New York Journal of Mathematics

New York J. Math. **6** (2000) 87–93.

An Example of a Conservative Exact Endomorphism which is Not Lim Sup Full

Julia A. Barnes and Stanley J. Eigen

Abstract. We show how to modify a construction of Hamachi to obtain a conservative exact endomorphism which is not lim sup full.

Contents

1. Introduction

In showing certain endomorphisms are exact, Rohlin [7] developed the notion of full and showed that in the finite measure preserving case these two notions are equivalent. However, there are open questions of conservativity and exactness when measure preserving is not known. To this end, Barnes [1] developed the notion of lim sup full, and showed that lim sup full, nonsingular, n-to-1 endomorphisms are conservative and exact whether or not the map is finite measure preserving. She applied this to certain classes of rational maps, obtaining conservativity and exactness when measure preserving is not known. The purpose of this note is to show by example that a conservative exact map need not be lim sup full.

The example is based on a construction (see Section 3) studied by Hamachi [3] and Krengel [6] and depends strongly on a theorem of Kakutani [5]. As a class, the general construction gives a family of maps which are conservative, exact shift maps on the one-sided symbol space with different product measures. By a construction of Bruin and Hawkins [2] conservative examples can be moved to the Riemann sphere where they are invariant for certain rational maps whose Julia set is the entire sphere.

Received April 30, 1999.
Mathematics Subject Classification. 37.
Key words and phrases. exact, full, lim sup full.

ISSN 1076-9803/00

2. Preliminaries

The transformation constructed by Hamachi was a Bernoulli shift on a two-sided product space. The example in this paper is a one-sided factor of such a map.

Set $\Omega = \prod_{i=-\infty}^{\infty} \{0,1\}_i$. The transformation T is the shift $(T\omega)_n = \omega_{n+1}$. Most of the work presented here deals with the construction of the product measure $P = \prod_{k=-\infty}^{\infty} P_k$, where $P_k(0) = \frac{1}{1+\lambda_k}$ and $P_k(1) = \frac{\lambda_k}{1+\lambda_k}$ for some $\lambda_k \geq 1$. Since Ω is fixed throughout the paper, we refer to the measure space (Ω, P, T) as P. We say T is **nonsingular** with respect to the measure P if the measures P and $P \circ T^{-1}$ are equivalent; **conservative** if for every set A of positive measure, there is a $k > 0$ such that $P(A \cap T^{-k}(A)) > 0$; **measure preserving** if the measures P and $P \circ T^{-1}$ are identical; **ergodic** if for any measurable set A, $P(A \triangle T^{-1}A) = 0 \Rightarrow P(A)$ is zero or one.

We are interested in a one-sided shift map defined on a factor of Ω. Let $X = \prod_{i=-\infty}^{0} \{0,1\}_i$ with the product measure $Q = \prod_{k=-\infty}^{0} P_k$. Define $S = T^{-1}$, i.e., $(\cdots x_{-2}x_{-1}x_0) \mapsto (\cdots x_{-3}x_{-2}x_{-1})$. As above, we refer to the measure space (X, Q, S) as S. We use $\mathcal{B}$ to denote the sigma algebra of measurable sets in X. The same definitions as above are used to define S as nonsingular, conservative, measure preserving, and ergodic. In addition, we call S **exact** if $\cap_0^\infty S^{-1}\mathcal{B}$ contains only sets of Q-measure one or zero; **full** if $\lim_{j\to\infty} Q(S^j B) = 1$ for all $B \in \mathcal{B}$ of positive measure; **lim sup full** if $\limsup_{j\to\infty} Q(S^j B) = 1$ for all $B \in \mathcal{B}$ of positive measure [1].

3. The Shape of the Measure *P*

The measure P is determined by a fixed probability distribution μ on $\{0,1\}$, a sequence of probability distributions $\nu_i = \{\frac{1}{1+\lambda_i}, \frac{\lambda_i}{1+\lambda_i}\}$, and two sequences of non-negative integers n_i and m_i. Hamachi assigns values to λ_i, n_i and m_i by induction. He does this in a way which guarentees that the resulting measure he obtains is nonsingular, conservative, ergodic and preserves no equivalent finite or infinite invariant measure for the two-sided shift. We present a simpler induction process in the next section. Our goal is for a nonsingular, conservative exact measure which is not lim sup full for the one-sided shift. Since it is not lim sup full, it follows that it has no equivalent finite invariant measure. It remains open whether it has an equivalent infinite invariant measure; it also remains open whether the one-sided version of Hamachi's example is lim sup full.

