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Pseudocharacters on a Class of Extensions of Free Groups

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Dedicated to Caroline and Natalie

Abstract. A *pseudocharacter* of a semigroup S is a real function φ on S satisfying the following conditions.

1) The set $\{\varphi(xy) - \varphi(x) - \varphi(y);\ \ x, y \in S\}$ is bounded.
2) For $x \in S$ and $n \in \mathbf{N}$ (and $n \in \mathbf{Z}$ if S is a group),

$$\varphi(x^n) = n\varphi(x).$$

A description of the space of pseudocharacters on some extensions of free groups is given.

Contents

1. Introduction

In 1940 S. M. Ulam posed the following problem. Given a group G_1, a metric group (G_2, d) and a positive number ε, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $T : G_1 \to G_2$ exists with $d(f(x), T(x)) < \varepsilon$ for all $x, y \in G_1$? See S. M. Ulam (1960) or (1974) for a discussion of such problems, as well as D. H. Hyers (1941, 1983), D. H. Hyers and S. M. Ulam (1945, 1947), Th. M. Rassias (1978), J. Aczèl and J. Dhombres (1989).

In case of a positive answer to the previous problem, we say that the homomorphisms $G_1 \to C_2$ are *stable* or that the Cauchy functional equation

$$\varphi(xy) = \varphi(x)\varphi(y) \tag{1}$$

is *stable*.

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The first affirmative answer was given by D. H. Hyers [14] in 1941. Consider the additive Cauchy equation

$$\varphi(xy) = \varphi(x) + \varphi(y). \tag{2}$$

Obviously this equation is exactly the same as equation (1), but with the use of the additive notation on the right-hand side we emphasize that the range of the function is in an additive group.

Theorem 1 (D. H. Hyers). *Let E_1, E_2 be Banach spaces and let $f : E_1 \to E_2$ satisfy the following condition: there is $\varepsilon > 0$ such that*

$$||\, f(x+y) - f(x) - f(y)\,|| < \varepsilon \quad \text{for all} \quad x, y \in E_1. \tag{3}$$

Then there exists $T : E_1 \to E_2$ such that

$$T(x+y) - T(x) - T(y) = 0 \quad \text{for all} \quad x, y \in E_1 \tag{4}$$

and

$$||\, f(x) - T(x)\,|| < \varepsilon \quad \text{for all} \quad x \in E_1. \tag{5}$$

If we carefully look at the proof of Hyers' Theorem, the existence of the additive function T uniformly approximating f, we easily recognize that the result remains true if we replace the additive group of the Banach space E_1 by a commutative semigroup S. So we can conclude that the homomorphisms from an abelian simigroup into the additive group of a Banach space are stable.

After Hyers' result a great number of papers on the subject have been published, generalizing Ulam's problem and Hyers' Theorem in various directions. See [11], [15]–[22], [26]–[28].

Definition 1. Let G be a semigroup and B a Banach space. We say that equation (2) is stable for the pair (G, B) if, for every function $f : G \to B$ such that

$$||f(xy) - f(x) - f(y)|| \leq \delta, \quad x, y \in G \quad \text{for some} \quad \delta \geq 0,$$

there exists a solution φ of (2) such that

$$||f(x) - \varphi(x)|| \leq \varepsilon, \quad \forall x \in G$$

for some ε depending only on δ.

In [12] it has been proved that B_1, B_2 are Banach spaces, then (2) is stable for (G, B_1) if and only if it is stable for (G, B_2).

Due to this remark we simply say that (2) is stable for a group or a semigroup G. Thus Hyers's Theorem says that (2) is stable for any commutative semigroup G. A remarkable achievement was that of L. Székelyhidi who in [28] replaced the original proof given by Hyers by a new one based on the use of invariant means.

Theorem 2 (L. Székelyhidi). *Let G be a left (right) amenable semigroup, then (2) is stable for G.*

Now a question naturally arises: do groups or semigroups exist for which equation (2) is not stable? In view of L. Székelyhidi's Theorem we must look among non-amenable groups or semigroups and in fact in [4, 5, 7, 8, 10] it was proved that on a free nonabelian group (or semigroup) the additive Cauchy equation (2) is not stable. We recall the example of Forti (see [10]). Let $F(\alpha, \beta)$ be the free group generated by the two elements α, β. Let each word $x \in F(\alpha, \beta)$ be written in reduced form, i.e., x does not contain pairs of the forms $\alpha\alpha^{-1}$, $\alpha^{-1}\alpha$, $\beta\beta^{-1}$, $\beta^{-1}\beta$ and has no exponents different from 1 and -1. Define the function $f : F(\alpha, \beta) \to \mathbf{R}$ as follows. If $r(x)$ is the number of pairs of the form $\alpha\beta$ in x and $s(x)$ is the number of pairs of the form $\beta^{-1}\alpha^{-1}$ in x, put $f(x) = r(x) - s(x)$.

It is easily seen that for all $x, y \in F(\alpha, \beta)$ we have $f(xy) - f(x) - f(y) \in \{-1, 0, 1\}$, i.e., f satisfies (3). Now, assume that there is $T : F(\alpha, \beta) \to \mathbf{R}$ such that the relations (4), and (5) hold. But T is completely determined by its values $T(\alpha)$ and $T(\beta)$, while f is identically zero on the subgroups A and B generated by α and β, respectively. Indeed, $T(\alpha^n) = nT(\alpha)$ and $f(\alpha^n) = 0$ for $n \in \mathbf{N}$. Since $T(\alpha^n) - f(\alpha^n) = nT(\alpha)$ for $n \in \mathbf{N}$, it follows that $T(\alpha) = 0$. Similarly we have $T(\beta) = 0$, so that T is identically zero on $F(\alpha, \beta)$. Hence, $f - T = f$ on $F(\alpha, \beta)$ where f is unbounded. This contradiction proves that there is no homomorphism $T : F(\alpha, \beta) \to \mathbf{R}$ such that the relation (5) holds.

It turns out that the existence of mappings that are "almost homomorphisms" but are not small perturbations of homomorphisms has an algebraic nature.

Definition 2. A *quasicharacter* of a semigroup S is a real-valued function f on S such that $\{f(xy) - f(x) - f(y) \,|\, x, y \in S\}$ is bounded.

Definition 3. By a *pseudocharacter* on a semigroup S (group S) we mean a quasicharacter f such that for $x \in S$ and $n \in \mathbf{N}$ (and for $n \in \mathbf{Z}$, if S is group),

$$f(x^n) = nf(x).$$

The set of quasicharacters of semigroup S is a vector space (with respect to the usual operations of addition of functions and multiplication by scalars), which will be denoted by $KX(S)$. The subspace of $KX(S)$ consisting of pseudocharacters will be denoted by $PX(S)$ and the subspace consisting of real additive characters of the semigroup S, will be denoted by $X(S)$.

We say that a pseudocharacter φ of the group G is *nontrivial* if $\varphi \notin X(G)$.

In connection with the example of Forti, note that his function is a quasicharacter of the free group $F(\alpha, \beta)$ but not a pseudocharacter of $F(\alpha, \beta)$. In [5, 7] the set of all pseudocharacters of free groups was described. In [4]–[9] a description of the spaces of pseudocharacters on free groups and semigroups, semidirect and free products of semigroups was given.

