b at p is $d - 1$.

Unfortunately, the moduli space of stable curves with dth roots of a fixed sheaf $\mathcal{K}$ is not smooth when d is not prime, and so we must consider not just roots of a bundle, but rather a coherent net of roots. This additional structure suffices to make the stack of stable curves with coherent root nets smooth [13].

Definition 2.4. Given a semi-stable n-pointed curve $(X, p_1, \ldots, p_n)$, and a rank-one, torsion-free sheaf $\mathcal{K}$ on X, and an n-tuple $\mathbf{m}$, a *coherent net of roots of type* $\mathbf{m}$ *for* $\mathcal{K}$ is a collection $\{\mathcal{E}_d, c_{d,d'}\}$ consisting of a rank-one torsion-free sheaf $\mathcal{E}_d$ for each d dividing r, and a homomorphism $c_{d,d'} : \mathcal{E}_d^{\otimes d/d'} \to \mathcal{E}_{d'}$ for each d' dividing d with the following properties:

- $\mathcal{E}_1 = \mathcal{K}$, and $c_{d,d} = 1$ for each d dividing r.
- For every pair of divisors d' and d of r such that d' divides d, let $\mathbf{m}'$ be the n-tuple $(m'_1, \ldots, m'_n)$ such that m'_i is the smallest, non-negative integer congruent to $m_i \bmod(d/d')$. The $\mathcal{O}_X$-module homomorphism $c_{d,d'} : \mathcal{E}_d^{\otimes d/d'} \to \mathcal{E}_{d'}$, must make $\mathcal{E}_d$ into a d/d'-root of $\mathcal{E}_{d'}$, of type $\mathbf{m}'$, such that all these maps are compatible. That is, the diagram

$$(\mathcal{E}_d^{\otimes d/d'})^{\otimes d'/d''} \xrightarrow{c_{d,d'}^{\otimes d'/d''}} \mathcal{E}_{d'}^{\otimes d'/d''}$$

commutes for every $d''|d'|d|r$.

If r is prime, then a coherent net of rth-roots is simply an rth root of $\mathcal{K}$. Moreover, if $\mathcal{E}_d$ is locally free, then up to isomorphism $\mathcal{E}_d$ uniquely determines all $\mathcal{E}_{d'}$ and all $c_{d,d'}$ such that $d'|d$.

Definition 2.5. A *stable, n-pointed, r-spin curve of type* $\mathbf{m} = (m_1, \ldots, m_n)$ is an n-pointed, stable curve $(X, p_1, \ldots, p_n)$ and a coherent net of rth roots of ω_X of

type $\mathbf{m}$, where ω_X is the canonical (dualizing) sheaf of X. An r-spin curve is called *smooth* if X is smooth.

Note that this definition of a smooth r-spin curve differs from that of Section 2.1, in that a spin curve carries the additional data of explicit isomorphisms $\mathcal{E}_d^{\otimes d/d'} \to \mathcal{E}_{d'}$; however, for smooth curves, and indeed, whenever $\mathcal{E}_r$ is locally free, $\mathcal{E}_r$ and $c_{r,1}$ completely determine the spin structure, up to isomorphism.

Definition 2.6. An *isomorphism of r-spin curves* from $(X, p_1, \ldots, p_n, \{\mathcal{E}_d, c_{d,d'}\})$ to $(X', p_1', \ldots, p_n', \{\mathcal{E}_d', c_{d,d'}'\})$ is an isomorphism of pointed curves

$$\tau : (X, p_1, \ldots, p_n) \xrightarrow{\sim} (X', p_1', \ldots, p_n')$$

and a system of isomorphisms $\{\beta_d : \tau^*\mathcal{E}_d' \xrightarrow{\sim} \mathcal{E}_d\}$, with β_1 the canonical isomorphism $\tau^*\omega_{X'}(-\sum_i m_i p_i') \xrightarrow{\sim} \omega_X(-\sum m_i p_i)$, and such that the β_d are compatible with all of the maps $c_{d,d'}$ and $\tau^* c_{d,d'}'$.

The definition of families of r-spin curves is relatively technical and unenlightening. For the details of those definitions see [13]. For our purposes, it will suffice to know the basic properties of the stack of r-spin curves from [13] as given in Section 2.3

2.2.2. ALTERNATE DESCRIPTION OF STABLE SPIN CURVES. The following characterization of spin curves in terms of line bundles on a partial normalization of the underlying curve is very useful and helps illustrate the nature of stable spin curves.

Consider a stable spin curve $(X, \{\mathcal{E}_d, c_{d,d'}\})$ and the partial normalization $\widetilde{X} \xrightarrow{\pi} X$ of X at each of the singularities of $\mathcal{E}_r$ (i.e., the nodes of X where $\mathcal{E}_r$ fails to be locally free). The completion $\hat{\mathcal{O}}_{X,p}$ of the local ring of X near a singularity p of $\mathcal{E}_r$ is isomorphic to $A = k[[x,y]]/xy$, and $\mathcal{E}_r$ corresponds to an A-module $E \cong xk[[x]] \oplus yk[[y]] = \langle \zeta_1, \zeta_2 \,|\, x\zeta_2 = y\zeta_1 = 0 \rangle$. The homomorphism $c_{r,1} : \mathcal{E}_r^{\otimes r} \to \omega$ corresponds to a homomorphism of A-modules $E^{\otimes r} \to A$ of the form $\zeta_1^r \mapsto x^u$, $\zeta_2^r \mapsto y^v$, and $\zeta_1^i \zeta_2^{r-i} \mapsto 0$ for $0 < i < r$. The condition on the cokernel (Definition 2.3(3)) implies that $u + v = r$. The pair $\{u, v\}$ is called the *order* of $c_{r,1}$ at p. If $\mathcal{E}_r$ is locally free at p, then $c_{r,1}$ is an isomorphism and the order is $\{0,0\}$.

Let $\widetilde{A} = k[[x]] \oplus k[[y]] \cong \hat{\mathcal{O}}_{\widetilde{X},p^+} \oplus \hat{\mathcal{O}}_{\widetilde{X},p^-}$, where $\{p^+, p^-\}$ is the inverse image $\pi^{-1}(p)$ of the normalized node. The pullback $\pi^*\mathcal{E}_r$ of $\mathcal{E}_r$ corresponds to $E \otimes_A \widetilde{A}$, which is no longer torsion-free; but if $\pi^\natural \mathcal{E}_r := \pi^*\mathcal{E}_r/torsion$, which corresponds to a free $\widetilde{A}$-module $\widetilde{E}$, then $c_{r,1}$ induces an isomorphism

$$\widetilde{c}_{r,1} : \pi^\natural \mathcal{E}_r^{\otimes r} \xrightarrow{\sim} \pi^*\omega_X(-up^+ - vp^-) = \omega_{\widetilde{X}}((1-u)p^+ + (1-v)p^-).$$

Conversely, given a partial normalization $\pi : \widetilde{X} \to X$ with $\pi^{-1}(q_i) = \{q_i^+, q_i^-\}$, the inverse images of the singular points q_i, and given integers $u_i \in (0, r)$ and $v_i \in (0, r)$ for each normalized singularity q_i such that $u_i + v_i = r$, consider an rth root $\mathcal{L}$ of the line bundle

$$\pi^*\omega_X \otimes \mathcal{O}_{\widetilde{X}}(-\sum(u_i q_i^+ + v_i q_i^-)) \cong \omega_{\widetilde{X}} \otimes \mathcal{O}_{\widetilde{X}}(-\sum((u_i - 1)q_i^+ + (v_i - 1)q_i^-)).$$

Of course for such an $\mathcal{L}$ to exist, u_i and v_i must be chosen to make the degree of the rth power of $\mathcal{L}$ divisible by r. From this rth root and partial normalization we

can create an rth root of ω_X on X by taking $\mathcal{E}_r = \pi_* \mathcal{L}$, and by taking $c_{r,1}$ to be the map induced by adjointness from the composite

$$\mathcal{L}^{\otimes r} \xrightarrow{\sim} \pi^* \omega_X \otimes \mathcal{O}_{\widetilde{X}}(-\sum u_i q_i^+ + v_i q_i^-) \hookrightarrow \pi^* \omega_X.$$

Thus, there is a one-to-one correspondence between the rth roots of ω_X and the rth roots of $\omega_{\widetilde{X}}(-(u-1)q_i^+ - (v-1)q_i^-)$ on $\widetilde{X}$.

Moreover, if $\{u, v\}$ is the order of $c_{r,1} : \mathcal{E}_r^{\otimes r} \to \omega_X$ at p, then for each d dividing r, the order of $(\mathcal{E}_d, c_{d,1})$ is $\{u_d, v_d\}$ where u_d and v_d are the least non-negative integers congruent to u and v respectively, modulo d. So $\mathcal{E}_d$ is locally free at p if and only if d divides u (and hence v). If u and v are relatively prime, then no $\mathcal{E}_d$ is locally free, and thus all $\mathcal{E}_d$ are completely determined (up to isomorphism) by $(\mathcal{E}_r, c_{r,1})$ (or $\pi^\natural \mathcal{E}_r$) by

$$\pi^\natural \mathcal{E}_d := \pi^\natural \mathcal{E}_r^{\otimes r/d} \otimes \mathcal{O}_{\tilde{X}}((1/d)(u - u_d)p^+ + (1/d)(v - v_d)p^-)$$

and $\mathcal{E}_d = \pi_* \pi^\natural \mathcal{E}_d$. However, if u and v are not relatively prime, but rather have $\gcd(u, v) = \ell > 1$, then the root $\mathcal{E}_\ell$ is locally free at p, and hence requires the additional gluing datum of an ℓth root of unity (non-canonically determined) to construct $\mathcal{E}_\ell$ from $\pi^\natural \mathcal{E}_\ell$. The remainder of the spin structure can clearly be reconstructed from the two pieces $c_{r,\ell} : \mathcal{E}_r^{\otimes r/\ell} \to \mathcal{E}_\ell$ and $c_{\ell,1} : \mathcal{E}_\ell^{\otimes \ell} \to \omega$.

When the spin structure has no singularity (i.e., $\mathcal{E}_r$ is locally free) at a node of the underlying curve, this corresponds to E. Witten's definition of the *Ramond sector* of topological gravity [22, 21]; whereas, when u and v are non-zero, the spin structure is what Witten calls a *generalized Neveu-Schwarz sector*. If $\gcd(u, v) = \ell > 1$ then we sometimes say that the spin structure is *semi-Ramond*, corresponding to the fact that $\mathcal{E}_\ell$ is locally free (Ramond) but $\mathcal{E}_r$ is not.

2.2.3. EXAMPLES. It is useful to consider a few examples of stable spin curves. Both of the examples in this section are relevant to the study of the Picard group of the stack and will be important in Section 3.2.