In general the distribution μ occurs on all positive coordinates and on each of the negative coordinates 0 through $-m_1+1$ of the sequence space. The distribution ν_1 occurs on each of the next n_1 negative coordinates. Then μ occurs on each of the next m_2 negative coordinates. Then ν_2 occurs on the next n_2 — and so on.

The following figures illustrate the measure P displayed on the coordinates of Ω and the measure Q displayed on the coordinates of X.

$$P = \cdots \underbrace{\overbrace{\cdots}^{\nu_3}}_{n_3}\underbrace{\overbrace{\cdots}^{\mu}}_{m_3}\underbrace{\overbrace{\cdots}^{\nu_2}}_{n_2}\underbrace{\overbrace{\cdots}^{\mu}}_{m_2}\underbrace{\overbrace{\cdots\ -m_1}^{\nu_1}}_{n_1}\underbrace{\overbrace{\cdots\ 0}^{\mu}}_{m_1}\underbrace{\overbrace{1\cdots\ \cdots\ \cdots}^{\mu}}_{\infty}$$

$$Q = \cdots \overbrace{\underbrace{\cdots}_{n_3}}^{\nu_3}\overbrace{\underbrace{\cdots}_{m_3}}^{\mu}\overbrace{\underbrace{\cdots}_{n_2}}^{\nu_2}\overbrace{\underbrace{\cdots}_{m_2}}^{\mu}\overbrace{\underbrace{\cdots\ -m_1}_{n_1}}^{\nu_1}\overbrace{\underbrace{\cdots\ 0}_{m_1}}^{\mu}.$$

Here $\mu = \{\frac{1}{2}, \frac{1}{2}\}$ is fixed, and $\nu_i = \{\frac{1}{1+\lambda_i}, \frac{\lambda_i}{1+\lambda_i}\}$, where $2 > \lambda_i > 1$ for all i with $\lim_{i\to\infty} \lambda_i = 1$.

As in Hamachi [3], the values of λ_i, n_i and m_i are chosen by an inductive process - though not quite in the obvious order. While doing this induction we want to control three things, the first two of which already appear in [3]: (i) T should be nonsingular with respect to the measure P; (ii) T should be conservative with respect to the measure P; (iii) S should be not lim sup full with respect to the measure Q. The nonsingularity and conservativity of S follow from the corresponding properties for T.

In order to guarantee the nonsingularity of T with respect to P, we use Hamachi's version [3] of Kakutani's theorem [5]. In particular, we choose the λ_i such that $\sum_{i=1}^{\infty} \left(\log(\lambda_i)\right)^2 < \infty$.

In order to guarantee the conservativity of T with respect to P, (again as in [3]) we force the Radon-Nikodym derivatives to sum to infinity, i.e., $\sum_{i=0}^{\infty} \frac{dPT^i}{dP}(\omega) = \infty$, almost everywhere mod P. The Radon-Nikodym deriatives for the measure P are analyzed by $\frac{dPT^i}{dP}(\omega) = \prod_{k=-\infty}^{\infty} \frac{dPT^i(\omega_k)}{dP(\omega_k)} = \prod_{k=-\infty}^{\infty} \frac{P_{k-i}(\omega_k)}{P_k(\omega_k)}$.

In order to guarantee that S is not lim sup full, we construct a distinguished set of positive measure E in X with the property that $\lim_{j\to\infty} Q(S^j E) = 0$. The set E will be the intersection of a collection of sets A_i based on the disjoint coordinates associated to the measures ν_i.

4. The Induction Process

We construct the measure P by finding appropriate values for the sequences n_i, m_i, and λ_i, as well as for sets A_i used in constructing the set E described above. Along the way, we construct some auxilary increasing sequences N_i, K_i, L_i of integers and set $m_i = N_i + N_{i+1}$, $n_i = \sum_{j=1}^{i} N_j$, and $L_i = \sum_{j=1}^{i} m_j + n_j$. We also inductively construct reals $I_{i,j} > 0$ which are used in controlling the Radon-Nikodym derivatives.

