

Geodesics on Riemann Surfaces with Ramification Points of Order Greater than Two

Mark Sheingorn

ABSTRACT. Most commonly, studies of geodesy on Riemann surfaces proceed on those surfaces without ramification. While it is true that every surface of finite type (stemming from a finitely generated Fuchsian group) has a finite-sheeted cover which is free of ramification, local geometric information is lost in this process.

We seek to analyze geodesics on ramified surfaces directly. After briefly reviewing (the by now well-understood) situation of ramification points of order 2, we turn to higher ramification. Examples are offered on the surface stemming from the squares of elements of the full modular group $\Gamma^2 \backslash \mathcal{H}$ of signature $(0; 3, 3, \infty)$. Already here we can see that a fundamental property of negatively curved surfaces fails: there are closed curves with apparently non-trivial homotopy, yet having no geodesic in their homotopy class. Next, we construct surfaces on which length L and number of self-intersections N of a closed bounded height geodesic are closely linked: There are constants a and b depending only the surface and said height such that $aL(\tau) < bN(\tau) < L(\tau)$. On such surfaces, long simple geodesic arcs cannot exist.

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1. Geodesics near efp2's

The geometry of geodesics near efp2's is well understood by reason of its relative tractability; its study is motivated by its centrality to the hyperbolic geometry of the Markoff spectrum near 3. (See, for example, [S]). Thus we just give a brief summary. A geodesic which hits an elliptic fixed point of order two (denoted hereafter *efp2*) simply bounces off it, returning along the same route it came in on, in the opposite direction. A geodesic passing near an efp2 swings around it, returning in nearly the opposite direction along a path that diverges from the incoming path. This is illustrated by the action of $T : z \mapsto -1/z$ on the vertical geodesic through $-\epsilon$. This is mapped to the one through 0 and $1/\epsilon$, with the portion below $y = 1$ on the incoming being mapped to the portion above $y = 1$ on the outgoing. In certain situations, notably $\Gamma^3 \setminus \mathcal{H}$ (see [SS]), this 'bouncing-without-local-intersection' accounts for all simple geodesics on the surface. (Here Γ^3 is generated by the cubes of the full modular group and has signature $(0; 2, 2, 2, \infty)$.)

(Notable previous work in a related area was undertaken by C. Judge, alone and with co-author E. Gutkin. In [J], the focus is on irrational rotations and so we leave the domain of Riemann surfaces. In [GJ], the *translation surfaces* are flat, apart from the elliptic points, so again we not on a Riemann surface. The paper of D. Gabai and W. Kazez ([GK]), deals with ramification no higher than two. Geodesy of individual curves is not the central concern of these studies, however.)

We now turn to a simple situation with efp3's on $\Gamma^2 \setminus \mathcal{H}$. (Here *efpn* denotes a ramification point of order n , while Γ^2 denotes the group generated by the squares in the Modular group $\Gamma(1)$.) All the characteristics elucidated in this paper are already present in this simple example

2. Geodesics near efp3's on $\Gamma^2 \setminus \mathcal{H}$

2.1. Basic facts about Γ^2 . The group Γ^2 has signature $(0; 3, 3; \infty)$. Γ^2 is of index two in $\Gamma(1)$ while $\Gamma(2)$, a normal subgroup, is index three in Γ^2 . Further, $\Gamma^2 = \langle P_1, P_2 \rangle$, where

$$P_1 =: \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad P_2 =: \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