Example 1 (Two irreducible components and one node)**.** First consider a stable curve X with two smooth, irreducible components C and D, of genus k and $g - k$ respectively, meeting in one double point p. In this case there exists a unique choice of u and v that makes the degree of

$$\pi^* \omega_X(-up^+ - vp^-) \cong \omega_{\widetilde{X}}(-(u-1)p^+ - (v-1)p^-)$$

divisible by r on both components. The resulting rth roots are locally free (Ramond) if and only if ω_X has an rth root, which is to say, if and only if u and v can be chosen to be 0, or $2k - 1 \equiv 0 \pmod{r}$. If $2k - 1 \not\equiv 0 \pmod{r}$ then the resulting (Neveu-Schwarz) rth root corresponds to an rth root of $\omega_C(-(u-1)p^+)$ on C and an rth root of $\omega_D(-(v-1)p^-)$ on D. If $\gcd(2k-1, r) = 1$ then all of the dth roots are Neveu-Schwarz. Even if $\gcd(2k-1, r) = \ell$ is bigger than 1, since the dual graph of X is a tree, all gluing data for constructing $\mathcal{E}_\ell$ from $\pi^\natural \mathcal{E}_\ell$ will yield (non-canonically again) isomorphic $\mathcal{E}_\ell$'s and hence isomorphic nets of roots, and the spin curve corresponds to an element in $\mathfrak{S}_{k,1}^{1/r, u-1} \times \mathfrak{S}_{g-k,1}^{1/r, v-1}$. Thus spin curves obey something like the splitting axiom of quantum cohomology (see [14, §4.1] for more details).

Example 2 (One irreducible component and one node)**.** The second example is given by an irreducible stable curve X with one node. In this case there are r different choices of u and v that permit spin structures: either $u = v = 0$, in which case the resulting spin structure is locally free (Ramond); or $u \in \{1, \ldots, r-1\}$ and $v = r - u$, in which case the rth root structure is not locally free (Neveu-Schwarz). In this second case, if $\gcd(u, r) = 1$, then the r-spin structure on X is induced by an rth root of the bundle $\omega_{\widetilde{X}}(-(u-1)p^+ - (v-1)p^-)$, and thus corresponds to an element of $\mathfrak{S}_{g-1,2}^{1/r,(u-1,v-1)}$. Since the points p^+ and p^- of the normalization are not ordered, this gives a degree-2 morphism $\overline{\mathfrak{S}}_{g,r}^{1/r,(n-1,v-1)} \to \overline{\mathfrak{S}}_g^{1/r}$. But if $\ell = \gcd(u, r) > 1$, then for a given $\pi^\natural \mathcal{E}_\ell$ there are ℓ distinct choices of gluing data, and ℓ choices of $\mathcal{E}_\ell$. This gives $\ell/2$ distinct morphisms $\overline{\mathfrak{S}}_{g,2}^{1/r,(u-1),(v-1)}$ into $\overline{\mathfrak{S}}_g^{1/r}$.

2.3. **Properties of the stack of spin curves.**

2.3.1. BASIC PROPERTIES. The main facts we need to know from [13] about the stack of stable spin curves are contained in the following theorem.

Theorem 2.7. [13, Thms 2.4.4, and 3.3.1]. *The stack $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$ of n-pointed r-spin curves of genus g and type $\mathbf{m}$ is a smooth, proper, Deligne-Mumford stack over $\mathbb{Z}[1/r]$, and the natural forgetful morphism $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}} \to \overline{\mathfrak{M}}_{g,n}$ is finite and surjective.*
Moreover, if $\ell_{g,r}(\mathbf{m})$ is defined as

$$\ell_{g,r}(\mathbf{m}) := \begin{cases} 1 & \text{if } g = 0 \text{ and } r | 2 + \sum m_i \\ \gcd(r, m_1, \ldots, m_n) & \text{if } g = 1 \text{ and } r | \sum m_i \\ \gcd(2, r, m_1, \ldots, m_n) & \text{if } g > 1 \text{ and } r | \sum m_i + 2 - 2g \\ 0 & \text{otherwise} \end{cases}$$

and if $d_{g,r}(\mathbf{m})$ is defined to be the number of (positive) divisors of $\ell_{g,r}(\mathbf{m})$ (including 1 and $\ell_{g,r}(\mathbf{m})$ itself), then $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$ is the disjoint union of $d_{g,r}(\mathbf{m})$ irreducible components.
Also, $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$ contains the stack of smooth spin curves $\mathfrak{S}_{g,n}^{1/r,\mathbf{m}}$ as an open dense substack, and the coarse moduli spaces of $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$ and $\mathfrak{S}_{g,n}^{1/r,\mathbf{m}}$ are normal and projective (respectively, quasi-projective).

Although the actual details of the definition of a family of r-spin curves are not necessary for this paper, we do need the description from [13] of the universal deformation of a stable spin curve.

Theorem 2.8. [13, Thm. 2.4.2]. *Given a stable spin curve $(X, \{\mathcal{E}_d, c_{d,d'}\})$ with singularities q_i of order $\{u_i, v_i\}$, the universal deformation space of $(X, \{\mathcal{E}_d, c_{d,d'}\})$ is the cover*

$$\operatorname{Spec} \mathfrak{o}[[\tau_1, \ldots, \tau_m, t_{m+1}, \ldots, t_{3g-3+n}]] \to \operatorname{Spec} \mathfrak{o}[[t_1, \ldots, t_{3g-3+n}]],$$

where $t_i = \tau_i^{r_i}$, and $r_i = r/\gcd(u_i, v_i)$, the scheme $\operatorname{Spec} \mathfrak{o}[[t_1, \ldots, t_{3g-3+n}]]$ is the universal deformation space for the underlying curve X, and the nodes $q_1, \ldots, q_m$ correspond to the loci of vanishing of $t_1, \ldots, t_m$, respectively.

2.3.2. RELATIONS BETWEEN THE DIFFERENT STACKS. There are several natural morphisms between the stacks.

1. There is a canonical isomorphism from $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$ to $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}'}$ where $\mathbf{m}'$ is an n-tuple whose entries are all congruent to $\mathbf{m}$ mod r; namely for any net $\{\mathcal{E}_d, c_{d,d'}\}$ of type $\mathbf{m}$, let $\{\mathcal{E}_d', c_{d,d'}'\}$ be the net given by $\mathcal{E}_d' = \mathcal{E}_d \otimes \mathcal{O}(1/d \sum (m_i - m_i')p_i)$ where p_i is the ith marked point, and $c_{d,d'}'$ is the obvious homomorphism. Because of this canonical isomorphism, we will often assume that all the m_i lie between 0 and $r-1$ (inclusive).

2. There is a morphism $\overline{\mathfrak{S}}_{g,n+1}^{1/r,\mathbf{m}'} \xrightarrow{\pi} \overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$, where $\mathbf{m}'$ is the $(n+1)$-tuple $(m_1, \ldots, m_n, 0)$; and π is the morphism which simply forgets the $(n+1)$st marked point. If m_{n+1} is not congruent to zero mod r, the degree of $\omega(\mathbf{m}')$ is $2g - 2 - \sum_{i=1}^{n+1} m_i$ and the degree of $\omega(\mathbf{m})$ is $2g - 2 - \sum_1^n m_i$. Since both cannot be simultaneously divisible by r at least one stack is empty, and there is no morphism. This morphism, sometimes called "forgetting tails," is not the universal curve over $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$, although it is birational to the universal curve. In particular, the two are isomorphic over the open stack $\mathfrak{S}_{g,n}^{1/r,\mathbf{m}}$.

3. If s divides r, then there is a natural map $[r/s] : \overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}} \to \overline{\mathfrak{S}}_{g,n}^{1/s,\mathbf{m}}$, which forgets all of the roots and homomorphisms in the rth-root net except those associated to divisors of s.

3. Picard group

3.1. **Definitions.** Throughout this section we will assume that $g \geq 2$ and $n = 0$, or that $g = 1 = n$.

By the term *Picard group* we mean the Picard group of the moduli functor; that is to say, the Picard group is the group of line bundles on the stack. By a *line bundle $\mathcal{L}$ on the stack* $\overline{\mathfrak{S}}_g^{1/r}$, we mean a functor that takes any family of spin curves $\mathfrak{Q} = (\mathcal{X}/S, \{\mathcal{E}_d, c_{d,d'}\})$ in $\overline{\mathfrak{S}}_g^{1/r}$ and assigns to it a line bundle $\mathcal{L}(\mathfrak{Q})$ on the scheme S, and which takes any morphism of spin curves $f : \mathfrak{Q}/S \to \mathfrak{P}/T$ and assigns to it an isomorphism of line bundles $\mathcal{L}(f) : \mathcal{L}(\mathfrak{Q}) \xrightarrow{\sim} f^*\mathcal{L}(\mathfrak{P})$, with the condition that the isomorphism must satisfy the cocycle condition (i.e., the isomorphism induced by a composition of maps agrees with the composition of the induced isomorphisms). The groups $\operatorname{Pic}\mathfrak{S}_g^{1/r}$, $\operatorname{Pic}\mathfrak{M}_g$, and $\operatorname{Pic}\overline{\mathfrak{M}}_g$ are defined similarly. For more details on Picard groups of moduli problems see [11, pg. 50] or [18, §5].

3.2. **Basic divisors and relations.**

3.2.1. THE TAUTOLOGICAL BUNDLES. Recall that for any family of stable curves $\pi : \mathcal{X} \to S$, the Hodge class $\lambda(\mathcal{X}/S)$ in $\operatorname{Pic}\mathfrak{M}_g$ is the determinant of the Hodge bundle (the push-forward of the canonical bundle)

$$\lambda(\mathcal{X}/S) := \det \pi_! \omega_{\mathcal{X}/S} = \wedge^g \pi_* \omega_{\mathcal{X}/S}.$$

It is well-known that $\operatorname{Pic}\mathfrak{M}_g$ is the free Abelian group generated by λ when $g > 1$ (cf. [2]) and for $g = 1$ and $g = 2$, $\operatorname{Pic}\mathfrak{M}_g$ is also generated by λ, but is cyclic of order 12, and 10, respectively [7, §5.4].

In a similar way we define a bundle $\mu = \mu^{1/r}$ in $\operatorname{Pic}\overline{\mathfrak{S}}_g^{1/r}$ as the determinant of the rth root bundle. In particular, if $\mathfrak{Q} = (\pi : \mathcal{X} \to S, \{\mathcal{E}_d, c_{d,d'}\})$ is a stable

spin curve, then $\mu(\mathfrak{Q}) := \det \pi_! \mathcal{E}_r = (\det R^0 \pi_* \mathcal{E}_r) \otimes (\det R^1 \pi_* \mathcal{E}_r)^{-1}$ on S. Similarly, define $\mu^{1/d} := \det \pi_! \mathcal{E}_d$ for each d dividing r. Note that the pullback of μ from $\overline{\mathfrak{S}}_{g,s}$ to $\overline{\mathfrak{S}}_g^{1/r}$ via $[r/s]^*$ is exactly $\mu^{1/s}$.