For a mapping f of the group G into the semigroup of linear transformations of a vector space, sufficient conditions for the coincidence of the solution of the functional inequality $||f(xy) - f(x) \cdot f(y)|| < c$ with the solution of the corresponding functional equation $f(xy) - f(x) \cdot f(y) = 0$ were studied in the papers [2, 3]. In the papers [13, 24], it was independently shown that if a continuous mapping f of a compact group G into the algebra of endomorphisms of a Banach space satisfies the relation $\|f(xy) - f(x) \cdot f(y)\| \le \delta$ for all $x, y \in G$ with a sufficiently small $\delta > 0$, then f is ε-close to a continuous representation g of the same group in the same Banach space (i.e., we have $\|f(x) - g(x)\| < \varepsilon$ for all $x \in G$).

Let H be a Hilbert space and let $U(H)$ be the group of unitary operators of H endowed by operator-norm topology. If H is n-dimensional, $n \in \mathbf{N}$, we denote $U(H)$ by $U(n)$.

Definition 4. Let $0 < \varepsilon < 2$. Let T be a mapping of a group G into $U(H)$. We say that T is an *ε-representation* if for any $x, y \in G$ the relation

$$\|T(xy) - T(x)T(y)\| < \varepsilon$$

holds.

V. Milman raised the following question: Let $\rho : G \to U(H)$ be an ε-representation with small ε. Is it true that ρ is near to an actual representation π of the group G in H, i.e., does there exist some small $\delta > 0$ such that $\|\rho(x) - \pi(x)\| < \delta$ for all $x \in G$? Answering this question Kazhdan in [24] obtained the following:

Theorem 3 (D. Kazhdan)**.** *There is a group Γ with the following property. For any $0 < \varepsilon < 1$ and any natural number $n > \frac{3}{\varepsilon}$ there exists an ε-representation ρ such that for any homomorphism $\pi : G \to U(n)$ the relation*

$$\|\rho - \pi\| = \sup\{\|\rho(x) - \pi(x)\| \,;\, x \in \Gamma\} > \frac{1}{10}$$

holds.

Note that the group Γ has the following presentation in terms of generators and relations: $\Gamma = \langle x, y, a, b \,\|\, x^{-1}y^{-1}xya^{-1}b^{-1}ab\rangle$.

By using pseudocharacters a strengthening of Kazhdan's Theorem was established in [9] as follows.

We say that a group G belongs to the class $\mathcal{K}$ if every nonunit quotient group of G has an element of order two.

Theorem 4 (V. Faĭziev)**.** *Let H be a Hilbert space and let $U(H)$ be its group of unitary operators. Suppose the groups A and B belong to the class $\mathcal{K}$ and the order of B is more than two. Then the free product $G = A * B$ has the following property. For any $\varepsilon > 0$ there exists a mapping $T : G \to U(H)$ satisfying the following conditions.*

1) $\|T(xy) - T(x) \cdot T(y)\| \le \varepsilon, \quad$ *for all* $\ x, y \in G$.
2) *For any representation $\pi : G \to U(H)$, we have*

$$\sup\{\|T(x) - \pi(x)\| \,;\, x \in G\} = 2.$$

There is the following connection of quasicharacters and pseudocharacters with the theory of Banach algebra cohomology: The definition of a quasicharacter coincides with that of a bounded 2-cocycle on the semigroup. Hence, if a semigroup S has a nontrivial pseudocharacter, i.e., $PX(S) \setminus X(S) \ne \emptyset$, then arguing as [23], Proposition 2.8, we obtain $H^2(S, \mathbf{C}) \ne 0$.

The aim of this paper is to establish an existence of nontrivial pseudocharacters on some classes of extensions of free groups and to describe the set of pseudocharacters on some groups.

2. Pseudocharacters on some extensions of free groups

Let G be a group and let α be an automorphism of G. For any $\varphi \in PX(G)$ we set $\varphi^\alpha(x) = \varphi(x^\alpha)\ \ \forall x \in G$. It is clear that φ^α is a pseudocharacter of the group G.

Definition 5. We shall say that φ is *invariant* under α if $\varphi^\alpha = \varphi$. If this relation is true for each a from $H \subseteq Aut\, G$, we shall say that φ is invariant under H.

Denote by $PX(G, H)$ the subspace of $PX(G)$ consisting of a pseudocharacters of the group G invariant under H.

In this article by F we mean a free group with a set of free generators X such that $|X| \geq 2$.

Recall that a word $v = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_n}^{\varepsilon_n}$ ($\varepsilon_i \in \{1, -1\}$) is reduced if $x_{i_k}^{\varepsilon_k} \neq x_{i_{k+1}}^{-\varepsilon_{k+1}}$ for $k = 1, 2, \ldots, n-1$.

By the length of v we mean the number n which we denote by $|v|$. If v is reduced and the relation $x_{i_1}^{\varepsilon_1} \neq x_{i_n}^{-\varepsilon_n}$ holds we say that v is cyclically reduced. Denote by $\sigma(X)$ the set of reduced words. And denote by $c(X)$ the set of cyclically reduced words. For a word v denote by $\sigma(v)$ an element from $\sigma(X)$ such that $v = \sigma(v)$. Let $\sigma(v) = u^{-1} z u$, where $z \in c(X)$. We set $c(v) = z$.

Each pseudocharacter of any group is invariant under its inner automorphisms (see [7] Lemma 15). Therefore if A is an automorphism group of the group F, then $PX(F, A) = PX(F, A \cdot Inn\, F)$. Hence below without loss of generality, we can assume that $Inn\, F \subseteq A$.

Denote by $\overline{a}$ the image of the element $a \in A$ under the natural epimorphism $A \to A/Inn\, F$ and denote by $\overline{A}$ the image of the group A under the same epimorphism.

Definition 6. Two elements u, v from F are called *A-conjugate* if there is $a \in A$ such that elements u^a and v are conjugate in F.

We denote the relation of A-conjugacy by $\sim_A$. It is clear that $\sim_A$ is an equivalence relation.

Definition 7. An element u of F is called *simple* if for each $v \in F$ and each $n \geq 2$ the relation $u \neq v^n$ holds.

Denote by $\mathcal{P}$ the set of simple elements of the group F. The set $\mathcal{P}$ is divided into classes of A-conjugacy.

Denote by $\overline{\mathcal{P}}$ the set of A-conjugacy classes of elements of $\mathcal{P}$. Denote by $\overline{\mathcal{P}}_0$ subset of $\overline{\mathcal{P}}$ consisting of classes such that in each of them there is a pair of mutually inverse elements.