As to fundamental region, in Figure 1 we offer two. If we let \mathcal{G} be the standard $\Gamma(1)$ fundamental region. Then two fundamental regions for Γ^2 are $\mathcal{G} \cup T(\mathcal{G})$ and $\mathcal{G} \cup S(\mathcal{G})$, where $S : z \mapsto z + 1$ and T (defined above) are the usual pair of $\Gamma(1)$ generators fixing ∞ and i respectively. Call these \mathcal{F}_T and \mathcal{F}_S , again respectively (see Figure 1). Note that in either region, the inequivalent efp3's are $-\bar{\rho}$ and ρ where as usual $\rho =: \frac{1+i\sqrt{3}}{2}$. Last, the facts that P_1 fixes $-\bar{\rho} = \rho - 1$, and P_2 fixes ρ , whilst $P_2^{-1}P_1 : z \mapsto z + 2$, and $P_2^{-1}(-\bar{\rho}) = \rho + 1$ give the side pairings of the two fundamental regions. The action of T (or equivalently S) constitutes a conformal isometry of $\Gamma^2 \setminus \mathcal{H}$. Also $z \mapsto -\bar{z}$ is an anti-conformal isometry of $\Gamma^2 \setminus \mathcal{H}$. (Equivalently, $\begin{pmatrix} \cdot & b \\ c & \cdot \end{pmatrix} \in \Gamma^2 \iff \begin{pmatrix} \cdot & -b \\ -c & \cdot \end{pmatrix} \in \Gamma^2$.) See R. Rankin [R], for further details.

By virtue of the Jordan curve theorem, there are no simple closed geodesics on $\Gamma^2 \setminus \mathcal{H}$. (Likewise there are none on $\Gamma(2) \setminus \mathcal{H}$ which has signature $(0; \infty, \infty)$ or on

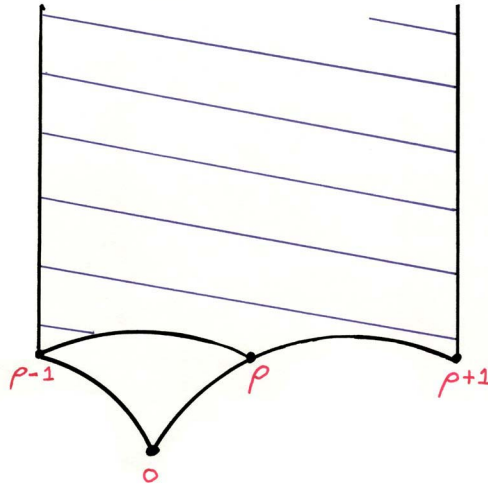


FIGURE 1. $T = \frac{z+1}{-z}$, $S = z + 2$, $R = \frac{z-1}{z}$, $T = S^{-1}R$, $\rho = \frac{1+i\sqrt{3}}{2}$.

$\Gamma(1)\backslash\mathcal{H}$, for that matter.) This is because such a geodesic would have to bound a disc containing precisely one of the special points and this is impossible.

2.2. A special geodesic (the unit circle) and some of its perturbations.

Call the unit circle v and examine it in \mathcal{F}_T . It is simple to see (since the P_i 's are rotations through $2\pi/3$) that v is equivalent (in \mathcal{F}_T) to the path that emanates from ∞ , goes down the left side to $-\bar{\rho}$, over to ρ (along v , of course) and then up to ∞ . Clearly the extent of any of the three h-lines involved project onto the same geodesic on $\Gamma^2\backslash\mathcal{H}$.

Let us perturb the left-hand vertical as follows: consider the action of $P_1^2 = P_1^{-1}$ on the h-line (f, ∞) , where $f = -1/2 + \epsilon$, $\epsilon > 0$. This maps to $(-1, 1 + 4\epsilon)$ to the first order in ϵ , with all remaining terms in the ϵ expansion having positive coefficients. (That this is near v illustrates the rigidity of the situation near the points ρ and $\rho - 1$). The shaded area in Figure 2 gives a loop about $\rho - 1$ formed by the perturbed geodesic. This we shall call a 3-loop and its interior (the component containing the efp3) a 3-disc. As we remarked above, no such structure exists about an efp2.

Next, the same phenomenon repeats over the point ρ . The action of P_2 maps the geodesic over v to $(-1/4\epsilon, 1/2)$, again forming a 3-loop, this time about ρ . This continues in \mathcal{F}_T until a 'high' intersection with the original perturbation.

Note that if a geodesic passes through an efp3 (as with v), then it is bounced off at an angle of $\pi/3$. This is a self intersection, (denoted si); it lifts to two different h-lines through ρ or $\rho - 1$, and also it is the homotopic limit of 3-loops all of which have an si at a regular point of \mathcal{F}_T , near the efp3.