3.2.2. BOUNDARY DIVISORS INDUCED FROM $\overline{\mathfrak{M}}_g$. In addition to λ and μ, there are elements of $\mathrm{Pic}\, \overline{\mathfrak{S}}_g^{1/r}$ that arise from the boundary divisors of $\overline{\mathfrak{S}}_g^{1/r}$. Recall that the boundary of $\overline{\mathfrak{M}}_g$ consists of the divisors δ_i where $i \in \{0, \ldots, \lfloor g/2 \rfloor\}$. Here, when i is greater than zero, δ_i is the closure of the locus of points in $\overline{\mathfrak{M}}_g$ corresponding to stable curves with exactly one node and two irreducible components, one of genus i and the other of genus $g - i$. And when i is zero, δ_0 is the closure of the locus of points corresponding to irreducible curves with a single node.

As we saw in Example 1 of Section 2.2.3, for any curve X with exactly one node and two irreducible components of genera i and $g - i$, there is a unique choice of integer $u(i)$ between 0 and $r - 1$ that determines a bundle $\mathcal{W}$ whose rth roots define the spin structures on X. If $2i \equiv 1 \pmod{r}$, then $u(i) = v(i) = 0$ and the bundle $\mathcal{W}$ is just the canonical bundle $\mathcal{W} = \omega_X$. In this case all the spin structures on X are locally free. If on the other hand, $2i \not\equiv 1 \pmod{r}$, then there is a unique choice of $u(i)$ with $r > u(i) > 0$, and such that $2i - 1 - u(i) \equiv 0 \pmod{r}$. In this case $v(i) = r - u(i)$, and $\mathcal{W}$ is not a line bundle on X, but rather a line bundle on the normalization $\nu : \widetilde{X} \to X$ at the node q. If $\nu^{-1}(q) = \{q^+, q^-\}$, then

$$\mathcal{W} = \omega_{\widetilde{X}}((1 - u(i))q^+ + (1 - v(i))q^-).$$

In this case the spin structures on X are constructed by pushing the rth roots of $\mathcal{W}$ on $\widetilde{X}$ down to X, i.e., using the rth roots of $\mathcal{W}$ on the irreducible components to make a torsion-free sheaf on the curve X. By Theorem 2.7, if i and $g - i$ are at least 2, and r is odd, there is one irreducible divisor α_i lying above δ_i. And when r is even, there are four divisors lying over δ_i (even-even, even-odd, odd-even, odd-odd).

In general, let $D_{k,r}(\mathbf{m})$ denote the set of positive divisors of $\ell_{k,r}(\mathbf{m})$, as in Theorem 2.7. These divisors index the set of irreducible components of $\overline{\mathfrak{S}}_{k,n}^{1/r,\mathbf{m}}$. For each $i \geq 1$, and each $a \in D_{i,r}(u(i) - 1)$ and $b \in D_{g-i,r}(v(i) - 1)$, let $\alpha_i^{(a,b)}$ denote the irreducible divisor consisting of the locus of spin curves lying over δ_i with a spin structure of index a on the genus-i component and of index b on the genus-$(g - i)$ component.

Over a stable curve in δ_0 there are also several choices of spin structure. Indeed, as we saw in Example 2 of Section 2.2.3, for any choice of order $\{u, v\}$, there is again a unique bundle $\mathcal{W}_u = \omega_{\widetilde{X}}((1 - u)q^+ + (1 - v)q^-)$ so that any rth root of $\mathcal{W}_u$ gives the pullback $\pi^\sharp \mathcal{E}_r$ to $\widetilde{X}$, and thus the entire spin structure *except for glue*. And conversely, any spin structure comes from an rth root of some $\mathcal{W}_u$ for $0 \leq u < r$ and a choice of glue (where the glue corresponds to an isomorphism $\eta : \mathcal{E}_\ell|_{q^+} \xrightarrow{\sim} \mathcal{E}_\ell|_{q^-}$ and $\ell = \gcd(u, v, r)$). Again for a particular choice of order $\{u, v\}$ of index $a \in D_{g-1,r}(u - 1, v - 1)$, and a particular choice of glue η, the corresponding divisor of spin curves of the given order, index and glue is irreducible. We denote these divisors by $\gamma_{j,\eta}^{(a)}$, where j is the smaller of u and v, η is the gluing datum, and a is the index. Of course, since the two points of the normalization are not distinguishable, we have $\gamma_{j,\eta}^{(a)} = \gamma_{r-j,\eta^{-1}}^{(a)}$. We denote the set of gluings for a

given choice of j by $\mathfrak{g}_j$, and we denote the set of gluings for a fixed j, modulo the equivalence $(r/2, \eta) \sim (r/2, \eta^{-1})$, by $\mathfrak{g}_j/S_2$. Let α_i denote the sum

$$\alpha_i := \sum_{D_i(u(i)-1) \times D_{g-i}(v(i)-1)} \alpha_i^{(a,b)}$$

and let γ_j denote the sum

$$\gamma_j := \sum_{\substack{\eta \in \mathfrak{g}_j/S_2 \\ a \in D_{g-1}(j-1, r-j-1)}} \gamma_{j,\eta}^{(a)}.$$

Pullback along the forgetful map $p : \overline{\mathfrak{S}}_g^{1/r} \to \overline{\mathfrak{M}}_g$ induces a map p^* from $\operatorname{Pic} \overline{\mathfrak{M}}_g$ to $\operatorname{Pic} \overline{\mathfrak{S}}_g^{1/r}$. And it is clear that the image of the boundary divisors (which we will denote by δ_i, regardless of whether we are working in $\operatorname{Pic} \overline{\mathfrak{M}}_g$ or $\operatorname{Pic} \overline{\mathfrak{S}}_g^{1/r}$) is still supported on the boundary of $\overline{\mathfrak{S}}_g^{1/r}$. Indeed, for i greater than one, δ_i is a linear combination of the terms $\alpha_i^{(a,b)}$ and δ_0 is a linear combination of the terms $\gamma_{j,\eta}^{(a)}$.

Proposition 3.1. *Let* $c_i := \gcd(2i-1, r) = \gcd(u(i), v(i))$. *If* $i \geq 1$ *then*

$$\delta_i = (r/c_i)\alpha_i.$$

And if $d_j := \gcd(j, r)$, *then we have*

$$\delta_0 = \sum_{0 \leq j \leq r/2} (r/d_j)\gamma_j.$$

Proof. The universal deformation of a spin curve with underlying stable curve in δ_i is dependent only upon u and v; in particular, if $c := \gcd(u, v)$ and $r' := r/c$ then the forgetful map from the universal deformation of the spin curve to the universal deformation of the underlying curve is of the form $\operatorname{Spec} \mathfrak{o}[[s]] \to \operatorname{Spec} \mathfrak{o}[[s^{r'}]]$ (see Theorem 2.8). Moreover, for $i \geq 1$ the morphism $\overline{\mathfrak{S}}_g^{1/r} \to \overline{\mathfrak{M}}_g$ is ramified along α_i to order $r' = r/c_i$. Similarly, over δ_0, the map $\overline{\mathfrak{S}}_g^{1/r} \to \overline{\mathfrak{M}}_g$ is ramified along γ_j to order $r' = r/d_j$. In either case the proposition follows. $\square$

3.2.3. BOUNDARY DIVISORS INDUCED FROM OTHER-ORDER SPIN CURVES. If s divides r, say $sd = r$, then the natural map $[d] : \overline{\mathfrak{S}}_g^{1/r} \to \overline{\mathfrak{S}}_g^{1/s}$ induces a homomorphism $[d]^* : \operatorname{Pic} \overline{\mathfrak{S}}_g^{1/s} \to \operatorname{Pic} \overline{\mathfrak{S}}_g^{1/r}$. Let $\alpha_i^{1/s}$, and $\gamma_j^{1/s}$ also indicate the images of the corresponding boundary divisors under the map $[d]^*$.

Proposition 3.2. *Let* $u(i)$ *indicate the unique integer* $0 \leq u(i) < r$ *such that* $2i - 1 - u(i) \equiv 0 \pmod{r}$. *For each divisor* s *of* r, *let* $c_i^{1/s}$ *be* $\gcd(u(i), v(i), s) = \gcd(u(i), s)$ *and let* $d_i^{1/s}$ *be* $\gcd(j, (r-j), s) = \gcd(j, s)$. *In* $\operatorname{Pic} \overline{\mathfrak{S}}_g^{1/r}$ *the boundary divisors* $\alpha_i^{1/s}$, *and* $\gamma_i^{1/s}$ *are given in terms of the elements* $\alpha_i^{1/r}$, *and* $\gamma_j^{1/r}$ *as follows*:

$$\alpha_i^{1/s} = \frac{r}{c_i^{1/r}} \frac{c_i^{1/s}}{s} \alpha_i^{1/r}.$$

And $\gamma_k^{1/s}$ *behaves similarly, but now the index* j *behaves like* $u \pmod{s}$, *and there are several choices of* k *between* 0 *and* $r/2$ *such that* $k \equiv \pm j \pmod{s}$ ($\gamma_k^{1/r}$ *is the same divisor as* $\gamma_{r-k}^{1/r}$). *This gives several divisors* $\gamma_k^{1/r}$ *over each* $\gamma_j^{1/s}$:

$$\gamma_j^{1/s} = \sum_{\substack{k \equiv \pm j \,(\mathrm{mod}\,s) \\ 0 \le k \le r/2}} \frac{r}{d_k^{1/r}} \frac{d_k^{1/s}}{s} \gamma_k^{1/r}.$$

Proof. The proof is straightforward and very similar to the proof of Proposition 3.1. Only note that $\gamma_k^{1/r}$ lies over $\gamma_j^{1/s}$ if $k \equiv j \pmod{s}$ *or if* $k \equiv -j$ (mod s). □

3.2.4. INDEPENDENCE OF SOME ELEMENTS OF $\operatorname{Pic}\overline{\mathfrak{S}}_g^{1/r}$. The following two results are generalizations of Cornalba's results [4, Proposition 7.2]. Their proofs are essentially the same as those in [4], and so will just be sketched here.