Let us verify that $\overline{\mathcal{P}}_0 \neq \overline{\mathcal{P}}$. Let $x, y \in X$; $x \neq y$, $m \geq 1$. We check that the element $v = x^m y x y^{-1}$ is not A-conjugate to v^{-1}. For this we show that for any $a \in Aut\, F$ the element v^a is not conjugate to v^{-1} in the group F. Indeed, suppose that for some $b \in Aut\, F$ the element v^b is conjugate to v^{-1} in F. Then there is $\alpha \in Aut\, F$ such that $v^\alpha = v^{-1}$. By Proposition 4.1 from [25] there is an automorphism β of F which is nonidentity only on a finite subset of X, and such that $v^\beta = v^{-1}$. Let us choose a finite subset $X_1' = \{x_1, x_2, \ldots, x_k\}$ of X such that $x, y \in X_1'$ and the relations $x^\beta = x \ \ \forall x \in X \setminus X_1'$ hold. Let us add if necessary to X_1' elements $x_{k+1}, \ldots, x_q$ from $X \setminus X_1'$ such that all the words x_i^β, $i \leq k$, can be written in alphabet $X_1 = \{x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_q\}$. Let $X_2 = X \setminus X_1$. And let F_i be the subgroups of F generated by X_i, $i = 1, 2$ respectively. It is clear that F is free product $F = F_1 * F_2$. Since β is an automorphism of the group F and $F_2^\beta = F_2$, $F_1^\beta \subseteq F_1$ we obtain that β is an automorphism of F_1 too. Direct calculation shows that if τ is a Whitehead automorphism of the group F_1 (see [25]) such that $|c(v^\tau)| = |c(v)|$, then either τ is a permutable automorphism or there is an element $a \in X_1^{\pm 1}$ such that $v^\tau = a^{-\varepsilon} v a^\varepsilon$, $\ \varepsilon \in \{-1, 1\}$. Similarly for the word v^{-1} from the equality $|c((v^{-1})^\tau)| = |c(v^{-1})|$ it follows that either τ is a permutable automorphism or for some $g \in X_1^{\pm 1}$ the relation $(v^{-1})^\tau = g^{-\varepsilon_i} v^{-1} g^{\varepsilon_i}$, $\varepsilon_i \in \{-1, 1\}$ is valid. It is clear that there is no Whitehead transformation that transforms the word v into some cyclic permutation of the word v^{-1}. Hence, by

Proposition 4.19 from [25] we find that there is no element α from $Aut\, F_1$ that transforms v into v^{-1}. Futhermore, there is no automorphism of the group F that transforms v into v^{-1}. Hence, the element $v = x^m yxy^{-1}$ is not A-conjugate to v^{-1}.

Let $v = x_{i_1}^{\varepsilon_{i_1}} x_{i_2}^{\varepsilon_{i_2}} \cdots x_{i_{n-1}}^{\varepsilon_{i_{n-1}}} x_{i_n}^{\varepsilon_{i_n}}$ be a reduced word. We recall (see [7]) that the set of "beginnings " $H(v)$ and the set of "ends" $K(v)$ of the word v is defined as follows: if $n \leq 1$, then $H(v) = K(v) = \{\wedge\}$, where $\wedge$ is empty word. If $n \geq 2$, then

$$H(v) = \{\wedge, x_{i_1}^{\varepsilon_{i_1}}, x_{i_1}^{\varepsilon_{i_1}} x_{i_2}^{\varepsilon_{i_2}}, \dots, x_{i_1}^{\varepsilon_{i_1}} x_{i_2}^{\varepsilon_{i_2}} \cdots x_{i_{n-1}}^{\varepsilon_{i_n}}\}$$
$$K(v) = \{x_{i_2}^{\varepsilon_{i_2}} \cdots x_{i_n}^{\varepsilon_{i_n}}, \dots, x_{i_{n-1}}^{\varepsilon_{i_{n-1}}} x_{i_n}^{\varepsilon_{i_n}}, x_{i_n}^{\varepsilon_{i_n}}, \wedge\}.$$

Let $Q = \overline{\mathcal{P}} \setminus \overline{\mathcal{P}}_0$.

Lemma 1. *There is a set of representatives T of the A-conjugacy classes belonging to Q such that the following conditions hold.*

1) $T \subset c(X)$
2) $T^{-1} = T$.
3) $H(w) \cap K(w) = \{\wedge\}$ *for all* $w \in T$.
4) *There exists* $T^+ \subset T$ *such that* $T^+ \cap (T^+)^{-1} = \emptyset$ *and* $T = T^+ \cup (T^+)^{-1}$.

Proof. The Lemmas 3 and 4 from [7] imply that there is a system of representatives P of classes of conjugacy in F satisfying 1)–4) with P in place of T.

It is clear if $q \in Q$,then q is the union of conjugacy classes in F. Let us choose as a representative of a class q an element from the set $q \cap P$. It is clear that we can choose the system of representatives such that w is a representative if and only if w^{-1} is a representative too. Hence, we can choose $T^+ \subseteq T$ such that $T^+ \subseteq P^+$, $T^+ \cap (T^+)^{-1} = \emptyset$ and $T = T^+ \cup (T^+)^{-1}$. □

Lemma 2. *Let φ be a pseudocharacter of F such that $|\varphi(uv) - \varphi(u) - \varphi(v)| \leq c$ for any $u, v \in F$ and $|\varphi(x)| \leq \delta$ for $x \in X$. Then for any v with $|v| \geq 1$, we have*

$$|\varphi(v)| \leq (|v| - 1)c + |v|\delta.$$

Proof. For any $u, v \in F$ the inequality $|\varphi(uv)| \leq |\varphi(u)| + |\varphi(v)| + c$ holds. Hence, by induction on the length of the word v we get $|\varphi(v)| \leq (|v| - 1)c + |v|\delta$. □

Denote by $BPX(F_X, A)$ the subspace of $PX(F, A)$ consisting of pseudocharacters that are bounded on the set X. Let P be the set from the proof of Lemma 1. Now define a system of measures on the set P. For any pair of reduced words u and v and for any three reduced words a, b, c such that the word abc is reduced, too, we define the measures $\mu_{u,v}$, $\mu_{a,b,c}$ $\lambda_{u,v}$ and $\lambda_{a,b,c}$ on the set P as follows.

It is easy to see that for any $u, v \in \sigma(X)$ there is a uniquely defined triple of words u_1, v_1, z from $\sigma(X)$ such that $u \equiv u_1 z$, $v \equiv z^{-1} v_1$, $u_1 v_1 \in \sigma(X)$. Now set $\gamma_1(u, v) = u_1$, $\gamma_2(u, v) = v_1$, $\alpha(u, v) = z$. We set $\mu_{u,v}(w) = 1$ if and only if there are nonempty words t and τ such that

$$t \in K(\gamma_1(u, v)), \qquad \tau \in H(\gamma_2(u, v)), \quad w = t\tau;$$

otherwise we set $\mu_{u,v}(w) = 0$. We set $\mu_{a,b,c}(w) = 1$ if and only if $b \neq \wedge$ and there are nonempty words t and τ such that

$$t \in K(a), \quad \tau \in H(c), \quad w = tb\tau;$$

otherwise we set $\mu_{a,b,c}(w) = 0$. Now we set

$$\lambda_{u,v}(w) = \mu_{u,v}(w) - \mu_{u,v}(w^{-1}), \quad \lambda_{a,b,c}(w) = \mu_{a,b,c}(w) - \mu_{a,b,c}(w^{-1}).$$