Fix the sequence $1 > \epsilon_i > 0$ such that $\prod_{i=1}^{\infty}(1 - \epsilon_i) > 0$. The sequence ϵ_i is used in choosing the sets A_i and eventually in the construction of E.

We use the following lemma to select variables A_i and N_i below.

Lemma 4.1. *Given measures $\nu \neq \mu$ on $\{0,1\}$, and given $1 > \epsilon > 0$, there is an $N > 0$, such that for any natural number $n > N$ there is a set*

$$A \subset \prod_{j=1}^{n} \{0,1\}_j$$

with the property that for the finite product measures,

$$(\prod_{1}^{n} \mu)(A) < \epsilon \quad (\prod_{1}^{n} \nu)(A) > 1 - \epsilon.$$

Proof. Without loss of generality, we assume that $\mu(1) < \nu(1)$. Put $\alpha = \frac{\mu(1)+\nu(1)}{2}$, and define $A_n = \{\omega \in \prod_{i=1}^n \{0,1\}_i : \frac{1}{n}\sum_{i=1}^n \omega_i > \alpha\}$. Then $(\prod_1^n \mu)(A_n) \to 0$ and $(\prod_1^n \nu)A_n \to 1$. Thus, for any natural number $n > N$ there is a set $A \subset \prod_{j=1}^n \{0,1\}_j$ with the property that for the finite product measures, $(\prod_1^n \mu)(A) < \epsilon$ and $(\prod_1^n \nu)(A) > 1-\epsilon$. □

Step 1. Choose $1 < \lambda_1 < 2$ to obtain ν_1. From Lemma 4.1, we obtain $N_1 > 0$ and a set A_1 satisfying $(\prod_1^{N_1} \mu)(A_1) < \epsilon_1$ and $(\prod_1^{N_1} \nu_1)(A_1) > 1-\epsilon_1$. Set $n_1 = N_1$, and initialize $L_0 = N_1$. Choose K_1 such that $K_1\lambda_1^{-n_1} > 1$. Define $I_{1,1} = K_1\lambda_1^{-n_1}$.

Note that we have specified n_1, λ_1, A_1 and K_1, but we have not yet specified values for m_1 or L_1. This is done in Step 2.

Step 2. Choose $1 < \lambda_2 < 1 + 1/2$ such that $\lambda_2^{-(L_0+K_1)} I_{1,1} > 1$, and define $I_{2,1} = \lambda_2^{-(L_0+K_1)} I_{1,1}$.

From Lemma 4.1, we obtain $N_2 > L_0+K_1$ and a set A_2 satisfying $(\prod_1^{N_2} \mu)(A_2) < \epsilon_2$ and $(\prod_1^{N_2} \nu_2)(A_2) > 1-\epsilon_2$. Now set $m_1 = N_2 + N_1$, and $n_2 = N_2 + N_1$. Let $L_1 = n_1 + m_1$ and observe that $L_1 > n_2$. Choose K_2 such that $K_2\lambda_2^{-n_2}\lambda_1^{-n_1} > 1$. Define $I_{2,2} = K_2\lambda_2^{-n_2}\lambda_1^{-n_1}$.

The values for m_2 and L_2 will be specified in the next step.

Step 3. Choose $1 < \lambda_3 < 1{+}1/3$ such that $\lambda_3^{-(L_0+K_1)} I_{2,1} > 1$, and $\lambda_3^{-(L_1+K_2)} I_{2,2} > 1$. This is possible since $I_{2,1}$ and $I_{2,2}$ are both greater than 1. Define $I_{3,1} = \lambda_3^{-(L_0+K_1)} I_{2,1}$, and $I_{3,2} = \lambda_3^{-(L_1+K_2)} I_{2,2}$. From Lemma 4.1, we obtain $N_3 > L_1+K_2$ and a set A_3 satisfying $(\prod_1^{N_3} \mu)(A_3) < \epsilon_3$ and $(\prod_1^{N_3} \nu_3)(A_3) > 1-\epsilon_3$. Set $m_2 = N_3 + N_2$, $n_3 = N_3 + N_2 + N_1$ and $L_2 = n_2 + m_2 + L_1$. Again $L_2 > n_3$. Choose K_3 such that $K_3\lambda_3^{-n_3}\lambda_2^{-n_2}\lambda_1^{-n_1} > 1$. Define $I_{3,3} = K_3\lambda_3^{-n_3}\lambda_2^{-n_2}\lambda_1^{-n_1}$.