Proposition 3.3. *The forgetful map* $\overline{\mathfrak{S}}_g^{1/r} \to \overline{\mathfrak{M}}_g$ *induces an injective homomorphism on the Picard groups.*

Proof. The method of proof is simply to recall from [2] that there are families of stable curves $\mathcal{X}/S$, with S a smooth and complete curve, such that the vectors $(\deg_S \lambda, \deg_S \delta_0, \ldots, \deg_S \delta_{\lfloor g/2 \rfloor})$ are independent. After suitable base change, one can install a spin structure on the families in question. (This follows, for example, from the fact that, after base change, a spin structure can be installed on the generic fibre of $\mathcal{X}/S$, and since the stack is proper over $\overline{\mathfrak{M}}_g$, this structure can be extended–again after possible base change–to the entire family). Since the effect of base change is to multiply the vectors' entries by a constant, the vectors are still independent, and thus the elements λ and δ_i are all independent in $\operatorname{Pic}\overline{\mathfrak{S}}_g^{1/r}$. □

J. Kollár pointed out to me the following alternate proof: Note that the stacks in question are both smooth. Thus the result follows from the fact that for any finite cover $f : X' \to X$ of a normal variety X and for any line bundle $\mathcal{L}$ on X, the line bundle $f_* f^* \mathcal{L}$ is a multiple of $\mathcal{L}$ (cf. [8, 6.5.3.2]). In other words, for some n we have

$$f_* f^* \mathcal{L} = \mathcal{L}^{\otimes n}.$$

But since the Picard group of $\overline{\mathfrak{M}}_g$ has no torsion, the pullback of any line bundle to $\overline{\mathfrak{S}}_g^{1/r}$ cannot be trivial.

Proposition 3.4. *The elements* λ, $\alpha_i^{(a,b)}$ *and* $\gamma_{j,\rho}^{(a)}$ *are independent in* $\operatorname{Pic}\overline{\mathfrak{S}}_g^{1/r}$ *when* $g > 1$.

Proof. Again the idea is to install a spin structure on a family of curves $\mathcal{X}/S$ over a smooth, complete curve, where the degrees of λ, and the δ_i are known. In particular, given two curves S and T of genera i and $g - i$, respectively, fix $t \in T$ and let s vary in S. Consider the family $\mathcal{X}/S$ constructed by joining the two curves at the points s and t [11, §7]. Then the degrees of λ and δ_j are all zero on S, except when $j = i$, and then $\deg_S \delta_i = 2 - 2i$. We can construct an rth root of $\omega_S((1 - u(i))s)$ on S and an rth root of $\omega_T((1 - v(i))t)$ on T, and thus an r-spin structure on $\mathcal{X}/S$. Moreover, the two rth roots can be chosen to be of any index in $D_{i,r}(u(i) - 1)$ or $D_{g-i,r}(v(i) - 1)$, respectively; and thus $\mathcal{X}/S$ can be endowed with a spin structure of any index (a, b) in $D_{i,r}(u(i) - i) \times D_{g-i,r}(v(i) - 1)$ along every

fibre. Because the $\alpha_i^{(a,b)}$ are disjoint, the degree of $\alpha_i^{(a',b')}$ for any other index (a',b') must be zero on S, but $\deg_S \delta_i \neq 0$ implies that $\deg(\sum_{(a,b)} \alpha_i^{(a,b)})$ is non-zero.

Consequently, in any relation of the form $0 = \ell\lambda + \sum e_i^{(a,b)} \alpha_i^{(a,b)} + \sum c_{j,\rho}^{(a)} \gamma_{j,\rho}^{(a)}$, the coefficients e_i must all vanish. And thus a relation must be of the form $0 = \ell\lambda + \sum c_{j,\rho}^{(a)} \gamma_{j,\rho}^{(a)}$. But a similar method shows that the coefficients $c_{j,\rho}^{(a)}$ must also be zero. In particular, consider the family $\mathcal{Y}/C$ constructed by taking a general curve C of genus $g-1$ and identifying one fixed point p with another, variable point q [11, §7]. Again one may produce an r-spin structure on $\mathcal{Y}/C$ of any type. And $\deg_C \lambda = \deg_C \gamma_{k,\psi}^{(b)} = 0$ whenever $\gamma_{k,\psi}^{(b)} \neq \gamma_{j,\rho}^{(a)}$, but $\deg_C \delta_0 = \sum_{j,\rho,a} r/d(j) \deg \gamma_{j,\rho}^{(a)}$ is equal to $2-2g$, which is non-zero (since $g > 1$). Thus $c_{j,\rho}^{(a)} = 0$, and so also $\ell = 0$. $\qquad\square$

3.3. **Less obvious relations and their consequences.** Another important question about these bundles is what relations exist between the bundles μ, λ, and the various boundary divisors. In this section we provide a partial answer.

3.3.1. MAIN RELATION. The main result of this section is Theorem 3.8, which provides relations between $\mu^{1/r}$, $\mu^{1/s}$, λ, and some boundary divisors. The proof will be given in Section 3.3.2. To state the result we need the following definitions. Motivation for the somewhat-peculiar notation will be made clear in the following section.

Definition 3.5. Let $u(i)$ denote, as in the previous section, the unique integer $0 \leq u(i) < r$ such that $2i - 1 \equiv u(i) \pmod{r}$. Let $v(i) = r - u(i)$, $c_i^{1/s} := \gcd(u(i), v(i), s) = \gcd(2i-1, s)$, and $d_j^{1/s} := \gcd(j, r-j, s) = \gcd(j, s)$ and let $c_i = c_i^{1/r}, d_j = d_j^{1/r}$. Then we define $\langle \widetilde{\mathcal{E}}_r, \mathfrak{E}_r \rangle$ to be the element in $\operatorname{Pic} \overline{\mathfrak{S}}_g^{1/r}$ defined by the following boundary divisor:

$$\langle \widetilde{\mathcal{E}}_r, \mathfrak{E}_r \rangle := \sum_{1 \leq i \leq g/2} (u(i)v(i)/c_i)\alpha_i + \sum_{0 \leq j \leq r/2} (j(r-j)/d_j)\gamma_j.$$

Similarly, for s dividing r we make the following definitions.

Definition 3.6. Let $u'(i)$ denote the unique integer $0 \leq u'(i) < s$ such that $2i - 1 \equiv u'(i) \pmod{s}$. Let $v'(i) = s - u'(i)$. And let $\langle \widetilde{\mathcal{E}}^{1/s}, \mathfrak{E}_s \rangle$ denote the following boundary divisor in $\operatorname{Pic} \overline{\mathfrak{S}}_g^{1/r}$:

$$\langle \widetilde{\mathcal{E}}^{1/s}, \mathfrak{E}_s \rangle = \sum_{1 \leq i \leq g/2} \frac{u'(i)v'(i)}{c_i^{1/s}} \alpha_i^{1/s} + \sum_{0 \leq j \leq s/2} \frac{j(s-j)}{d_j^{1/s}} \gamma_j^{1/s}$$

$$= \frac{r}{s} \sum_{1 \leq i \leq g/2} \frac{(u'(i)v'(i))}{c_i^{1/r}} \alpha_i^{1/r} + \frac{r}{s} \sum_{\substack{1 \leq j \leq s/2 \\ 1 \leq k \leq r/2 \\ k \equiv \pm j \pmod{s}}} \frac{(j(s-j))}{d_k^{1/r}} \gamma_k^{1/r}.$$

Definition 3.7. Let δ indicate the element of $\operatorname{Pic} \overline{\mathfrak{S}}_g^{1/r}$ defined by the sum of the boundary divisors pulled back from $\operatorname{Pic} \overline{\mathfrak{M}}_g$:

$$
\begin{aligned}
\delta &= \sum_{i=0}^{g/2} \delta_i \\
&= \sum_{0 \leq j \leq r/2} (r/d_j)\gamma_j + \sum_{1 \leq i \leq g/2} (r/c_i)\alpha_i \\
&= \sum_{0 \leq k \leq s/2} \left(\frac{s}{d_k^{1/s}}\right) \gamma_k^{1/s} + \sum_{1 \leq i \leq g/2} \left(\frac{s}{c_i^{1/s}}\right) \alpha_i^{1/s}.
\end{aligned}
$$

With all of this notation in place we can now state the main theorem.

Theorem 3.8. *In terms of the notation defined above, the following relations hold in* $\operatorname{Pic} \overline{\mathfrak{S}}_g^{1/r}$:

$$
\begin{aligned}
r\langle \widetilde{\mathcal{E}}_r, \mathfrak{E}_r \rangle &= (2r^2 - 12r + 12)\lambda - 2r^2\mu + (r-1)\delta \\
s\langle \widetilde{\mathcal{E}}_s, \mathfrak{E}_s \rangle &= (2s^2 - 12s + 12)\lambda - 2s^2\mu_s + (s-1)\delta.
\end{aligned}
$$

This may also be written in terms of α and γ instead of $\langle \widetilde{\mathcal{E}}_r, \mathfrak{E}_r \rangle$, $\langle \widetilde{\mathcal{E}}_s, \mathfrak{E}_s \rangle$, and δ. But the final expression is greatly simplified if we define

$$
\sigma_k^{1/s} := \sum_{\substack{1 \leq i \leq g/2 \\ i \equiv k \pmod{s}}} \alpha_i^{1/s}.
$$

Let $\sigma_k = \sigma_k^{1/r}$, and if k is not an integer, then let $\sigma_k^{1/s} = 0$.

Also, we will continue to use the notation $c_j^{1/s} = \gcd(2i-1, s)$, $c_i = c_i^{1/r}$, $d_j^{1/s} = \gcd(j, s)$, and $d_j = d_j^{1/r}$.

Theorem 3.8.bis. *In terms of the notation defined above, the following relation holds in* $\operatorname{Pic} \overline{\mathfrak{S}}_g^{1/r}$:

$$
\begin{aligned}
(2r^2 - 12r + 12)\lambda - 2r^2\mu = {}& (1-r)(\sigma_{\frac{r+1}{2}} + \gamma_0) \\
& + \sum_{1 < k < \frac{r}{2}} 2(r/c_k)(rk - 2k^2 + 2k - r)\sigma_k \\
& + \sum_{\frac{r}{2}+1 < k < r} 2(r/c_k)(3rk - 2k^2 + 2k - 2r - r^2)\sigma_k \\
& + \sum_{1 < j \leq r/2} (r/d_j)(j(r-j) - (r-1))\gamma_j.
\end{aligned}
$$

Similarly,

$$(2s^2 - 12s + 12)\lambda - 2s^2\mu^{1/s} = (1-s)(\sigma_{\frac{s+1}{2}} + \gamma_0^{1/s})$$

$$+ \sum_{1 < k < \frac{s}{2}} 2(s/c_k^{1/s})(sk - 2k^2 + 2k - s)\sigma_k^{1/s}$$

$$+ \sum_{\frac{s}{2}+1 < k < s} 2(s/c_k^{1/s})(3sk - 2k^2 + 2k - 2s - s^2)\sigma_k^{1/s}$$

$$+ \sum_{1 < j \leq s/2} (s/d_j^{1/s})(j(s-j) - (s-1))\gamma_j^{1/s}.$$

The proof of Theorem 3.8 will be postponed until the next section. The proof of Theorem 3.8.bis is a straightforward, but tedious calculation, accomplished simply by writing out the definitions of all of the different terms, and applying Theorem 3.8.