Let $v \in \sigma(X)$. Then there is a uniquely defined pair of words $\overline{v}$, $z(v)$ from σ such that $v \equiv z(v)^{-1}\overline{v}z(v)$ and the word $\overline{v}$ is cyclically reduced. Furthermore, for each pair of reduced words u, v from the group F we define three measures $\Delta_{u,v}$, $\nu_{u,v}$, $\Theta_{u,v}$ on P as follows:

$$\Delta_{u,v}(w) = \lambda_{\gamma_1(u,v),\gamma_2(u,v)}(w) - \lambda_{\gamma_1(u,v),\alpha(u,v)}(w) - \lambda_{\alpha(u,v)^{-1},\gamma_2(u,v)}(w);$$

$$\begin{aligned}\nu_{u,v}(w) = \lambda_{\overline{\sigma(uv)},\overline{\sigma(uv)}}(w) &- \lambda_{z(\sigma(uv))^{-1},\overline{\sigma(uv)}}(w) - \lambda_{\overline{\sigma(uv)},z(\sigma(uv))}(w)\\ &- \lambda_{z(\sigma(uv))^{-1},\overline{\sigma(uv)},z(\sigma(uv))}(w) - \lambda_{\overline{u},\overline{u}}(w) - \lambda_{z(u)^{-1},\overline{u}}(w)\\ &- \lambda_{\overline{u},z(u)}(w) - \lambda_{z(u)^{-1},\overline{u},z(u)}(w) - \lambda_{\overline{v},\overline{v}}(w)\\ &- \lambda_{z(v)^{-1},\overline{v}}(w) - \lambda_{\overline{v},z(v)}(w) - \lambda_{z(v)^{-1},\overline{v},z(v)}(w);\end{aligned}$$

$$\Theta_{u,v}(w) = \Delta_{u,v}(w) + \nu_{u,v}(w).$$

For any two words u, v we set $\Theta_{u,v}(w) = \Theta_{\sigma(u),\sigma(v)}(w)$.

Let $w \in P^+$ and let $v \in \sigma(X)$. Then either v has no subwords equal to w or w^{-1}, or

$$v \equiv t_1 w^{i_1} t_2 w^{i_2} \cdots t_k w^{i_k} t_{k+1}. \tag{6}$$

Here $\equiv$ denotes graphical equality of the words, $i_j \in \mathbf{Z} \setminus \{0\}$, and the words t_j have no occurrences of subwords equal to w or w^{-1}. And in this presentation each occurrence of the words w and w^{-1} in v is fixed.

The presentation of v in the form (6) we shall call its w-decomposition.

Now for each element $w \in P$ define a function e_w on the set of words in the group alphabet X. First we define e_w on cyclically reduced words. Let u be a cyclically reduced word. Suppose that u has occurrences of w^ε, $\varepsilon \in \{1, -1\}$ and $u = t_1 w^{i_1} \dots t_k w^{i_k} t_{k+1}$ is its w-decomposition, then we set

$$e_w(u) = \sum_{j=1}^{k} i_j + \lambda_{t_{k+1},t_1}(w).$$

Suppose that u has no occurrences of w^ε, $\varepsilon \in \{1, -1\}$. Consider two cases.

1) If among the cyclic permutations of u there are no words containing occurrences of w or w^{-1}, we set $e_w(u) = 0$.
2) If among the cyclic permutations of u there is a word containing w^ε, $\varepsilon \in \{1, -1\}$, we set $e_w(u) = \varepsilon$.

Now let $v \in \sigma(X)$; then v is uniquely representable in the form $v \equiv z(v)^{-1}\overline{v}z(v)$, where $\overline{v} \in c(X)$. In this case we set $e_w(v) = e_w(\overline{v})$. Finally for an arbitrary word v we set $e_w(v) = e_w(\sigma(v))$.

Lemma 3. *The system of pseudocharacters $\{e_w\,;\, w \in P^+\}$ has the following properties.*

1) $|e_w(uv) - e_w(u) - e_w(v)| \le 15$ *for any* $u, v \in F$ *and* $w \in P^+$.
2) *If* $|w_1| < |w_2|$, *then* $e_{w_2}(w_1) = 0$.
3) *If* $|w_1| = |w_2|$ *and* $w_1 \ne w_2$, *then* $e_{w_2}(w_1) = 0$.
4) $e_w(w) = 1$ *for each* $w \in P^+$.

Proof. See [7], Theorem 1. □

From these properties we obtain that if $w_1, w_2 \in T^+$, $w_1 \ne w_2$ and $|w_1| \le |w_2|$, then $e_{w_2}(w_1) = 0$.

Definition 8. Let $w \in T^+$. We shall say that w satisfies the *condition of boundness* (a) if for any $v \in F$ there is $c(w,v) > 0$ such that

$$card\{\alpha \in \overline{A} \mid e_w(v^\alpha) \neq 0\} \leq c(w,v). \tag{7}$$

Let M be a subset of T^+. We shall say that M satisfies the *condition of boundness* (a), if for any $w \in M$ and any $v \in F$ there is $c(w,v) > 0$ such that the relation (7) holds.

For $w \in T^+$ satisfying the condition of boundness (a) we define a function δ_w by setting:

$$\delta_w(v) = \sum_{\alpha \in C} e_w(v^\alpha) \qquad v \in F,$$

where C denotes a system of representatives of cosets of the quotient group A by $Inn\,F$. Since any pseudocharacter is invariant under inner automorphisms we obtain that the definition of the function δ_w does not depend on the system of representatives of the cosets of the quotient group A by $Inn\,F$. Hence, we can write

$$\delta_w(v) = \sum_{\alpha \in \overline{A}} e_w(v^\alpha).$$

By formula 43 from [7] we have

$$\Theta_{u,v}(w) = e_w(uv) - e_w(u) - e_w(v) \quad \text{for all } w \in P^+ \text{ and } u, v \in F. \tag{8}$$

Starting from the measures $\Theta_{u,v}$ we define measures $\Theta^A_{u,v}$ and $\overline{\Theta}^A_{u,v}$ on the set T^+ as follows:

$$\Theta^A_{u,v}(w) = \sum_{\alpha \in \overline{A}} \Theta_{u^\alpha, v^\alpha}(w), \qquad \overline{\Theta}^A_{u,v} = \frac{1}{\delta_w(w)} \Theta^A_{u,v}.$$

Definition 9. Let $w \in T^+$. We shall say that w satisfies the *condition of boundness* (b) if there is $d_w > 0$ such that

$$card\{\alpha \in \overline{A} \mid w \in supp\, \Theta_{u^\alpha, v^\alpha}\} \leq d_w \quad \text{for all } u, v \in F. \tag{9}$$

We shall say that $M \subseteq T^+$ satisfies the *condition of boundness* (b) if for each $w \in M$ there is $d_w > 0$ such that the relation (9) holds.

Lemma 4. *Let $w \in T^+$ satisfy the conditions of boundness (a) and (b). Then the function δ_w is an element of $PX(F, A)$.*

Proof. The condition of boundness (a) is used in the definition of the function δ_w. From the conditions of boundness (b) we have

$$\begin{aligned}
|\delta_w(uv) - \delta_w(u) - \delta_w(v)| &= |\sum_{a \in \overline{A}} e_w((uv)^\alpha) - \sum_{a \in \overline{A}} e_w(u^\alpha) - \sum_{a \in \overline{A}} e_w(v^\alpha)| \\
&= |\sum_{a \in \overline{A}} [e_w(u^\alpha v^\alpha) - e_w(u^\alpha) - e_w(v^\alpha)]| \\
&= \sum_{a \in \overline{A}} |\Theta_{u^\alpha, v^\alpha}(w)| \leq 15 \cdot d_w
\end{aligned}$$

for any $u, v \in F$. ☐

Definition 10. Let $T_n^+ = \{w\,;\, w \in T^+,\, |w| = n\}$.