Step $t+1$. Suppose that $m_{t-1}, L_{t-1}, n_t, \lambda_t, A_t$, and K_t have been fixed. Also, $I_{t,1}, I_{t,2}, ..., I_{t,t}$ are all defined and are greater than 1. Choose λ_{t+1} such that $1 < \lambda_{t+1} < 1 + 1/(t+1)$ and $\lambda_{t+1}^{-(L_{j-1}+K_j)} I_{t,j} > 1$ for all $j \in \{1,2,...,t\}$, and define $I_{t+1,j} = \lambda_{t+1}^{-(L_{j-1}+K_j)} I_{t,j}$. Use Lemma 4.1 to obtain an integer $N_{t+1} > L_{t-1}+K_t$ and a set A_{t+1} satisfying $(\prod_1^{N_{t+1}} \mu)(A_{t+1}) < \epsilon_{t+1}$ and $(\prod_1^{N_{t+1}} \nu_{t+1})(A_{t+1}) > 1-\epsilon_{t+1}$. Set $m_t = N_{t+1} + N_t$, $L_t = n_t + m_t + L_{t-1}$, and $n_{t+1} = N_{t+1} + n_t$. Choose K_{t+1} such that $K_{t+1}\prod_{j=1}^{t+1} \lambda_j^{-n_j} > 1$. Let $I_{t+1,t+1} = K_{t+1}\prod_{j=1}^{t+1} \lambda_j^{-n_j}$.

Remark 4.2. From the definitions, for fixed $i \geq 1$ the sequence $I_{t,i}$ is a decreasing sequence and $\lim_{t\to\infty} I_{t,i} \geq 1$.

5. Nonsingularity, Conservativity, and Not Lim Sup Full

In this section, we show that the one-sided shift map with the previously constructed measure P is a conservative, exact 2-to-1 endormorphism which is not lim sup full.

Lemma 5.1. *T is nonsingular with respect to P.*

Proof. From the construction of the measure P we have a sequence λ_t with the property that $\lambda_t < 1 + 1/t$ for all $t > 0$. Therefore,

$$\sum_{i=1}^{\infty}(\log(\lambda_i))^2 \leq \sum_{i=1}^{\infty}(\log(1+1/i))^2 < \sum_{i=1}^{\infty}(1/i)^2 < \infty.$$

Thus T is nonsingular with respect to P [5]. $\square$

It follows that S is also nonsingular with respect to Q.

Lemma 5.2. *S not lim sup full with respect to Q.*

Proof. We now construct a set E from the collection of sets A_i obtained in the previous section.

Let A_i' denote the set of all sequences in X with A_i placed on the leftmost N_i coordinates of n_i. This can be observed by the following illustration:

$$P = \cdots \underbrace{\overbrace{\cdots}^{\mu}}_{N_4+N_5}\underbrace{\overbrace{A_3\cdots}^{\nu_3}}_{N_3+N_2+N_1}\underbrace{\overbrace{\cdots}^{\mu}}_{N_3+N_4}\underbrace{\overbrace{A_2\cdots}^{\nu_2}}_{N_2+N_1}\underbrace{\overbrace{\cdots}^{\mu}}_{N_2+N_3}\underbrace{\overbrace{A_1}^{\nu_1}}_{N_1}\underbrace{\overbrace{\cdots}^{\mu}}_{N_1+N_2}\underbrace{\overbrace{\cdots\ \cdots\ \cdots}^{\mu}}_{\infty}.$$

Hence, for $N_1 \le j \le N_1 + N_2$, $Q(S^j A_1') < \epsilon_1$; for $N_1 + N_2 \le j \le N_1 + N_2 + N_3$, $Q(S^j A_2') < \epsilon_2$; and in general, $Q(S^j A_k') < \epsilon_k$ for $\sum_{i=1}^{k} N_i \le j \le \sum_{i=1}^{k+1} N_i$.

Let $E = \cap_{i=1}^{\infty} A_i'$. Since A_i' forms a disjoint collection of sets based in ν_i blocks, $Q(E) = \prod_{i=1}^{\infty} Q(A_i') > \prod_{i=1}^{\infty}(1-\epsilon_i) > 0$.