The main thing to note about the relations of Theorems 3.8 and 3.8.bis is that they hold in $\operatorname{Pic} \overline{\mathfrak{S}}_g^{1/r}$ and not just in $\operatorname{Pic} \overline{\mathfrak{S}}_g^{1/r} \otimes \mathbb{Q}$—that is to say, not just modulo torsion.

We also have the following immediate corollaries:

Corollary 3.9. *If s divides r the following relations hold in* $\operatorname{Pic} \mathfrak{S}_g^{1/r}$*:*

$$(2r^2 - 12r + 12)\lambda = 2r^2\mu^{1/r}$$

$$(2s^2 - 12s + 12)\lambda = 2s^2\mu^{1/s}$$

$$2r^2(s^2 - 6s + 6)\mu^{1/r} = 2s^2(r^2 - 6r + 6)\mu^{1/s}.$$

Corollary 3.10. *The following special cases of the relations in Theorems* 3.8 *and* 3.8.bis *hold*:

$r = 2$	$4\lambda + 8\mu^{1/2}$	$=$	γ_0
$r = 3$	$6\lambda + 18\mu^{1/3}$	$=$	$2\gamma_0 + 2\sigma_2$
$r = 4$	$4\lambda + 32\mu^{1/4}$	$=$	$3\gamma_0 - 2\gamma_2$
	$4\lambda + 8\mu^{1/2}$	$=$	$\gamma_0 + 2\gamma_2$
$r = 5$	$2\lambda - 50\mu^{1/5}$	$=$	$-4\gamma_0 + 10(\gamma_2 + \sigma_2 + \sigma_4) - 4\sigma_3$
$r = 6$	$12\lambda - 72\mu^{1/6}$	$=$	$-5\gamma_0 + 9\gamma_2 + 8(\gamma_3 + \sigma_2 + \sigma_5)$
	$4\lambda + 8\mu^{1/2}$	$=$	$\gamma_0 + 3\gamma_2$
	$6\lambda + 18\mu^{1/3}$	$=$	$2\gamma_0 + 4\gamma_3 + 4(\sigma_2 + \sigma_5)$
$r = 7$	$26\lambda - 98\mu^{1/7}$	$=$	$-6(\gamma_0 + \sigma_4) + 28(\gamma_2 + \sigma_3 + \sigma_5) + 42(\gamma_3 + \sigma_2 + \sigma_6)$
$r = 8$	$44\lambda - 128\mu^{1/8}$	$=$	$-7\gamma_0 + 20\gamma_2 + 18\gamma_4 + 64(\gamma_3 + \sigma_2 + \sigma_3 + \sigma_6 + \sigma_7)$
	$4\lambda - 32\mu^{1/4}$	$=$	$3\gamma_0 + 6\gamma_4 - 4\gamma_2$
	$4\lambda - 8\mu^{1/2}$	$=$	$\gamma_0 + 4\gamma_2 + 2\gamma_4$

Note that in the case when $r = 2$, the relation $4\lambda + 8\mu^{1/2} = \gamma_0$ is exactly the content of Theorem (3.6) in [5].

3.3.2. PROOF OF THE MAIN RELATION AND SOME OTHER RELATIONS. Although some of the proof of Theorem 3.8 resembles that of Theorem (3.6) in [5], the proof of Theorem 3.8 requires many involved calculations for the case $r > 2$ that do not arise when $r = 2$.

To accomplish the proof without annihilating torsion elements, we need some tools other than the usual Grothendieck-Riemann-Roch; in particular, we'll use the following construction of Deligne [6].

Definition 3.11. Given two line bundles $\mathcal{L}$ and $\mathcal{M}$ on a family of semi-stable curves $f : \mathcal{X} \to S$, define a new line bundle, the Deligne product $\langle \mathcal{L}, \mathcal{M} \rangle$ on S, as

$$\langle \mathcal{L}, \mathcal{M} \rangle := (\det f_! \mathcal{L} \mathcal{M}) \otimes (\det f_! \mathcal{L})^{-1} \otimes (\det f_! \mathcal{M})^{-1} \otimes \det f_! \mathcal{O}_\mathcal{X}.$$

Here, as earlier, $\det f_! \mathcal{L}$ indicates the line bundle $(\det R^0 f_* \mathcal{L}) \otimes (\det R^1 f_* \mathcal{L})^{-1}$. We will write this additively as $\langle \mathcal{L}, \mathcal{M} \rangle = \det f_! \mathcal{L}, \mathcal{M} - \det f_! \mathcal{L} - \det f_! \mathcal{M} + \det f_! \mathcal{O}$. Deligne shows that this operation is symmetric and bilinear. It is also straightforward to check that when the base S is a smooth curve then the degree on S of the Deligne product $\langle \mathcal{L}, \mathcal{M} \rangle$ is just the intersection number of $\mathcal{L}$ with $\mathcal{M}$; that is to say, if $(-.-)$ is the intersection pairing on $\mathcal{X}$, then

$$\deg_S \langle \mathcal{L}, \mathcal{M} \rangle = (\mathcal{L}.\mathcal{M}).$$

Using this notation, and writing ω for the canonical bundle of $\mathcal{X}/S$, Serre duality takes the form

$$\langle \omega, \mathcal{M} \rangle = \det f_! (\mathcal{M}^{-1}) - (\det f_! \mathcal{M}),$$

or

$$\det f_! (\mathcal{L}\omega) = \det f_! (\mathcal{L}^{-1}).$$

Deligne proves the following variant of Grothendieck-Riemann-Roch that holds with integer, rather than just rational coefficients.

Lemma 3.12 (Deligne-Riemann-Roch). *Let $f : \mathcal{X} \to S$ be a family of stable curves, and let $\mathcal{L}$ be any line bundle on $\mathcal{X}/S$. If ω denotes the canonical sheaf of $\mathcal{X}/S$, then*

$$2(\det f_! \mathcal{L}) = \langle \mathcal{L}, \mathcal{L} \rangle - \langle \mathcal{L}, \omega \rangle + 2(\det f_! \omega).$$

We also need some additional line bundles associated to a family of r-spin curves $(\mathcal{X}/S, \{\mathcal{E}_d, c_{d,d'}\})$ that intuitively amount to measuring the difference between $\mathcal{E}_r$ and a "real" rth root of the canonical bundle. To construct these bundles we first recall from [12] some canonical constructions related to any rth root $(\mathcal{E}_r, c_{r,1})$ on a curve $\mathcal{X}/S$

1. There is a semi-stable curve $\pi : \widetilde{\mathcal{X}}_{\mathcal{E}_r} \to \mathcal{X}$, uniquely determined by $\mathcal{X}$ and $\mathcal{E}_r$, so that the stable model of $\widetilde{\mathcal{X}}$ is $\mathcal{X}$.
2. On $\widetilde{\mathcal{X}}$, there is a unique line bundle $\mathcal{O}_{\widetilde{\mathcal{X}}}(1)$, and a canonically-determined, injective map $\beta : \mathcal{O}_{\widetilde{\mathcal{X}}}(1)^{\otimes r} \to \omega_{\widetilde{\mathcal{X}}/S}$, the push forward $\pi_* \mathcal{O}_{\widetilde{\mathcal{X}}}(1)$ is $\mathcal{E}_r$, and the map b is induced from β by adjointness.
3. Moreover, the degree of $\mathcal{O}_{\widetilde{\mathcal{X}}}(1)$ is one on any exceptional curve in any fibre.

To simplify, we will denote $\mathcal{O}_{\widetilde{\mathcal{X}}_{\mathcal{E}_r}}(1)$ by $\widetilde{\mathcal{E}}_r$. And we define a new bundle $\mathfrak{E}$ by

$$\mathfrak{E}_r = \omega_{\widetilde{\mathcal{X}}} \otimes \widetilde{\mathcal{E}}_r^{\otimes -r}.$$

The advantage of using $\mathfrak{E}_r$ is that it is completely supported on the exceptional locus, and thus it is easy to describe explicitly; and its Deligne product with other line bundles on $\widetilde{\mathcal{X}}$ easy to compute.

Similarly, if s divides r, then there is a family $(\mathcal{E}_s, c_{s,1})$ of sth roots on $\mathcal{X}$. And corresponding to this family, there is a line bundle $\widetilde{\mathcal{E}}_s$ on $\widetilde{\mathcal{X}}_{\mathcal{E}_s}$ which pushes down to $\mathcal{E}_s$, and another line bundle $\mathfrak{E}_s = \omega_{\widetilde{\mathcal{X}}_s} \otimes (\widetilde{\mathcal{E}}')^{\otimes -s}$.

Theorem 3.13. *The Deligne products $\langle \widetilde{\mathcal{E}}_r, \mathfrak{E}_r \rangle$ and $\langle \widetilde{\mathcal{E}}_s, \mathfrak{E}_s \rangle$ agree with Definition 3.5. Also, the Deligne product of the canonical bundle ω with $\mathfrak{E}$ is trivial. In particular, we have*

1. *The Deligne product $\langle \widetilde{\mathcal{E}}_r, \mathfrak{E}_r \rangle$ is exactly*

$$\sum_{1 \le i \le g/2} (u(i)v(i)/c_i)\alpha_i^{1/r} + \sum_{0 \le j \le r/2} (j(r-j)/d_j)\gamma_j^{1/r}.$$

2. *The Deligne product $\langle \widetilde{\mathcal{E}}_s, \mathfrak{E}_s \rangle$ is exactly*

$$r/s \sum_{1 \le i \le g/2} (u''(i)v''(i)/c_i^{1/r})\alpha_i^{1/r} + r/s \sum_{\substack{1 \le j \le s/2 \\ 1 \le k \le r/2 \\ k \equiv j \pmod s}} j(s-j)/d_j^{1/r}\gamma_{k^{1/r}}.$$

3. *And $\langle \omega_{\widetilde{\mathcal{X}}_r}, \mathfrak{E}_r \rangle = \mathcal{O} = \langle \omega_{\widetilde{\mathcal{X}}_s}, \mathfrak{E}_s \rangle$.*

Proof. For $\mathcal{X}$ smooth, $\mathfrak{E}_r$ is canonically isomorphic to $\mathcal{O}$, and thus in the smooth case, $\langle \widetilde{\mathcal{E}}_r, \mathfrak{E}_r \rangle$ and $\langle \omega_{\widetilde{\mathcal{X}}}, \mathfrak{E}_r \rangle$ are trivial.