Returning to the perturbation, limiting cases are $\epsilon = 0$ which gives v and $\epsilon = 1/2$ which gives a 3-loop with its node at ∞ that comprises half of $\Gamma^2\backslash\mathcal{H}$ (the limit of region shaded in Figure 3, where here and in what follows π projects \mathcal{H} to the surface). This is the largest 3-loop as beyond this we are coming down the back

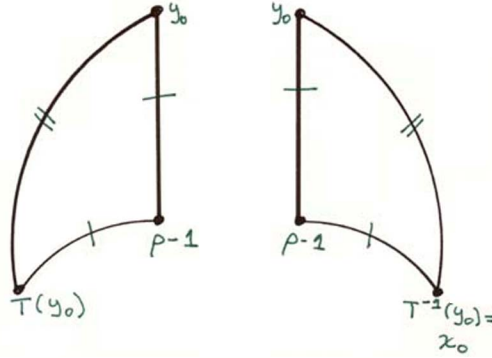


FIGURE 4

symmetric about the line connecting the node to the efp3. Here we write simply P for P_1^2 of the previous section, it rotates counter-clockwise about $\rho - 1$ through $\pi/3$ radians. The point x_0 is at our disposal in this picture and we can choose it anywhere between the extremes y_0 and $P^{-1}(y_0)$, as illustrated in Figure 4. The triangles are congruent and so $\angle b = \angle c$. Also, now we may choose x_0 so that the angle there is $\pi/2$. The congruence of the two resulting triangles now shows that the reflection through $(y_0, \rho - 1, x_0)$, which has lifts on a single h-line in \mathcal{H} , fixes the 3-disc. Next, note that no use was made of the fact that $(y_0, P^{-1}(y_0))$ is vertical, so this symmetry exists for any 3-loop.

2.4. Absence of long low height simple geodesic arcs on $\Gamma^2 \setminus \mathcal{H}$. We work in \mathcal{F}_S . The presence of the hypothesis ‘low height’ is due to the fact that vertical and nearly vertical arcs within \mathcal{F}_S are obviously simple and may be arbitrarily long.

To begin with, the geometric underpinning of this absence is that to continue on without closing, low height geodesics are forced to swing about an efp3 after a length depending only on $\Gamma^2 \setminus \mathcal{H}$ and the height, but not on the geodesic. But that turning path creates an si.

Consider the lift of a geodesic in \mathcal{F}_S that has feet at the ends of the interval $[a, b]$. WLOG we may assume that $a \in [-1/2, 1/2]$, because $z \mapsto z + 1$ is a global isometry of $\Gamma^2 \setminus \mathcal{H}$. Clearly the arc emanating from a hits a side of \mathcal{F}_S paired by P , since $a < b$. Let us call these sides σ_i and σ_{i+1} , for the efp2’s embedded in them.

Let us begin with the assumption that upon leaving a the arc hits σ_i at $\alpha + i\sqrt{1 - \alpha^2}$, and then σ_{i+1} in the point $\beta + i\sqrt{2\beta - \beta^2}$. This forces $a < 0$. If $\beta \leq 1$, then it is easy to see that $\alpha < \beta$. Applying P to the arc $(a, \alpha + i\sqrt{1 - \alpha^2})$ gives us an arc leaving σ_{i+1} between ρ and $\beta + i\sqrt{2\beta - \beta^2}$ and ending at $1 - 1/a$ which is greater than 2. This forces an si with the original arc, and we see the resulting 3-loop in \mathcal{F}_S . Now this argument works just as well if $\beta \geq 1$ as long as $\alpha \geq 0$. But if $-1/2 \leq \alpha < 0$ then the radius of such an arc is at least $5/4$, and the diameter is at least $5/2$, greater than 2 by a definite amount. This forces a subsequent si at height at least $3/4$, and so the simple portion of the arc is of bounded length.