Let $\epsilon > 0$. Since $\lim_{i\to\infty} \epsilon_i = 0$, there is a natural number K such that for all $k > K$, $\epsilon_k < \epsilon$. Then for all $j > N_K$, there is a $k > K$ such that $\sum_{i=1}^{k} N_i \le j \le \sum_{i=1}^{k} N_{i+1}$. So, $Q(S^j(E)) = \sum_{i=1}^{\infty} Q(S^j(A_i')) \le Q(S^j(A_k')) < \epsilon_k < \epsilon$. Thus, $\lim_{j\to\infty} Q(S^j(E)) = 0$, and S is not lim sup full. $\square$

Lemma 5.3. *T is conservative with respect to P.*

Proof. To show conservativity we demonstrate that the sum of the Radon-Nikodym derivatives sum to infinity.

It is clear that

$$\sum_{i=1}^{\infty} \frac{dPT^i}{dP}(\omega) \ge \sum_{i=L_0}^{L_0+K_1-1} \frac{dPT^i}{dP}(\omega) + \cdots + \sum_{i=L_t}^{L_t+K_{t+1}-1} \frac{dPT^i}{dP}(\omega) + \cdots.$$

We will show that for all $t \ge 0$, $\sum_{i=L_t}^{L_t+K_{t+1}-1} \frac{dPT^i}{dP}(\omega) \ge 1$. For each i we have $\frac{dPT^i}{dP}(\omega) = \prod_{k=-\infty}^{\infty} \frac{P_{k-i}(\omega_k)}{P_k(\omega_k)}$. Since the sum $\sum_{i=L_t}^{L_t+K_{t+1}-1} \frac{dPT^i}{dP}(\omega)$ has K_{t+1} terms, we will show that for all $t \ge 0$ the product $\prod_{k=-\infty}^{\infty} \frac{P_{k-i}(\omega_k)}{P_k(\omega_k)} \ge K_{t+1}^{-1}$.

Let a_j denote the left-end coordinate of the ν_j block and b_j denote the right-end coordinate of the ν_j block. Then

$$\prod_{k=-\infty}^{\infty} \frac{P_{k-i}(\omega_k)}{P_k(\omega_k)} = \cdots \prod_{b_{j+1}+i+1}^{a_j-1} \frac{P_{k-i}(\omega_k)}{P_k(\omega_k)} \prod_{k=a_j}^{b_j+i} \frac{P_{k-i}(\omega_k)}{P_k(\omega_k)} \cdots$$
$$\cdots \prod_{b_2+i+1}^{a_1-1} \frac{P_{k-i}(\omega_k)}{P_k(\omega_k)} \prod_{k=a_1}^{b_1+i} \frac{P_{k-i}(\omega_k)}{P_k(\omega_k)} \prod_{k=b_1+i+1}^{\infty} \frac{P_{k-i}(\omega_k)}{P_k(\omega_k)}.$$

It is immediate that $\prod_{k=b_1+i+1}^{\infty} \frac{P_{k-i}(\omega_k)}{P_k(\omega_k)} = 1$ since all the $P_l(\omega_k)$ with $k > b_1 + i + 1$ are the same, i.e., $\{\frac{1}{2}, \frac{1}{2}\}$.

Assume $L_0 \le i < L_0 + K_1$.

Remark 5.4. This means that there are at most $L_0 + K_1$ coordinates of each ν_j block moving to the left into a μ block, and there are at most $L_0 + K_1$ coordinates from the μ block on the right moving into the ν_j block. For the ν_1 block there are only $n_1 < L_0 + K_1$ coordinates to move and these move completely into the μ block to the left.

It is again immediate that for each j, $\prod_{b_{j+1}+i+1}^{a_j-1} \frac{P_{k-i}(\omega_k)}{P_k(\omega_k)} = 1$, since all the $P_l(\omega_k)$ with $b_{j+1} + i + 1 \leq k \leq a_{j-1}$ are the same, i.e., $\{\frac{1}{2}, \frac{1}{2}\}$. The term $\prod_{k=a_1}^{b_1+i} \frac{P_{k-i}(\omega_k)}{P_k(\omega_k)} \geq \lambda_1^{-n_1} = I_{1,1}K_1^{-1} > K_1^{-1}$ because the entire ν_1 block, which is of length n_1, is moved into a μ block and simultaneously covered by a μ block. For $j > 1$, the terms $\prod_{k=a_j}^{b_j+i} \frac{P_{k-i}(\omega_k)}{P_k(\omega_k)} \geq \lambda_j^{-i} \geq \lambda_j^{-(L_0+K_1)}$. This follows from the above remark. Hence, the product $\prod_{k=-\infty}^{\infty} \frac{P_{k-i}(\omega_k)}{P_k(\omega_k)} \geq \lim_{t\to\infty} I_{t,1}K_1^{-1} \geq K_1^{-1}$.