In the general case these products are all integral linear combinations of boundary divisors. To compute the coefficients we evaluate degrees on families of curves $\mathcal{X}/S$ parameterized by a smooth curve S and having smooth generic fibre. In this case, since $\mathfrak{E}$ is supported on the exceptional locus of $\widetilde{\mathcal{X}}$, whereas the canonical bundle has degree zero on the exceptional locus, the product $\langle \omega_{\widetilde{\mathcal{X}}}, \mathfrak{E}_r \rangle$ must be trivial.

Computing the coefficients of $\langle \widetilde{\mathcal{E}}_r, \mathfrak{E}_r \rangle$ requires that we consider the local structure of $\mathfrak{E}_r$ near an exceptional curve. Recall from [13] that for any singularity where $\mathcal{E}$ has order $\{u, v\}$ with $u, v > 0$, and such that if $c := \gcd(u, v)$, $u' := u/c$, $v' := v/c$, and $r' := r/c$, the underlying singularity of $\mathcal{X}/S$ is analytically isomorphic to $\mathrm{Spec}\,(\hat{\mathcal{O}}_{S,s}[[x,y]]/xy - \tau^{r'})$, where τ is an element of $\hat{\mathcal{O}}_{S,s}$. $\mathcal{E}_r$ is generated by two elements, say ν and ξ, with the relations $x\nu = \tau^{v'}\xi$ and $y\xi = \tau^{u'}\nu$. Moreover, over such a singularity, $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$ is locally given as

$$\pi : \mathrm{Proj}\,_A(A[\nu, \xi]/(\nu x - \tau^{v'}\xi, \nu\tau^{u'} - \xi y)) \to \mathrm{Spec}\, A,$$

where A is the local ring of $\mathcal{X}$ at the singularity. The exceptional curve, call it D, is defined by the vanishing of x and y, and we have a situation like that depicted in Figure 1.

We need to express $\mathfrak{E}_r$ in terms of a divisor, but this is easy since it is supported completely on the exceptional locus. $\mathfrak{E}_r$ is locally of the form $\mathcal{O}_{\widetilde{\mathcal{X}}}(nD)$, for some integer n. And any Weil divisor of the form nD is Cartier if and only if u' and v' both divide n. Moreover, it is easy to see that if nD is Cartier, then when restricted to the exceptional curve D, the degree of nD is $-n/u' - n/v'$. Finally, since $\mathfrak{E}_r$ has degree $-r$ on D, we have $n = u'v'c = uv/c$ so that

$$\mathfrak{E}_r = \mathcal{O}_{\widetilde{\mathcal{X}}}((uv/c)D).$$

Now, to compute the coefficients of $\langle \widetilde{\mathcal{E}}_r, \mathfrak{E}_r \rangle$ just note that for families $f : \mathcal{X} \to S$ over a smooth base curve S with smooth generic fibre, the degree of $\langle \widetilde{\mathcal{E}}_r, \mathfrak{E}_r \rangle$ is just the intersection number $(\widetilde{\mathcal{E}}_r, \mathfrak{E}_r)$. Thus if D_p indicates the exceptional curve over a
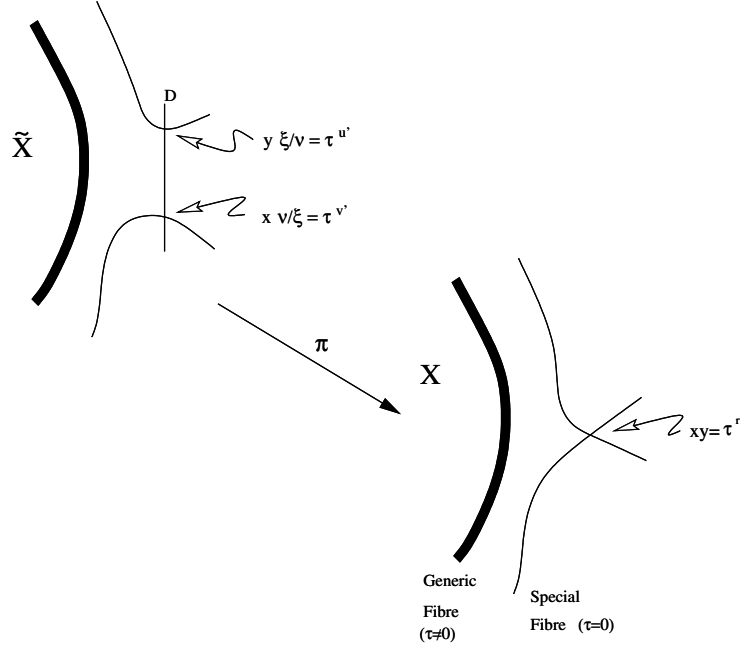
FIGURE 1.   Local structure of $\pi : \widetilde{\mathcal{X}} \to \mathcal{X}$ over a singularity of $\mathcal{E}$.

singularity p, and if $\{u_p, v_p\}$ indicates the order of $(\mathcal{E}_r, c_{r,1})$ near p, then

$$
\deg_S \langle \widetilde{\mathcal{E}}_r, \mathfrak{E}_r \rangle = \sum_{\substack{p \text{ a sing-} \\ \text{singularity of } \mathcal{E}}} (u_p v_p / c_p) \deg_{D_p} \widetilde{\mathcal{E}}_r
$$

$$
= \sum_{\substack{p \text{ of} \\ \text{type } \alpha_i}} (u(i)v(i)/c_i) + \sum_{\substack{p \text{ of} \\ \text{type } \gamma_j}} (j(r-j)/d_j).
$$

The second line follows because $\deg_{D_p} \widetilde{\mathcal{E}}_r = 1$ and because over γ_j we have $u = j$ and $v = r - j$. This proves the theorem for $\langle \widetilde{\mathcal{E}}_r, \mathfrak{E}_r \rangle$, and the result for s is just the pullback of the relation for $\langle \widetilde{\mathcal{E}}_s, \mathfrak{E}_s \rangle$ from $\operatorname{Pic} \overline{\mathfrak{S}}_g^{1/s}$. $\qquad\qquad\square$

Now we can prove the main theorem.

Proof of Theorem 3.8. Since $\mathfrak{E}_r \otimes \widetilde{\mathcal{E}}_r^{\otimes r} = \omega$, we have

$$
r\langle \widetilde{\mathcal{E}}_r, \mathfrak{E}_r \rangle = -\langle \widetilde{\mathcal{E}}_r^{\otimes r}, \mathfrak{E}_r^{-1} \rangle - \langle \widetilde{\mathcal{E}}_r^{\otimes r}, \omega \rangle + \langle \widetilde{\mathcal{E}}_r^{\otimes r}, \omega \rangle
$$

$$
= \langle \widetilde{\mathcal{E}}_r^{\otimes r}, \omega \rangle - \langle \widetilde{\mathcal{E}}_r^{\otimes r}, \widetilde{\mathcal{E}}_r^{\otimes r} \rangle
$$

$$
= \langle \widetilde{\mathcal{E}}_r^{\otimes r}, \omega \rangle - r^2 \langle \widetilde{\mathcal{E}}_r, \widetilde{\mathcal{E}}_r \rangle.
$$

Deligne-Riemann-Roch and $\mu := \det f_!\widetilde{\mathcal{E}}_r$ now give

$$
\begin{aligned}
&= \langle \widetilde{\mathcal{E}}_r^{\otimes r}, \omega \rangle - r^2(2\mu - 2\lambda + \langle \mathcal{E}, \omega \rangle) \\
&= 2r^2\lambda - 2r^2\mu - (r-1)\langle \widetilde{\mathcal{E}}_r^{\otimes r}, \omega \rangle \\
&= 2r^2\lambda - 2r^2\mu - (r-1)\langle \omega \otimes \mathfrak{E}_r^{-1}, \omega \rangle \\
&= 2r^2\lambda - 2r^2\mu - (r-1)\langle \omega, \omega \rangle \\
&= 2r^2\lambda - 2r^2\mu - (r-1)(12\lambda - \delta)
\end{aligned}
$$

Here the last equality follows from the well-known Mumford isomorphism: $\langle \omega, \omega \rangle = 12\lambda - \delta$ (see [11]). □

3.4. **Torsion in** $\mathbf{Pic}\,\mathfrak{S}_g^{1/r}$. The Picard group of $\mathfrak{M}_g$ is known to be freely generated by λ when g is greater than 2 (see [2]). And Harer [10] has shown that for $r = 2$, the rational Picard group $\operatorname{Pic}\mathfrak{S}_g^{1/2} \otimes \mathbb{Q}$ has rank one for $g \geq 9$. So one might expect that $\operatorname{Pic}\mathfrak{S}_g^{1/2}$ is freely generated by μ or λ, but Cornalba showed in [5] that $\operatorname{Pic}\mathfrak{S}_{g,2}$ has 4-torsion, and one of the consequences of Theorem 3.8 is that whenever 2 or 3 divides r, there are torsion elements in $\operatorname{Pic}\mathfrak{S}_g^{1/r}$. In particular, the following proposition holds.

Proposition 3.14. *If r is not relatively prime to 6 and $g > 1$, then* $\operatorname{Pic}\mathfrak{S}_g^{1/r}$ *has torsion elements*:

1. *If r is even, then $r^2\mu - (r^2 - 6r + 6)\lambda \neq 0$, and thus $\frac{1}{2}(r^2\mu - (r^2 - 6r + 6)\lambda)$ is an element of order 4 in* $\operatorname{Pic}\mathfrak{S}_g^{1/r}$.
2. *If 3 divides r, then $\frac{2}{3}(r^2\mu - (r^2 - 6r + 6)\lambda) \neq 0$, and thus $\frac{1}{3}(r^2\mu - (r^2 - 6r + 6)\lambda)$ is an element of order 3 or 6.*
3. *If $r = sd$ and d is even, then $r^2(\mu_s - \mu_r) - 6(d^2 - rd + r + 1)\lambda \neq 0$, and thus $\frac{1}{2}r^2(\mu_s - \mu_r) - 3(d^2 - rd + r + 1)\lambda$ has order 4.*
4. *If $r = sd$ and 3 divides d, then $\frac{2}{3}r^2(\mu_s - \mu_r) - 4(d^2 - rd + r + 1)\lambda \neq 0$, and thus $\frac{1}{3}r^2(\mu_s - \mu_r) - 2(d^2 - rd + r + 1)\lambda$ has order 3 or 6.*

Corollary 3.15. *If 6 divides r then $\frac{1}{6}(r^2\mu - (r^2 - 6r + 6)\lambda)$ is an element of order 12.*

Proof of Proposition 3.14. If the proposition were false and the element in question were zero, then in $\operatorname{Pic}\overline{\mathfrak{S}}_g^{1/r}$ this element would be a sum of boundary divisors. In particular, the element in question would be of the form $\sum e_i^{(a,b)}\alpha_i^{(a,b)} + \sum c_{k,\rho}^{(a)}\gamma_{k,\rho}^{(a)}$. In the first case, multiplication by two, and in the second case, multiplication by three, allows us to replace the element in question with a sum (from Theorem 3.8) consisting exclusively of boundary divisors. Thus for the first case we have a relation between boundary divisors where the sum on the right has all coefficients divisible by two:

$$
(1 - r)\delta + r\langle \widetilde{\mathcal{E}}, \mathfrak{E} \rangle = 2\sum e_i^{(a,b)}\alpha_i^{(a,b)} + 2\sum c_{k,\rho}^{(a)}\gamma_{k,\rho}^{(a)}.
$$

And for the second case, the sum on the right has all coefficients divisible by three:

$$
(1 - r)\delta + r\langle \widetilde{\mathcal{E}}, \mathfrak{E} \rangle = 3(\text{boundary divisors}).
$$

Thus by Theorem 3.4 the coefficients on the left must also be divisible by 2 or 3, respectively. However, in both cases the coefficient of $\gamma_{0,\rho}^{(a)}$ for any ρ and a is $1 - r$, which has no divisors in common with r; a contradiction.