The remaining case is that upon leaving a , we hit σ_{i+1} first. This means that $0 < a < 1/2$. Consider the action of P^{-1} on the arc $(a, \beta + i\sqrt{2\beta - \beta^2})$. This

connects a point on σ_i with a point lying in $[1, 2]$, and that gives us an si after a bounded length, just as in the previous paragraph. We have established:

Proposition 1. *For each height h there is an effective Γ^2 constant C_h so that any geodesic arc on $\Gamma^2 \setminus \mathcal{H}$ of length at least C_h has a self-intersection.*

The following Proposition provides a lower bound:

Proposition 2. *There is a constant δ depending only on Γ^2 and h such that a $\Gamma^2 \setminus \mathcal{H}$ geodesic segment including any pair of self-intersections with at least one of the points of intersection occurring at a point of height less than h , has length at least δ .*

Proof. Begin at an arbitrary point p on the geodesic and proceed until arriving at the first si , call it s_1 . We have created a loop on $\Gamma^2 \setminus \mathcal{H}$ which divides the surface into two discs, one of which contains exactly one of the special points. Now continue from s_1 to the next intersection, s_2 . It is the length of the arcs between s_1 and s_2 that concerns us.

There are two cases: First, assume that the path from s_1 to s_2 itself contains another loop about another special point. In this case consider the path from s_1 about its loop and then on through s_2 , about its loop, finally returning to s_2 . This has a definite length because the $efp3$'s are at some remove from each other and the cusp — for example, the geodesic connecting ρ and $\rho + 1$ is shorter than the path between s_1 and s_2 containing the two aforementioned loops.

The other possibility is that s_2 lies on the path from p to s_1 . In this case, we have either wrapped around s_1 's special point a second time, or formed a bigon containing another special point. The latter case reduces to the one above, *mutatis mutandis*. In the former, if the twice wrapped special point is the cusp, then the height bound h forces the arc in question to have a definite length. If it is an $efp3$, then we are done unless the second loop is short. But then the original loop is even shorter as it lies wholly surrounded. This then forces the original 3-loop to close near its $efp3$. And that means the angle at s_2 is near $\pi/3$. This forces the first containing loop to be large. \square

The two propositions combine to prove:

Theorem 1. *Let $L(\tau)$ give the length of a geodesic arc τ and $N(\tau)$ the number of si 's created on τ as we traverse it. (There may be further si 's on τ created 'later'.) Then if τ has height less than h , there are constants a and b depending only $\Gamma^2 \setminus \mathcal{H}$ and h such that $aL(\tau) < bN(\tau) < L(\tau)$.*

2.5. The paths of short $\Gamma^2 \setminus \mathcal{H}$ simple closed geodesics on $\Gamma^3 \setminus \mathcal{H}$. Let scg denote simple closed geodesic. This section illustrates the point that quite uncomplicated geodesic paths on a surface may be very involved on a closely commensurate surface. In [SS], the author together with T. Schmidt analyzed pairs of $scgs$ and $pssi$ (proper single self-intersection) geodesics of D. Crisp and W. Moran ([CM]). As we described above, each scg connects to two $efp2$'s on $\Gamma^3 \setminus \mathcal{H}$. Associated with each scg , there is a unique closed geodesic connecting the remaining $efp2$ with itself and looping around the cusp. The only si is at the $efp2$, thus a $pssi$ is formed. All [CM] $pssi$'s are of this kind.

On $\Gamma^3 \setminus \mathcal{H}$ (up to isometry), the shortest scg is the h -line connecting i with $i + 1$. Its $pssi$ connects $i + 2$ with $i - 1$. Figure 5a (based on routine computations), shows

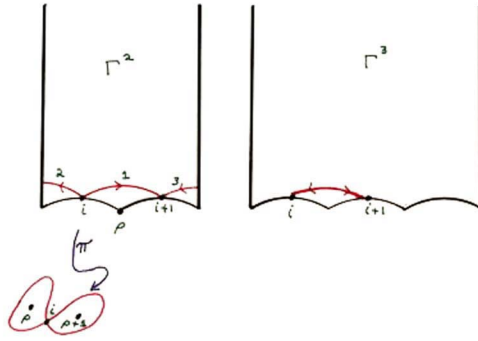


FIGURE 5A. $(i, i + 1)$ on Γ^2 and Γ^3 .