1) We shall say that the set T_n^+ satisfies the *condition of boundness* (b') if there is $d(n) > 0$ such that for each $w \in T_n^+$,

$$card\{\alpha \in \overline{A} \,|\, w \in supp\,\Theta_{u^\alpha, v^\alpha}\} \leq d(n) \quad \text{for all } u, v \in F.$$

2) We shall say that the set T_n^+ satisfies the *condition of boundness* (c) if there is $c(n) > 0$ such that

$$|T_n^+ \cap supp\,\Theta_{u,v}^A| \leq c(n) \quad \text{for all } u, v \in F.$$

3) We shall say that the set T^+ satisfies the *condition of boundness* (d) if for any $w \in T^+$ and any $v \in F$ such that $v \sim_A w$,

$$|c(v)| \leq |w|.$$

Lemma 5. *Let w_1, w_2 be distinct elements from the set T^+ satisfying the conditions of boundness (a), (c) and (d). Then $\delta_{w_2}(w_1) = 0$, $\delta_{w_2}(w_2) \geq 1$.*

Proof. Let $|w_1| < |w_2|$. Then the conditions of the Lemma imply that $|c(w_1^\alpha)| < |w_2|$ for all $\alpha \in A$. Hence, from Lemma 3, assertion 2), and from the fact that any pseudocharacter is invariant under inner automorphisms it follows that $\delta_{w_2}(w_1) = 0$. Now suppose that $|w_1| = |w_2|$ and $w_1 \neq w_2$. Then for each α from A the element w_1^α is conjugate in F neither to w_2, nor to w_2^{-1}. Moreover, the relation $|c(w_1^\alpha)| \leq |c(w_2)|$ holds. Hence, from Lemma 3, assertions 2) and 3), we get $e_{w_2}(w_1^\alpha) = 0$ for all $\alpha \in A$. The definition of the set T^+ implies that elements w_2^α, w_1^{-1} are not conjugate in F and the estimation $|c(w_2^\alpha)| \leq |c(w_2)|$ holds. Hence, from Lemma 3, assertions 2) and 3), we get that if for some α_0 from A the inequality $e_{w_2}(w_2^{\alpha_0}) \neq 0$ holds, then $e_{w_2}(w_2^{\alpha_0}) = 1$. Now from relation $e_{w_2}(w_2) = 1$ we have $\delta_{w_2}(w_2) \geq 1$. $\square$

Note that the set $\{\delta_w\,;\, w \in T^+\}$ is linearly independent. Indeed, let $w_1, \ldots, w_n$ be pairwise distinct elements from T^+ and let $\lambda_1, \ldots, \lambda_n$ be nonzero numbers such that $\sum_{i=1}^n \lambda_i \delta_{w_i} \equiv 0$. We may assume that $|w_1| \leq \cdots \leq |w_n|$. Then Lemma 5 implies $\psi(w_n) = \lambda_n \delta_{w_n}(w_n)$ and we obtain a contradiction to the relation $\lambda_n \neq 0$. Now set

$$\overline{\delta}_w(v) = \frac{1}{\delta_w(w)} \cdot \delta_w(v).$$

It is clear that $\overline{\delta}_w(w) = 1$. In general the function $\overline{\delta}_w$ is not an integer-valued pseudocharacter.

Lemma 6. *Let $w \in T^+$ and $u, v \in F$. Then*

$$\delta_w(uv) - \delta_w(u) - \delta_w(v) = \Theta_{u,v}^A(w), \tag{10}$$

$$\overline{\delta}_w(uv) - \overline{\delta}_w(u) - \overline{\delta}_w(v) = \overline{\Theta}_{u,v}^A(w). \tag{11}$$

Proof. The equality (11) follows from (10). Let us verify (10). From

$$e_w(u^\alpha v^\alpha) - e_w(u^\alpha) - e_w(v^\alpha) = \Theta_{u^\alpha, v^\alpha}(w),$$

(see [7] formula 43) we get

$$\begin{aligned}\delta_w(uv)-\delta_w(u)-\delta_w(v) &= \sum_{\alpha\in\overline{A}}[e_w^\alpha(uv)-e_w^\alpha(u)-e_w^\alpha(v)]\\ &= \sum_{\alpha\in\overline{A}}[e_w(u^\alpha v^\alpha)-e_w(u^\alpha)-e_w(v^\alpha)]\\ &= \sum_{\alpha\in\overline{A}}\Theta_{u^\alpha,v^\alpha}(w)\\ &= \overline{\Theta}_{u,v}(w).\end{aligned}$$

□

Theorem 5. *Let n be a positive integer. Let the set T_n^+ satisfy the conditions of boundness (a), (b′) and (c) and let t be a bounded function on the set T_n^+. Then:*

1) *The functions*

$$\varphi_t = \sum_{w\in T_n^+} t(w)\delta_w$$

$$\psi_t = \sum_{w\in T_n^+} t(w)\overline{\delta}_w \tag{12}$$

belong to the space $PX(F,A)$.

2) *$\psi_t(w)=t(w)$, for all $w\in T_n^+$.*

Proof. It is obvious that for each α from A, each integer n and each v from F, the equalities

$$\varphi_t^\alpha=\varphi_t \quad,\quad \psi_t^\alpha=\psi_t \quad,\quad \varphi_t^\alpha(v^n)=n\varphi_t(v) \quad,\quad \psi_t^\alpha(v^n)=n\psi_t(v)$$

hold. We verify that the set $\{\varphi_t(uv)-\varphi_t(u)-\varphi_t(v)\,;\,u,v\in F\}$ is bounded. Let $c>0$ be such that $\sup\{|t(w)|\,;\,w\in T_n^+\}\le c$. Then for any elements u,v from F we have

$$\varphi_t(uv)-\varphi_t(u)-\varphi_t(v) = \sum_{w\in T_n^+} t(w)\Theta_{u,v}^A \tag{13}$$

$$\psi_t(uv)-\psi_t(u)-\psi_t(v) = \sum_{w\in T_n^+} t(w)\overline{\Theta}_{u,v}^A. \tag{14}$$

Indeed,

$$\begin{aligned}\varphi_t(uv)-\varphi_t(u)-\varphi_t(v) &= \sum_{w\in T_n^+}t(w)\delta_w(uv)-\sum_{w\in T_n^+}t(w)\delta_w(u)-\sum_{w\in T_n^+}t(w)\delta_w(v)\\ &= \sum_{w\in T_n^+}t(w)[\delta_w(uv)-\delta_w(u)-\delta_w(v)]\\ &= \sum_{w\in T_n^+}t(w)\Theta_{u,v}^A.\end{aligned}$$

Similarly (14) is established. Further, from the conditions of boundness (b'),(c) and (13) we have

$$|\varphi_t(uv)-\varphi_t(u)-\varphi_t(v)| \le |\sum_{w\in T_n^+} t(w)\Theta_{u,v}^A| \le c\cdot d(n)c(n).$$

Similarly

$$|\psi_t(uv) - \psi_t(u) - \psi_t(v)| \le c \cdot d(n)c(n).$$

Thus, $\varphi_t, \psi_t \in PX(F, A)$. Let $w_0 \in T_n^+$, then Lemma 5 and the definition of $\overline{\delta}_w$ imply

$$\psi_t(w_0) = \sum_{w \in T_n^+} t(w)\overline{\delta}_w(w_0) = \sum_{w \in T_n^+} t(w)\frac{\delta_w(w_0)}{\delta_w(w)} = \frac{\delta_{w_0}(w_0)}{\delta_{w_0}(w_0)} = t(w_0).$$

□

Definition 11. We shall say that the set T^+ satisfies the *condition of boundness* (c') if for any $n \ge 1$ the set T_n^+ satisfies the *condition of boundness* (c).