The third and fourth cases are similar, but first we must subtract d^2 times the second equation of Theorem 3.8.bis from the first. Now reduction mod 2 and mod 3 give the necessary contradictions in the third and fourth cases, respectively.   □

4. **Examples**

4.1. **Genus 1 and index 1.** In the case of $g = 1$, the only boundary divisors of the stack $\overline{\mathfrak{S}}_1^{1/r} := \overline{\mathfrak{S}}_{1,1}^{1/r,\mathbf{0}}$ are those lying over δ_0; that is, the Ramond divisors $\gamma_{0,\rho}$, corresponding to the different gluings $\{\rho\}$ of $\mathcal{O}_{\mathbb{P}^1}$ at the unique node, and the Neveu-Schwarz divisors γ_j. In particular, if we write $\overline{\mathfrak{S}}_1^{1/r}$ as the disjoint union of its irreducible components

$$\overline{\mathfrak{S}}_1^{1/r} = \coprod_{d|r} \overline{\mathfrak{S}}_1^{1/r,(d)}$$

where a generic geometric point of $\overline{\mathfrak{S}}_1^{1/r,(d)}$ has $\mathcal{E}_r$ isomorphic to a dth root of $\mathcal{O}$, then the only boundary divisor in $\overline{\mathfrak{S}}_1^{1/r,(1)}$ is $\gamma_{0,1}$ corresponding to the trivial bundle $\mathcal{O}_X = \mathcal{E}_r$ on the singular curve X. Moreover, $\overline{\mathfrak{S}}_1^{1/r,(1)} \to \overline{\mathfrak{M}}_{1,1}$ is unramified, and so $\delta = \delta_0 = \gamma_0 = \gamma_{0,1}$ in $\operatorname{Pic} \overline{\mathfrak{S}}_1^{1/r,(1)}$. The boundary divisor $\langle \widetilde{\mathcal{E}}, \mathfrak{E} \rangle$ reduces to zero in this case, and so Theorem 3.8 gives

$$2r^2\mu = (2r^2 - 12r + 12)\lambda + (r-1)\delta$$
$$= (2r^2 - 12r + 12)\lambda + (r-1)\gamma_0.$$

We can give a much more complete description of the Picard group in this case using Edidin and Graham's equivariant intersection theory [7] and the following explicit construction of $\mathfrak{S}_1^{1/r,(1)}$.

Proposition 4.1. *The stack* $\mathfrak{S}_1^{1/r,(1)}$ *is isomorphic to the quotient* $\mathbb{A}^2_{c_4,c_6} - \Delta :=$ $\{(c_4, c_6)|c_4^3 - c_6^2 \neq 0\}$ *by the action of* $\mathbb{G}_m$, *given by* $v \cdot (c_4, c_6) = (v^{-4r}c_4, v^{-6r}c_6)$. *Similarly,* $\overline{\mathfrak{S}}_1^{1/r,(1)}$ *is* $\mathbb{A}^2_{c_4,c_6} - D := \{(c_4, c_6)|c_4 \neq 0 \neq c_6\}$ *modulo the same action of* $\mathbb{G}_m$.

Proof. First recall that if Δ is the locus $\{c_4^3 - c_6^2\}$ and D is the locus $\{c_4 = c_6 = 0\}$, then the stacks $\mathfrak{M}_{1,1}$ and $\overline{\mathfrak{M}}_{1,1}$ have a representation as the space of cubic forms $\{y^2 = x^3 - 27c_4x - 54c_6\}$, that is $\mathbb{A}^2_{c_4,c_6} - \Delta$ and $\mathbb{A}^2_{c_4,c_6} - D$ respectively, modulo the "standard" $\mathbb{G}_m$ action $v \cdot (c_4, c_6) = (v^{-4}c_4, v^{-6}c_6)$ [7, Remark following §5.4]. We denote this action by $s : \mathbb{G}_m \times \mathbb{A}^2 \to \mathbb{A}^2$ and the action of the proposition by $b : \mathbb{G}_m \times \mathbb{A}^2 \to \mathbb{A}^2$.

We have commutative diagrams of stacks

$$
\begin{array}{ccc}
(\mathbb{A}^2 - \Delta)/b & \longrightarrow & \mathfrak{S}_1^{1/r,(1)} \\
\downarrow & & \downarrow \\
(\mathbb{A}^2 - \Delta)/s & \longrightarrow & \mathfrak{M}_{1,1}
\end{array}
$$

and

$$
\begin{array}{ccc}
(\mathbb{A}^2 - D)/b & \longrightarrow & \overline{\mathfrak{S}}_1^{1/r,(1)} \\
\downarrow & & \downarrow \\
(\mathbb{A}^2 - D)/s & \longrightarrow & \overline{\mathfrak{M}}_{1,1}
\end{array}
$$

where the top morphism is given by the fact that there is a b-equivariant choice of a line bundle $\mathcal{E}_r$ on the curve $y^2 = x^3 - 27c_4 - 54c_6 \subseteq \mathbb{P}^2 \times (\mathbb{A}^2 - D)$ and a b-equivariant isomorphism $\mathcal{E}_r^{\otimes r} \xrightarrow{\sim} \omega$. The bundle $\mathcal{E}_r$ is generated by an rth root of dx/y, the invariant differential. This makes sense because dx/y has no zeros or poles.

Alternately, we may simply take the trivial line bundle $\mathcal{N}$ on $\mathbb{A}^2$ generated by an element ζ, with the action b defined as $v \cdot \zeta = v^{-1}\zeta$. If $\pi : \{y^2 = x^3 - 27c_4 - 54c_6\} \to \mathbb{A}^2$ is the projection to $\mathbb{A}^2$, then we define $\mathcal{E}_r$ to be $\pi^*\mathcal{N}$, and the homomorphism $\mathcal{E}_r^{\otimes r} \to \omega$ to be $\zeta^r \mapsto \frac{dx}{y}$. This homomorphism is b-equivariant since $v \cdot (\frac{dx}{y}) = v^{-r}\frac{dx}{y}$; and it is well-known that for this family $\frac{dx}{y}$ generates the relative dualizing sheaf ω_π.

The proposition now follows since both the left and right-hand vertical morphisms are étale of degree $1/r$, and the bottom morphism is an isomorphism. Thus the top morphism is étale of degree 1. It is clearly an isomorphism on geometric points, thus also an isomorphism of stacks. $\square$

It is easy to see that the line bundle $\mathcal{N}$ induces the pushforward $\pi_*\mathcal{E}_r$ on $\overline{\mathfrak{S}}_1^{1/r,(1)}$ and $\mathfrak{S}_1^{1/r,(1)}$. We denote this bundle by μ^+. Similarly the bundle $\mu^- := R^1\pi_*\mathcal{E}_r = -\pi_*(\omega \otimes \mathcal{E}_r^{-1})$ can be seen to be $-\lambda + \mu^+$, thus $\mu = \mu^+ - \mu^- = \lambda$, which is compatible with $12\lambda = \delta$ and with Theorem 3.8. Moreover, the explicit description of $\mathcal{N}$ shows that $r\mu^+ = \lambda$. So the order of μ^+ is $12r$ in $\mathrm{Pic}\,\mathfrak{S}_1^{1/r,(1)}$.

Corollary 4.2. *The Chow rings* $A^*(\mathfrak{S}_1^{1/r,(1)})$ *and* $A^*(\overline{\mathfrak{S}}_1^{1/r,(1)})$ *are isomorphic to* $\mathbb{Z}[t]/12rt$ *and* $\mathbb{Z}[t]/24r^2t^2$, *respectively. Consequently,* $\mathrm{Pic}\,\mathfrak{S}_1^{1/r,(1)} = \langle \mu^+ \rangle \cong \mathbb{Z}/12r\mathbb{Z}$ *and* $\mathrm{Pic}\,\overline{\mathfrak{S}}_1^{1/r,(1)} = \langle \mu^+ \rangle \cong \mathbb{Z}$.

Proof. By [7, Prop. 18 and 19] for any smooth quotient stack $\mathcal{F} = [X/G]$ the Chow ring $A^*(\mathcal{F})$ is the equivariant Chow ring $A_G^*(X) \cong A_*^G(X)$, and $\mathrm{Pic}\,\mathcal{F} = A_G^1(X)$. Thus it suffices to compute $A_*^G(X)$, where X is the $\mathbb{A}^2_{c_4,c_6} - \Delta$ or $\mathbb{A}^2_{c_4,c_6} - D$, and G in $\mathbb{G}_m$ with the action $b : \mathbb{G}_m \times X \to X$ of Proposition 4.1.

Choosing an $N + 1$-dimensional representation V of G with all weights -1, and letting $U = V - \{0\}$ be the open set where G acts freely, then the diagonal action of G on $(\mathbb{A}^2_{c_4,c_6} - \{0\}) \times U$ is free, and $A_i^G(\mathbb{A}^2_{c_4,c_6} - \{0\})$ is defined [7, Def. 1] to be the usual Chow group $A_i((\mathbb{A}^2 - \{0\}) \times U/G)$ of the quotient scheme $((\mathbb{A}^2 - \{0\}) \times U)/G$, which is isomorphic to the complement of the zero section of the vector bundle $\mathcal{O}(4r) \oplus \mathcal{O}(6r)$ over $\mathbb{P}^N$. Thus $A_G^*(\mathbb{A}^2_{c_4,c_6} - \{0\}) = \mathbb{Z}[t]/24r^2t^2$.