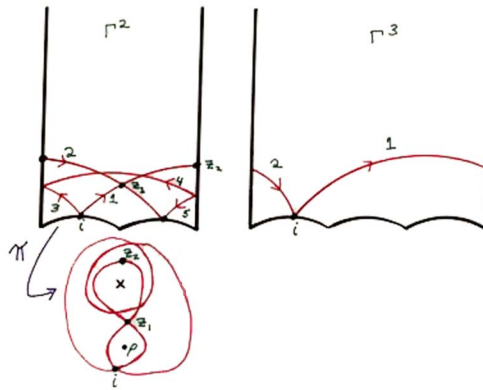


FIGURE 5B. $(i, i + 3)$ on Γ^2 and Γ^3 .

the paths of these scgs on $\Gamma^3 \setminus \mathcal{H}$, $\Gamma^2 \setminus \mathcal{H}$, and their fundamental regions. (Here and in what follows the numbers over the arcs gives the sequence in which they are traversed. Also, a double-headed arc ‘bounces’ back and forth between its (efp2) endpoints.)

Figure 5b, the h-line connecting i with $i + 3$, illustrates the contention of the abstract that (even) an uncomplicated closed geodesic on one surface may be very involved on another closely commensurable surface. The former, featuring a 3-loop heading off into a second 3-loop, the two comprising a figure 8 on $\Gamma^2 \setminus \mathcal{H}$, suggested the following curio¹ :

Proposition 3. *On $\Gamma^2 \setminus \mathcal{H}$ there is no geodesic in the homotopy class of a figure 8 with a cusp in one head and an efp3 in the other.*

Proof. First of all, it is not difficult to see that such a geodesic must consist of two arcs in \mathcal{F}_S , one connecting σ_i with $\sigma_{\rho+1}$ and the other connecting σ_{i+1} with $\sigma_{\rho-1}$.

¹The words “figure 8” in this sentence refer to a geodesic, not a figure (illustration) in this paper, which has only seven. “Figure” will be capitalized in references to illustrations.

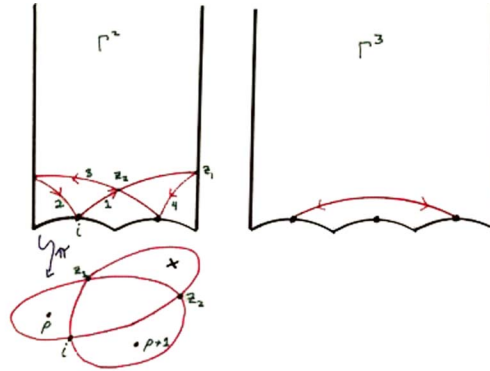


FIGURE 5C. $(i, i + 2)$ on Γ^2 and Γ^3 .

This because winding about either $\pi(\rho)$ or $\pi(\infty)$ causes multiple σ 's. Say we leave σ_i at the point $x + i\sqrt{1 - x^2}$ and arrive at $3/2 + iy$. We continue to $-1/2 + iy$ and then to $(1 - x) + i\sqrt{1 - x^2}$ on σ_{i+1} . Here of course $-1/2 \leq x \leq 1/2$ and $y \geq \sqrt{3}/2$. Now this figure 8 must be a closed geodesic. Thus (extensions of) the two arcs must be paired by S^2 and also P_2 . The S^2 condition forces $y^2 = 11/4 - 3x$ (and the bound on y gives the additional condition $5/12 \geq x$, but this plays no role). Taking this into account and computing the P_2 condition forces (using MATHEMATICA until the end of this proof)

$$-2905 + 8406x - 9288x^2 + 4608x^3 - 864x^4 = 0$$

which has no real roots at all. □

Here is another way to think about this. It isn't difficult to draw a curve in the homotopy class of the figure 8 containing the cusp in one head and ρ in the other. For example, consider the h-arc connecting $-1/2 + 50i$ with $1 + i$, and then the h-arc connecting $i = P_2(1 + i)$ with $3/2 + 50i = S^2(-1/2 + 50i)$. If this could be pulled tight to a geodesic, the (hyperbolic!) word for it would be $P_2S^{-2} = \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}$. But this is just the product of the $\Gamma(1)$ elliptics of order two fixing $1 + i$ and then $2 + i$. But we have seen such a product is not a figure 8 on $\Gamma^2 \backslash \mathcal{H}$ of the above description.