Definition 12. We shall say that the set T^+ satisfies the *condition of boundness* if it satisfies the conditions of boundness (a), (c'), (d) and for any $n \in \mathbf{N}$ the set T_n^+ satisfies the condition of boundness (b').

Let the set T^+ satisfy the condition of boundness. Denote by E the set of functions φ on the group F that satisfy the following conditions.

1) $\varphi(x^n) = n\varphi(x)$ for each $x \in F$ and each $n \in \mathbf{Z}$.
2) $\varphi(x^\alpha) = \varphi(x)$ for any $x \in F$ and each $\alpha \in A$.
3) $\varphi(xy) = \varphi(yx)$ for all $x, y \in F$.
4) φ is bounded on T_i^+ for each $i \in \mathbf{N}$.

Obviously, E is a linear space under the usual operations. Let $L(T)$ be a linear space of a real functions t on T^+, satisfying the following condition:

$$t \text{ is bounded on } T_i^+ \text{ for each } i \in \mathbf{N}.$$

Let us construct an isomorphism π between the linear spaces E and $L(T)$. Let $\varphi \in E$. For any $i \in \mathbf{N}$ let us define a function $t_i : T_i^+ \to R$ as follows. We set $t_1 = \varphi\big|_{T_1^+}$. The function t_1 is bounded. By Theorem 5 the function

$$\psi_{t_1} = \sum_{w \in T_1^+} t_1(w)\overline{\delta}_w$$

belongs to the space $BPX(F_X, A)$. Now define t_2 as follows: for any w from T_2^+ we set

$$t_2(w) = (\varphi - \psi_{t_1})(w).$$

It is clear that the function t_2 is bounded. Further, the functions t_i are defined by induction: if $t_1, \ldots, t_n$ have been already defined and are bounded, then for each $w \in T_{n+1}^+$ we set

$$t_{n+1}(w) = \varphi(w) - \sum_{i=1}^{n} \psi_{t_i}(w) \tag{15}$$

where the functions ψ_{t_i} are pseudocharacters, which are constructed in Theorem 5 by the formula (12). It is obvious that the function t_{n+1} is bounded. Now define a function $\pi(\varphi)$, which we denote by t, as follows: if $w \in T_i^+$, then we set $t(w) = t_i(w)$. It is clear that $t \in L(T^+)$ and that the mapping π is linear. Let us show that the following equality holds:

$$\varphi = \sum_{i=1}^{\infty} \psi_{t_i}. \tag{16}$$

If $w \in T_1^+$, then (12) and Lemma 5 imply that for each $i \geq 2$ the equality $\psi_{t_i}(w) = 0$ holds. Hence, $\varphi(w) = \psi_{t_1}(w) = t_1(w)$ and the equality (16) is valid. Suppose the equality (16) has been already established for all w from $\cup_{i=1}^n T_i^+$. Let us prove it for w from T_{n+1}^+. Suppose that $w \in T_{n+1}^+$. Then from (15) and the definition of the functions t_{n+1} and $\psi_{t_{n+1}}$ we obtain

$$\psi_{t_{n+1}}(w) = \varphi(w) - \sum_{i=1}^{n} \psi_{t_i}(w),$$

i.e.,

$$\varphi(w) = \sum_{i=1}^{n+1} \psi_{t_i}(w).$$

Now from the relation $\psi_{t_{n+1}}(w) = 0$ for $i > n+1$ we get

$$\varphi(w) = \sum_{i=1}^{\infty} \psi_{t_i}(w).$$

Thus the formula (16) is true for all w from T. The functions from the left and right sides of the equality (16) satisfy conditions 1) and 2) from the definition of the space E. Hence this equality will be true for all elements of F. Note that if the τ_i are bounded functions on T_i^+, $i \in \mathbf{N}$ and ψ_{τ_i} are pseudocharacters of F, that are defined by formula (12), then the function $\varphi = \sum_{i=1}^{\infty} \psi_{\tau_i}$ belongs to E and the following equation

$$\pi(\varphi)\big|_{T_i^+} = \tau_i, \quad \text{for each} \quad i \in \mathbf{N} \tag{17}$$

holds. Indeed, suppose that β_i are functions defined on the set T_i^+, $i \in \mathbf{N}$ such that $\beta_1 = \varphi\big|_{T_1^+}$ and that for $n \geq 1$ the function β_{n+1} is defined by the formula

$$\beta_{n+1} = (\varphi - \sum_{i=1}^{n} \psi_{\tau_i})\big|_{T_{n+1}^+}.$$

Then $\beta_1 = \varphi\big|_{T_1^+} = \tau_1$, $\beta_2(w) = \varphi(w) = \psi_{\tau_2}(w) = \tau_2(w)$ for each $w \in T_2^+$. Suppose that $\beta_i \equiv \tau_i$ for $i \leq n$. Then using the relation $\tau_i(w) = 0$ for $i > n+1$, for $w \in T_{n+1}^+$ we obtain

$$\begin{aligned}
\beta_{n+1}(w) &= \varphi(w) - \sum_{i=1}^{n} \psi_{\tau_i}(w) \\
&= \sum_{i=1}^{\infty} \psi_{\tau_i}(w) - \sum_{i=1}^{n} \psi_{\tau_i}(w) \\
&= \sum_{i=n+1}^{\infty} \psi_{\tau_i}(w) \\
&= \psi_{\tau_{n+1}}(w) \\
&= \tau_{n+1}(w).
\end{aligned}$$

Thus the equality (17) is established. In particular the equality (17) implies that the mapping π is an epimorphism. Let us show that $\ker\pi = 0$. Indeed, suppose that $\varphi \in \ker\pi$; then from (16) and (17) we obtain

$$\pi(\varphi)\big|_{T_k^+} = \sum_{i=1}^{\infty} \psi_{t_i}\big|_{T_k^+} = t_k = 0 \quad \text{for all } k \in \mathbf{N}.$$

Hence, for any $k \in \mathbf{N}$ we have $\varphi\big|_{T_k^+} \equiv 0$ and $\varphi \equiv 0$. Thus, $\ker\pi = 0$ and π is an isomorphism.

Denote by $K(T^+)$ the space of real function β on the set T^+ satisfying the following conditions.

1) $\beta\big|_{T_I^+}$ is a bounded function for all $i \in \mathbf{N}$.
2) There is an $\varepsilon = \varepsilon(\beta) > 0$ such that for each pair of reduced words u, v from F the following inequality holds:

$$\Big|\int_{T^+} \beta\, d\,\overline{\Theta}_{u,v}^A\Big| \le \varepsilon.$$

It is clear that $K(T^+)$ is a subspace of $L(T^+)$. Let us verify that $BPX(F_X, A)$ is subspace of E. It is clear from the conditions defining the space E that we must verify only 3). Let us show that this follows from Lemma 2. Indeed, if $\varphi \in BPX(F_X, A)$, then the function $\varphi\big|_{T_1^+}$ is bounded. From the fact that φ is a pseudocharacter it follows that there is $c > 0$ such that for any u, v from F the inequality $|\varphi(uv) - \varphi(u) - \varphi(v)| \le c$ holds. Let $|\varphi(x)| \le \delta$ for all $x \in X$. Then Lemma 2 implies that $|\varphi(v)| \le |v|\cdot\delta + (|v|-1)\cdot c$ for all $v \in F$. Hence, $BPX(F_X, A)$ is a subspace of E.