Moreover, since the form $c_4^3 - c_6^2$ has weighted degree $-12r$ with respect to the action, the G equivariant fundamental class $[\Delta]_G$ of Δ is $12rt$, and $A_*^G(\mathbb{A}^2 - \Delta) = A_*^G((\mathbb{A}^2 - \{0\} \times U)/[\Delta]_G = \mathbb{Z}[t]/12rt$. Similarly, the class $[D]_G$ is the intersection of $[c_4 = 0]_G$ and $[c_6 = 0] = (4rt) \cdot (6rt) = 0$. The result follows. $\square$

4.2. **Other components in genus 1.** Because we have no explicit representation of $\mathfrak{S}_1^{1/r,(d)}$ as a quotient stack for $d > 1$, this case is more difficult. Moreover, $\mathcal{E}_r$ has no global sections, nor any higher cohomology, so the bundle μ is trivial (and $\mu^+ = \mu^- = 0$). The only other obvious bundles on the stack are those induced by pullback along $[d] : \mathfrak{S}_1^{1/r,(d)} \to \mathfrak{S}_1^{d/r,(1)}$ from $\mathrm{Pic}\,\mathfrak{S}_1^{d/r,(1)}$. In particular, we have the bundles $\mu^{d/r,+}$ and λ. With $\lambda = r\mu^{d/r,+}/d$ and since $12\lambda = \delta$, the relation of Theorem 3.8 gives

$$2r^2\lambda = 0 \text{ in } \mathrm{Pic}\,\mathfrak{S}_1^{1/r,(d)}.$$

In particular, if 6 does not divide r, then the homomorphism

$$[d]^* : \mathrm{Pic}\,\mathfrak{S}_1^{d/r,(1)} \to \mathrm{Pic}\,\mathfrak{S}_1^{1/r,(d)}$$

is not injective.

In the case of $d = 2$ or 3 we can follow Mumford [18] and construct r-spin curves of index d which have non-trivial automorphisms, and these will give homomorphisms $\mathrm{Pic}\,\mathfrak{S}_1^{1/r,(d)} \to \mathbb{G}_m$. In particular, in the case that $d = 2$, consider the curve $E_{1728} : y^2 = x(x^2 - 1)$, and the two-torsion point $p = (0,0)$. Associated to p is the line bundle $\mathcal{E}_r := \{f\psi | f \in k(E_{1728}), (f) \geq p - \infty\}$, and the isomorphism $c_{r,1} : \mathcal{E}_r^{\otimes r} \xrightarrow{\sim} \omega$ defined by

$$g\psi^r \mapsto \frac{gdx}{x^{r/2}y}.$$

This is an isomorphism because x is a global section of $\mathcal{O}(-2p + 2\infty)$ giving an isomorphism to $\mathcal{O}$, $\frac{dx}{y}$ is a global section of ω giving an isomorphism to $\mathcal{O}$, and 2 divides r.

An automorphism σ of order 4 of the underlying curve E_{1728} can be defined as $\sigma(x,y) = (-x, iy)$, and σ can be extended to an automorphism of the entire spin curve $(E_{1728}, (\mathcal{E}_r, c_{r,1}))$ by letting ζ be a primitive $4r$th root of unity such that $\zeta^r = (-1)^{r/2}i$ and then defining

$$\sigma(\psi) = \zeta\psi.$$

$c_{r,1}$ is preserved by σ since $\sigma(\frac{dx}{yx^{r/2}}) = (-1)^{r/2}i(\frac{dx}{yx^{r/2}}) = \zeta^r(\frac{dx}{yx^{r/2}})$.

$(\mathcal{E}_r, c_{r,1})$ is clearly of index (2), and although $\pi_*\mathcal{E}_r$, $R^1\pi_*\mathcal{E}_r$, and $\pi_!\mathcal{E}_r$ are all trivial (zero or $\mathcal{O}$), on $\mathfrak{S}_1^{1/r,(2)}$, the bundle $\mathcal{E}_{r/2} = \mathcal{E}_r$ is always isomorphic to $\mathcal{O}_E$, and so the sheaves $\mu^{2/r,+} := \pi_*\mathcal{E}_{r/2}$ and $\mu^{2/r,-} := R^1\pi_*\mathcal{E}_{r/2}$ are always line bundles on $\mathfrak{S}_1^{1/r,(2)}$ (actually they are just the pullbacks of μ^+ and μ^-, respectively, from $\mathfrak{S}_1^{2/r,(1)}$ via the morphism $[2] : \mathfrak{S}_1^{1/r} \to \mathfrak{S}_1^{2/r}$).

For every line bundle $\mathcal{L} \in \mathrm{Pic}\,\mathfrak{S}_1^{1/r,(2)}$, the geometric point $(E_{1728}, (\mathcal{E}_r, c_{r,1}))$ of $\mathfrak{S}_1^{1/r,(2)}$ associates a one-dimensional vector space $\mathcal{L}(E_{1728}, (\mathcal{E}_r, c_{r,1})) \cong k$, and σ induces an automorphism $\mathcal{L}(\sigma) \in \mathbb{G}_m$ of $\mathcal{L}(E_{1728}, (\mathcal{E}_r, c_{r,1}))$. Moreover, $\mathcal{L}(\sigma)$ has order dividing $4r$ (the order of σ). Thus we have a homomorphism

$$\mathrm{Pic}\,\mathfrak{S}_1^{1/r,(2)} \to \langle\zeta\rangle \cong \mathbb{Z}/4r.$$

Moreover, the explicit description of $\mathcal{E}_r$ shows that $\mu^{2/r,+}$ maps to ζ^2 and $\mu^{2/r,-}$ maps to ζ^{2-r}.

In the case of $d = 3$ we can use a similar argument, applied to the curve E_0 : $y^2 + y = x^3$, the line bundle $\mathcal{E}_r = \{f\psi | f \in k(E_0), (f) \geq p - \infty\}$ and the isomorphism

$$c_{r,1} : \psi^r \mapsto \frac{dy}{x \cdot y^{r/3}}.$$

Choose a primitive $3r$th root of unity ξ, and define the automorphism

$$\tau : (x, y) \mapsto (\xi^{-r} x, y)$$

$$\psi \mapsto \xi\psi.$$

This is compatible with $c_{r,1}$, and so is an automorphism of $(E_0, (\mathcal{E}_r, c_{r,1}))$. This gives a homomorphism $\text{Pic}\,\mathfrak{S}_1^{1/r,(3)} \to \langle \xi \rangle \cong \mathbb{Z}/3r$ and the elements $\mu^{3/r,+} := \pi_* \mathcal{E}_{r/3} (= [3]^* \mu^+)$ and $\mu^{3/r,-} := R^1 \pi_* \mathcal{E}_{r/3}$ map to ξ^3 and ξ^{3-r}, respectively. Thus we have the following commutative diagrams.

$$
\begin{array}{ccc}
\text{Pic}\,\mathfrak{S}_1^{3/r,(1)} & \xrightarrow{\sim} & \langle \mu^{3/r,+} \rangle = \mathbb{Z}/12r \\
\downarrow & & \downarrow \\
\text{Pic}\,\mathfrak{S}_1^{1/r,(3)} & \longrightarrow & \mathbb{Z}/3r
\end{array}
$$

and

$$
\begin{array}{ccc}
\text{Pic}\,\mathfrak{S}_1^{1/r,(1)} & \xrightarrow{\sim} & \langle \mu^{2/r,+} \rangle = \mathbb{Z}/12r \\
\downarrow & & \downarrow \\
\text{Pic}\,\mathfrak{S}_1^{1/r,(2)} & \longrightarrow & \mathbb{Z}/4r
\end{array}
$$

where the right-hand vertical morphisms take $\mu^{3/r,+}$ to 3 and $\mu^{2/r,+}$ to 2, respectively, and $\lambda = r\mu^{3/r,+}$ or $r\mu^{2/r,+}$, respectively.

For $d > 3$ this strategy does not work as well, since no automorphisms of the underlying curve preserve a spin structure of type d. Nevertheless, we still have for any $(E, (\mathcal{E}_r, c_{r,1}))$ the automorphism defined by taking $\mathcal{E}_r$ to $\eta\mathcal{E}_r$, where η is any rth root of unity. This gives, for η a primitive rth root of unity,

$$\text{Pic}\,\mathfrak{S}_1^{1/r,(d)} \to \langle \eta \rangle = \mathbb{Z}/r$$

and the bundle $\mu^{d/r,+} := \pi_* \mathcal{E}_{r/d}$ maps to η^d.

This inspires the following conjecture:

Conjecture 4.3.

$$
\text{Pic}\,\mathfrak{S}_1^{1/r,(d)} = \langle \mu^{d/r,+} \rangle =
\begin{cases}
\mathbb{Z}/2r & \text{if } d = 2 \\
\mathbb{Z}/r & \text{if } d = 3 \\
\mathbb{Z}/(r/d) & \text{if } d > 3.
\end{cases}
$$

4.3. **General genus, $r = 2$.** For $g > 2$, and $r = 2$, we have that $2\mu + \lambda$ is an element of order 4, and λ (and hence μ) is an element of infinite order. Moreover, Harer has proved for $g > 9$ that $H_1(\mathfrak{S}_g^{1/2,even}, \mathbb{Z}) = H_1(\mathfrak{S}_g^{1/2,odd}, \mathbb{Z}) = \mathbb{Z}/4$, and $H^2(\mathfrak{S}_g^{1/2,even}, \mathbb{Q}) = H^2(\mathfrak{S}_g^{1/2,odd}, \mathbb{Q}) = \mathbb{Q}$, so for $g > 9$ we have

$$\operatorname{Pic}\mathfrak{S}_g^{1/2,odd} = \operatorname{Pic}\mathfrak{S}_g^{1/2,even} \cong \mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}.$$

What is not yet completely clear, but seems reasonable to expect, is the following presentation for that group.

Conjecture 4.4. For all $g > 2$,

$$\operatorname{Pic}\mathfrak{S}_g^{1/2,even} = \operatorname{Pic}\mathfrak{S}_g^{1/2,odd} = \langle \mu, \lambda \,|\, 8\mu + 4\lambda = 0 \rangle.$$

Conclusion

We have worked out many relations between the elements of $\operatorname{Pic}\overline{\mathfrak{S}}_g^{1/r}$ and $\operatorname{Pic}\mathfrak{S}_g^{1/r}$. This generalizes the work of Cornalba [4, 5], whose results hold in the case where $r = 2$. One of the interesting consequences of these relations is the existence of elements in $\operatorname{Pic}\mathfrak{S}_g^{1/r}$ of elements of order 4 if 2 divides r, and elements of order 3, if 3 divides r. Somehow 2 and 3 seem to be special, however, and when r is relatively prime to 6 and $g > 2$, there do not appear to be any torsion elements in $\operatorname{Pic}\mathfrak{S}_g^{1/r}$.

Corresponding results for $\operatorname{Pic}\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$ are given in [14], where they are used to prove the genus-zero case of the generalized Witten conjecture [22].

Acknowledgments

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