Figure 5c illustrates the paths of the h-line connecting i with $i+2$, an intermediate case.

3. $q > 3$ on the Hecke triangle surfaces

The Hecke triangle groups G_q , $q \geq 3$, are defined as $G_q = \langle S, T \rangle$ where

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } S = \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix} \text{ with } \lambda_q = 2 \cos \pi/q.$$

Each G_q is a Fuchsian group of the first kind of signature $(0; 2, q, \infty)$.

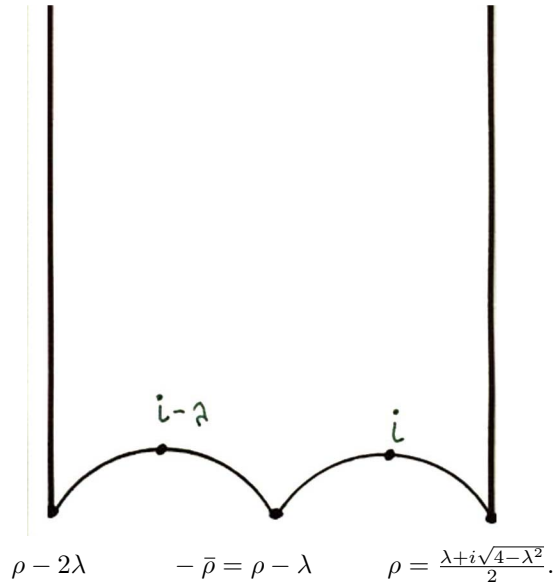


FIGURE 6. $G_q^2 = \langle E, S^2 \rangle = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2\lambda \\ 0 & 1 \end{pmatrix} \right\rangle$, $\lambda = 2 \cos \pi/q$

The analog of Γ^2 is G_q^2 , generated by S^2 and $E := \begin{pmatrix} 0 & -1 \\ 1 & \lambda_q \end{pmatrix}$. A simple application of Poincaré’s theorem shows this group is Fuchsian of signature $(0; q, q, \infty)$. Note that $E^{-1}S^2$ fixes $\rho - 2\lambda_q$ where $\rho := \frac{\lambda_q + i\sqrt{4 - \lambda_q^2}}{2}$. A fundamental region (call in \mathcal{F}_q) for G_q^2 is given in Figure 6.

Let us first consider the path in \mathcal{F}_q of the vertical line through $-\lambda_q/2$. Note that E fixes $\rho - \lambda_q$ which lies on this h-line. It is not hard to compute that $E^{[q/2]+1}(-\lambda_q/2) = \infty$ or 1 according as q is even or odd. In particular, if q is odd, the vertical from $\rho - \lambda_q$ to ∞ (when extended below $\rho - \lambda_q$) has an si at $\rho - \lambda_q$ (we saw this when $q = 3$ previously) but ‘bounces back’ to ∞ (when extended below $\rho - \lambda_q$) when q is even. In that case this arc is simple.

Now let us perturb the vertical to be based at $-\lambda_q/2 - \epsilon$. Obviously $E(\infty) = 0$ and $E(-\lambda_q/2) = -2/\lambda_q$. Since E rotates clockwise through $\rho - \lambda_q$, we have that $E(-\lambda_q/2 - \epsilon) = -2/(\lambda_q - 2\epsilon) < -2/\lambda_q$ as long as $\epsilon > 0$. Further, $-\lambda_q - 1 < -2/(\lambda_q - 2\epsilon)$ provided that $\epsilon < \frac{(\lambda_q + 2)(\lambda_q - 1)}{2(\lambda_q + 1)}$.