Using a similar argument, we see that for $s > 1$, $L_{s-1} \leq i \leq L_{s-1} + K_s - 1$, $\prod_{k=a_s}^{b_1+i} \frac{P_{k-i}(\omega_k)}{P_k(\omega_k)} \geq \prod_{j=1}^{s} \lambda_j^{-n_j} = I_{s,s}K_s^{-1}$. For all $t > s$, $\prod_{k=a_t}^{b_t+i} \frac{P_{k-i}(\omega_k)}{P_k(\omega_k)} \geq \lambda_t^{-i} \geq \lambda_t^{-(L_{s-1}+K_s)}$. Therefore, $\prod_{-\infty}^{\infty} \frac{P_{k-i}(\omega_k)}{P_k(\omega_k)} \geq \lim_{t\to\infty} I_{t,s}K_s^{-1} \geq K_s^{-1}$. □

Lemma 5.5. *S exact with respect to Q.*

Proof. Associated to (S, X, Q) is the usual odometer map R except that the coordinates of X are negative: if x is a point of X with $x_{-i} = 1$ for $i = 0, 1, 2, \cdots, k-1$ and $x_{-k} = 0$ then $(Rx)_{-i} = 0$, $i = 0, \cdots, k-1$, $(Rx)_{-k} = 1$ and $(Rx)_{-j} = x_{-j}$ for all $j > k$.

From the construction of Q, it follows that (R, X, Q) is a nonsingular ergodic transformation [4]. It then follows from the Kolmogorov zero-one law that S is exact. □

References

[1] J. Barnes, *Conservative exact rational maps of the sphere*, Journal of Mathematical Analysis and Applications **230** (1999), 350–374, MR 2000d:37048, Zbl 919.28012.

[2] H. Bruin and J. Hawkins, *Examples of expanding C^1 maps having no σ-finite measure equivalent to Lebesgue*, Israel J. Math. **108** (1998), 83–107, CMP 1 669 400.

[3] T. Hamachi, *On a Bernoulli shift with non-identical factor measures*, Ergod. Th. & Dyanam. Sys. **1** (1981), 273–283, MR 83i:28025, Zbl 597.28022.

[4] T. Hamachi and M. Osikawa, *Ergodic Groups of Automorphisms and Krieger's Theorems*, Seminar on Mathematical Sciences, no. 3, Keio University, Yokohama, 1981, MR 84d:46099, Zbl 472.28015.

[5] S. Kakutani, *On equivalence of infinite product measures*, Ann. of Math. **49** (1948), 214–224, MR 9,340e, Zbl 030.02303.

[6] U. Krengel, *Transformations without finite invariant measures have finite strong generators*, Contributions to Ergodic Theory and Probability (Ohio State Univ., Columbus, OH, 1970), Lecture Notes in Math., no. 160, Springer-verlag, Berlin, 1970, pp. 135–157, MR 42 #4703, Zbl 201.38303.

[7] V. Rohlin, *Exact endomorphisms of a Lebesgue space*, Amer. Math. soc. Transl. Ser. 2, **39** (1964), 1–36, Zbl 154.15703.

[8] C. Silva and P. Thieullen, *A skew product entropy for nonsingular transformations*, J. London Math. Soc. **52** (1995), 497–516, MR 97a:28015, Zbl 867.28012.

[9] P. Walters, *Roots of $n : 1$ measure preserving transformations*, J. London Math. Soc. **44** (1969), 7–14, MR 38 #1236, Zbl 159.07701.

Western Carolina University, Cullowhee, NC 28723
jbarnes@wpoff.wcu.edu http://wcuvax1.wcu.edu/~jbarnes/

Northeastern University, Boston MA 02115
eigen@neu.edu http://www.math.neu.edu/~eigen/

This paper is available via http://nyjm.albany.edu:8000/j/2000/6-5.html.