Theorem 6. *Let the set T^+ satisfy the condition of boundness. Then:*

1) *The mapping π is an isomorphism between the spaces $BPX(F_X, A)$ and $K(T^+)$.*
2) *Each element φ from $BPX(F_X, A)$ is uniquely representable in the form*

$$\varphi = \sum_{w \in T^+} \beta(w)\overline{\delta}_w,$$

where $\beta \in K(T^+)$.

Proof. 1) Suppose that $\varphi \in BPX(F_X, A)$ and that $|\varphi(uv) - \varphi(u) - \varphi(v)| \le \varepsilon$ for some $\varepsilon > 0$ and any $u, v \in F$. Let $\pi(\varphi) = \beta$ and $\beta_i = \beta\big|_{T_i^+}$ for $i \in \mathbf{N}$. Using (11) we obtain the following equation that is valid for any $u, v \in F$.

$$\begin{aligned}
\int_{T^+} \beta\, d\,\overline{\Theta}^A_{u,v} &= \sum_{i=1}^{\infty} \int_{T_i^+} \beta\, d\,\overline{\Theta}^A_{u,v} \\
&= \sum_{i=1}^{\infty} \sum_{w\in T_i^+} \beta(w)\overline{\Theta}^A_{u,v}(w) \\
&= \sum_{i=1}^{\infty} \sum_{w\in T_i^+} \beta(w)[\overline{\delta}_w(uv) - \overline{\delta}_w(u) - \overline{\delta}_w(v)] \\
&= \sum_{i=1}^{\infty} \sum_{w\in T_i^+} \beta(w)\overline{\delta}_w(uv) - \sum_{i=1}^{\infty} \sum_{w\in T_i^+} \beta(w)\overline{\delta}_w(u) \\
&\quad - \sum_{i=1}^{\infty} \sum_{w\in T_i^+} \beta(w)\overline{\delta}_w(v).
\end{aligned} \tag{18}$$

Now using formula (16), the definition of pseudocharacters ψ_{β_i} and formula (18) we get

$$\begin{aligned}
\Big| \int_{T^+} \beta\, d\,\overline{\Theta}^A_{u,v} \Big| &= \Big| \sum_{i=1}^{\infty} \int_{T_i^+} \beta\, d\,\overline{\Theta}^A_{u,v} \Big| \\
&= \Big| \sum_{i=1}^{\infty} \psi_{\beta_i}(uv) - \sum_{i=1}^{\infty} \psi_{\beta_i}(u) - \sum_{i=1}^{\infty} \psi_{\beta_i}(v) \Big| \\
&= |\varphi(uv) - \varphi(u) - \varphi(v)| \le \varepsilon.
\end{aligned}$$

Thus, $\pi(\varphi) \in K(T^+)$. Now let $\beta \in K(T^+)$ and $\beta_i = \beta\big|_{T_i^+}$. Then as was shown above, if ψ_{β_i} are the pseudocharacters defined by the formula (12) and if

$$\varphi = \sum_{i=1}^{\infty} \psi_{\beta_i},$$

then $\pi(\varphi) = \beta$, i.e., $\varphi = \pi^{-1}(\beta)$. Let us show that φ belongs to $BPX(F_X, A)$. Let $\varepsilon > 0$ such that $\big| \int_{T^+} \beta\, d\,\overline{\Theta}^A_{u,v} \big| \le \varepsilon$ for any $u, v \in F$. Then

$$\begin{aligned}
|\varphi(uv) - \varphi(u) - \varphi(v)| &= \Big| \sum_{i=1}^{\infty} \psi_{\beta_i}(uv) - \sum_{i=1}^{\infty} \psi_{\beta_i}(u) - \sum_{i=1}^{\infty} \psi_{\beta_i}(v) \Big| \\
&= \Big| \sum_{i=1}^{\infty} [\psi_{\beta_i}(uv) - \psi_{\beta_i}(u) - \psi_{\beta_i}(v)] \Big| \\
&= \Big| \sum_{i=1}^{\infty} \sum_{w\in T_i^+} \beta_i(w)[\overline{\delta}_w(uv) - \overline{\delta}_w(u) - \overline{\delta}_w(v)] \Big| \\
&= \Big| \sum_{i=1}^{\infty} \sum_{w\in T_i^+} \beta_i(w)\overline{\Theta}^A_{u,v}(w) \Big| \\
&= \Big| \int_{T^+} \beta\, d\,\overline{\Theta}^A_{u,v} \Big| \le \varepsilon.
\end{aligned}$$

Hence, $\varphi \in BPX(F_X, A)$ and π is an isomorphism between $BPX(F_X, A)$ and $K(T^+)$

The assertion 2) follows from assertion 1). $\square$

The following result was obtained in [6].

Theorem 7. *Suppose that f is a quasicharacter of a semigroup S and that $c > 0$ is such that*

$$|f(xy) - f(x) - f(y)| < c$$

for all $x, y \in S$. Then the function

$$\widehat{f}(x) = \lim_{n\to\infty} \frac{1}{2^n} f(x^{2^n}) \tag{19}$$

is well-defined and is a pseudocharacter of S such that

$$|\widehat{f}(xy) - \widehat{f}(x) - \widehat{f}(y)| < 4c \quad \textit{for all } x, y \in S.$$

Corollary 1. *Suppose that f is a quasicharacter of a group G and that $c > 0$ is such that*

$$|f(xy) - f(x) - f(y)| < c$$

for all $x, y \in G$. Then the function

$$\widehat{f}(x) = \lim_{n\to\infty} \frac{1}{2^n} f(x^{2^n})$$

is well-defined and is a pseudocharacter of G such that

$$|\widehat{f}(xy) - \widehat{f}(x) - \widehat{f}(y)| < 4c \quad \textit{for all } x, y \in G.$$

Proof. Theorem 7 implies that in order to prove that $\widehat{f}$ is a pseudocharacter of group G it remains to verify that for each $x \in G$ the equality $\widehat{f}(x^{-1}) = -\widehat{f}(x)$ holds. From the relation $\widehat{f}(x^n) = n\widehat{f}(x)$ for all $x \in G$ and $n \in \mathbf{N}$, we obtain $n\widehat{f}(1) = \widehat{f}(1^n) = \widehat{f}(1)$. The latter is possible only if $\widehat{f}(1) = 0$. Hence, $|\widehat{f}(1) - \widehat{f}(x) - \widehat{f}(x^{-1})| < 4c$ for all $x \in G$ and $|\widehat{f}(x) + \widehat{f}(x^{-1})| < 4c$ for all $x \in G$. Whence follows the inequality $n|\widehat{f}(x) + \widehat{f}(x^{-1})| = |\widehat{f}(x^n) + \widehat{f}((x^{-1})^n)| < 4c$ for all $x \in G$ and $n \in \mathbf{N}$. This is possible only if $\widehat{f}(x^{-1}) = -\widehat{f}(x)$. Now let $k > 0$. Then we have $\widehat{f}(x^{-k}) = \widehat{f}((x^k)^{-1}) = -\widehat{f}(x^k) = -k\widehat{f}(x)$. $\square$

Proposition 1. *Let $\overline{A}$ be a finite group. Then there exists T^+ satisfying the condition of boundness.*

Proof. Let k be the order of the group $\overline{A}$. It is clear that each class of A-conjugacy is the union of at most k classes of conjugacy in F. From this fact we find that the set $q \cap P$ contains at most k elements. Hence, one can choose an element of maximal length in the set $q \cap P$. Hence, there is a set of representatives T of classes of A-conjugacy belonging to q such that the following relations hold.