Now if $\lambda_q = 1$ (e.g., $q = 3$), this is impossible. But if $q > 3$ and ϵ is small enough, it is possible, and we have the E -image of the perturbed base $-\lambda_q/2 - \epsilon$ lying between $-\lambda_q - 1$ (the left foot of a fundamental region side) and $-2/\lambda_q$, the left foot of the unperturbed base image (which contains $\rho - \lambda_q$, of course). Regardless of the parity of q , this forces a q -loop about $\rho - \lambda_q$. It is not hard to compute that, for larger q , we may wind about the efpq for some spins before creating our q -loop. I.e., the geodesic arcs in \mathcal{F}_q contain a non-intersecting consecutive pair each requiring an application of E to remain in \mathcal{F}_q . This is similar to the more familiar ‘barber pole’ circumstance at cusps. That is, if we consider the h-line between 0 and $2M$

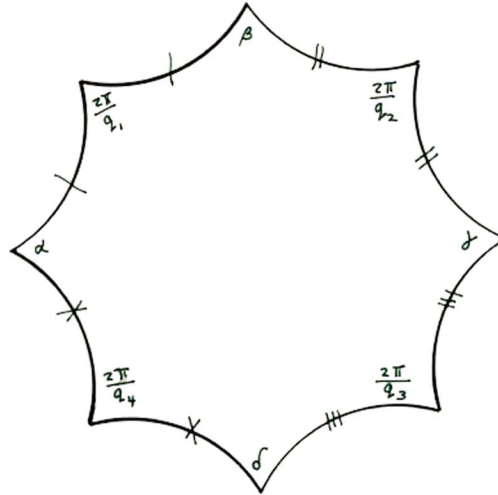


FIGURE 7. $\alpha + \beta + \gamma + \delta = 2\pi$
 $(0; q_1, \dots, q_4, \infty)$.
 \uparrow
 $?$

as it rises past $y = 2$, we see wrapping while rising in the cusp funnel to $\Re z = M$, after which we get wrapping while falling, with the production of numerous si's as we intersect the rising segment repeatedly.

None of the perturbation geometry used that fact that ∞ was a cusp, or that the vertical $(\rho - \lambda_q, \infty)$ lay in a fundamental region. These q -loops are locally present wherever we encounter higher ramification. We summarize this as follows:

Proposition 4. *Let $q > 2$. A geodesic passing near an $efpq$ makes a q -loop about it. This q -loop may have some winding about the $efpq$ before closing, if the passage is near enough. If the geodesic hits the $efpq$, then it bounces back on itself if q is even and bounces off at angle π/q if q is odd.*

4. Additional sphere groups

The author is indebted to B. Maskit for suggestions concerning the construction of these Fuchsian groups. Consider the fundamental region (due to Poincaré's theorem) pictured in Figure 7. There are two cases according as the cycle is accidental (i.e., comprised of ordinary points) or a cusp cycle (all vertices are real or ∞). If the cycle is cusp, then there is no condition on the angles (which are all zero anyway!) and the group's signature is $(0; q_1, \dots, q_n; \infty)$. If the cycle is accidental, then we must have the cycle angle sum is 2π and the signature is $(0; q_1, \dots, q_n)$. The genus is zero in each case, accounting for the appellation.

Here are some facts about these curious surfaces. The proofs mimic those of the previous sections; there are no new issues involved.

- If so many as two of the q 's are even, then an scg exists — the one that bounces between them. This h-line is a hyperbolic axis projecting to an scg consisting of a single arc within this fundamental region. The word for the associated hyperbolic is the product of the two elliptics that fix these endpoints.

- If all but at most one of the q 's is odd, then scgs do not exist as using the $\text{efp}q$'s to turn forces s 's. (We did this computation explicitly above when $\Gamma = G_q^2$. As we remarked there, no use was made of the fact that ∞ was a cusp, etc. Thus in this case it is easy to draw simple homotopy classes that cannot be 'pulled tight'.)
- Γ^3 is a sphere group with 8 edges and 7 vertices (3 $\text{efp}2$'s, 1 cusp, 3 in accidental cycle) but Γ^2 is of a different sort as we haven't the interleaving of $\text{cusp}/\text{efp}3$'s with vertices in the accidental cycle. At this writing we cannot attribute any significance to this circumstance.

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CUNY - BARUCH COLLEGE, NEW YORK, NY 10010

marksh@alum.dartmouth.org <http://www.panix.com/~marksh/>

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