1) $w \in T$ if and only if $w^{-1} \in T$.
2) Every element $w \in T$ has maximal length among the elements belonging to the set $c(X)$ and A-conjugated to w.

Hence, in the set T we can choose subset T^+ such that $T^+ \subseteq P^+$, $T^+ \cap (T^+)^{-1} = \emptyset$ and $T = T^+ \cup (T^+)^{-1}$. Now it easy to verify that the set T^+ satisfies the condition of boundness. $\square$

Proposition 2. *Let H be an invariant subgroup of finite index k in a group G. Then $PX(G) = PX(H,G)$.*

Proof. Let us show that each element from $PX(H,G)$ may be uniquely extended to a pseudocharacter of the group G. Let φ be an element from $PX(H,G)$ and $c > 0$ be such that

$$|\varphi(xy) - \varphi(x) - \varphi(y)| \leq c$$

for all $x, y \in H$. Suppose that A is a system of representatives of cosets of G by H, then each element $g \in G$ is uniquely representable in the form

$$g = \alpha(g)h(g)$$

where $\alpha(g) \in A$, $h(g) \in H$. By hypothesis the set

$$\{\alpha(\alpha_1\alpha_2)\,;\, \alpha_1, \alpha_2 \in A\}$$

is finite. Hence, there is $\delta > 0$ such that

$$|\varphi(\alpha(\alpha_1\alpha_2))| \leq \delta \quad \text{for all } \alpha_1, \alpha_2 \in A. \tag{20}$$

Now let φ_1 the function on G defined by formula $\varphi_1(\alpha h) = \varphi(h), \quad h \in H, \quad \alpha \in A$. Let us verify that φ_1 belongs to $KX(G)$. We have

$$\begin{aligned}
&|\varphi_1(\alpha_1h_1\alpha_2h_2) - \varphi_1(\alpha_1h_1) - \varphi_1(\alpha_2h_2)| \\
&\qquad = |\varphi_1(\alpha_1\alpha_2h_1^{\alpha_2}h_2 - \varphi_1(\alpha_1h_1) - \varphi_1(\alpha_2h_2)| \\
&\qquad = \varphi_1(\alpha(\alpha_1\alpha_2)h(\alpha_1\alpha_2)h_1^{\alpha_2}h_2) - \varphi_1(\alpha_1h_1) - \varphi_1(\alpha_2h_2)| \\
&\qquad = \varphi_1(h(\alpha_1\alpha_2)h_1^{\alpha_2}h_2) - \varphi_1(h_1) - \varphi_1(h_2)| \\
&\qquad = \varphi(h(\alpha_1\alpha_2)h_1^{\alpha_2}h_2) - \varphi(h_1) - \varphi(h_2)| \\
&\qquad = \varphi(h(\alpha_1\alpha_2)h_1^{\alpha_2}h_2) - \varphi(h_1^{\alpha_2}) - \varphi(h_2) - \varphi(h(\alpha_1\alpha_2)) \\
&\qquad\quad + \varphi(h(\alpha_1\alpha_2)) - \varphi(h_1^{\alpha_2}h_2) + \varphi(h_1^{\alpha_2}h_2)| \\
&\qquad \leq |\varphi(h(\alpha_1\alpha_2)h_1^{\alpha_2}h_2) - \varphi(h(\alpha_1\alpha_2)) - \varphi(h_1^{\alpha_2}h_2)| \\
&\qquad\quad + |\varphi(h_1^{\alpha_2}h_2) - \varphi(h_1^{\alpha_2}) - \varphi(h_2)| + |\varphi(h(\alpha_1\alpha_2))|
\end{aligned}$$

Now from (20) we get

$$|\varphi_1(\alpha_1h_1\alpha_2h_2) - \varphi_1(\alpha_1h_1) - \varphi_1(\alpha_2h_2)| \leq 2c + \delta.$$

Thus $\varphi_1 \in KX(G)$, hence $\varphi' = \widehat{\varphi}_1$ is a pseudocharacter of the group G. Here $\widehat{\varphi}_1$ is defined by (19). It is clear that $\varphi'\big|_H = \varphi$. Let us verify that the mapping $\varphi \to \varphi'$ is one-to-one and maps $PX(H,G)$ "onto" $PX(G)$. Indeed, if $f \in PX(G)$, then $\varphi = f\big|_H \in PX(H,G)$ and φ' coincides with f on subgroup H. Hence, the pseudocharacter $\psi = f - \varphi'$ vanishes on H. From the equality $\psi(g^k) = k\psi(g)$ for all $g \in G$ we obtain $\psi = 0$ on G and $f = \varphi'$. Similarly we verify that if $\varphi_1, \varphi_2 \in PX(H,G)$ and $\varphi_1 \neq \varphi_2$, then $\varphi_1' \neq \varphi_2'$. □

Corollary 2. *If a group G is a finite extension of a free group F of finite rank, then its space of pseudocharacters is described by Theorem* 6.

Proof. Let $X = \{x_1, x_2, \ldots, x_k\}$ be a set of free generators of the group F and let T^+ be the set from Proposition 1. Let A be the group of automorphisms of the group F that are induced by conjugation by elements from G. From the condition we have that the group $\overline{A} = A/Inn\,A$ is finite and we can apply Proposition 1.

From the finiteness of the set X it is clear that every element of $PX(F,G)$ is bounded on the set X. Hence $PX(F,G) = BPX(F_X, G)$. By Proposition 2 we have $PX(G) = PX(F,G) = BPX(F_X, G)$ and we obtain that the space of pseudocharacters of the group G is described by Theorem 6. □

We note that Theorem 6 may be used to describe the space of pseudocharacters of certain infinite extensions of free groups of infinite rank.

Example. Suppose that F is the free group with free generators $X = \{x_i\, ;\, i \in \mathbf{Z}\}$, and that A is an infinite cyclic group with generator a. Let $G = A \cdot F$ be the semidirect product such that $F \triangleleft G$ where A acts on F as follows:

$$x_i^a = x_{i+1}, \quad i \in \mathbf{Z}. \tag{21}$$

By Theorem 2 from [6] we have $PX(G) = X(A) \dot{+} PX(F, A)$, where $X(A)$ is a space of additive characters of the group A. Let φ be an arbitrary element from the space $PX(F, A)$. From (21) it follows that φ is constant on the set X. Hence, we have $PX(F,A) = BPX(F_X, A)$ and the problem of describing of the space $PX(G)$ is reduced to the problem of describing of $BPX(F_X, A)$. Let T^+ be some set satisfying the conditions of Lemma 1 and belonging to P^+. It is easy to verify that T^+ satisfies the condition of boundness. From this fact we obtain that the Theorem 6 describes the space $BPX(F_X, A)$ and therefore the space of pseudocharacters of the group G too.

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