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# Doi-Koppinen Hopf Modules Versus Entwined Modules

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Abstract. A Hopf module is an $A$-module for an algebra $A$ as well as a $C$-comodule for a coalgebra $C$, satisfying a suitable compatibility condition between the module and comodule structures. To formulate the compatibility condition one needs some kind of interaction between $A$ and $C$. The most classical case arises when $A = C =: H$ is a bialgebra. Many subsequent variants of this were unified independently by Doi and Koppinen; in their version an auxiliary bialgebra $H$, over which $A$ is a comodule algebra and $C$ a module coalgebra, governs the compatibility. Another very general type of interaction between $A$ and $C$ is an entwining map as studied by Brzeziński — without an auxiliary bialgebra.

Every Doi-Koppinen datum induces an entwining structure, so Brzeziński's notion of an entwined module includes that of a Doi-Koppinen Hopf module. This paper investigates whether the inclusion is proper.

By work of Tambara, every entwining structure can be obtained from a suitable Doi-Koppinen datum whenever the algebra under consideration is finite dimensional.

We show by examples that this need not be true in general.

Contents



## 1. The definitions

Throughout the paper $k$ is a field, and all algebras, coalgebras etc. are meant to be over $k$. We denote the multiplication map of an algebra $A$ by $\nabla\colon A \otimes A \to A$, and the comultiplication of a coalgebra $C$ by $\Delta\colon C \to C \otimes C$, using Sweedler's notation in the form $\Delta(c) = c_{(1)} \otimes c_{(2)}$. We refer to [4] for the general background on Hopf algebra theory.

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Let $H$ be a Hopf algebra. A (right-right) Hopf module over $H$ is a right $H$-module as well as right $H$-comodule $M$ with the property that the comodule structure map $\rho\colon M \to M\otimes H$ is an $H$-module map; here $M\otimes H$ is endowed with the diagonal right $H$-module structure. This means that $(mh)_{(0)}\otimes(mh)_{(1)} = m_{(0)}h_{(1)}\otimes m_{(1)}h_{(2)}$ holds for $m \in M$ and $h \in H$, when we write the comodule structure as $\rho(m) = m_{(0)}\otimes m_{(1)}$. Equivalently, the $H$-module structure $\mu\colon M\otimes H \to M$ is an $H$-comodule map. This ur-notion of Hopf module has seen far-reaching generalizations. First of all, it makes perfect sense to replace the $H$-module structure by an $A$-module structure for an $H$-comodule algebra $A$, or, dually, to replace the $H$-comodule structure by a $C$-comodule structure for an $H$-module coalgebra $C$. Moreover, one has reason to study Hopf modules which are $A$-modules for an $H$-comodule algebra $A$, but only $\overline{H}$-comodules for some quotient coalgebra and right module $\overline{H}$ of $H$. To unify all these situations Doi [2] and Koppinen [3] have introduced what we will call Doi-Koppinen Hopf modules with respect to a Doi-Koppinen datum. By definition, a Doi-Koppinen datum is a triple $(A, C, H)$ in which $H$ is a bialgebra, $A$ is a right $H$-comodule algebra, and $C$ is a right $H$-module coalgebra whose module structure we denote by a dot. A Doi-Koppinen Hopf module with respect to the Doi-Koppinen datum $(A, C, H)$ is a right $A$-module as well as right $C$-comodule $M$ satisfying the compatibility condition $\rho(ma) = m_{(0)}a_{(0)} \otimes m_{(1)} \cdot a_{(1)}$ for all $a \in A$ and $m \in M$. Again, we may formulate this as the condition that the module structure is a $C$-colinear map, or, equivalently, the comodule structure is an $A$-module map, by making the obvious definitions.

As Brzeziński [1] realized, all that is really needed to write down a Hopf module-like compatibility between a right $A$-module and right $C$-comodule structure is a so-called entwining structure: This is by definition a triple $(A, C, \psi)$ in which $\psi$ is a map $\psi\colon C \otimes A \to A \otimes C$ satisfying

$$(A \otimes \Delta)\psi = (\psi \otimes C)(C \otimes \psi)(\Delta \otimes A)\colon C \otimes A \to A \otimes C \otimes C,$$
$$\psi(C \otimes \nabla) = (\nabla \otimes C)(A \otimes \psi)(\psi \otimes A)\colon C \otimes A \otimes A \to A \otimes C,$$

$(A \otimes \varepsilon)\psi = \varepsilon \otimes A$, and $\psi(c \otimes 1) = 1 \otimes c$. An entwined module with respect to the entwining structure $\psi$ is by definition a right $A$-module and right $C$-comodule $M$ such that

$$\rho\mu = (\mu \otimes C)(M \otimes \psi)(\rho \otimes A)\colon M \otimes A \to M \otimes C.$$

Every Doi-Koppinen datum induces an entwining structure, namely

$$\psi\colon C \otimes A \ni c \otimes a \mapsto a_{(0)} \otimes c \cdot a_{(1)} \in A \otimes C.$$

The entwined modules with respect to this entwining structure are precisely the Doi-Koppinen Hopf modules for the Doi-Koppinen datum in consideration.

While entwining structures are sometimes harder to cope with notationally, they offer a clear conceptual advantage over Doi-Koppinen data: The auxiliary bialgebra that is needed to formulate the compatibility condition for a Doi-Koppinen Hopf module is no longer needed for entwining structures. However, the obvious question arises whether entwining structures are a truly more general notion than Doi-Koppinen data, or whether in fact every entwining structure is induced by a suitable Doi-Koppinen datum.

## 2. The finite dimensional case

If the algebra $A$ in an entwining structure $(A, C, \psi)$ is finite dimensional, then work of Tambara [5] shows that there is in fact a suitable bialgebra $H$ and Doi-Koppinen datum $(A, C, H)$ inducing the entwining map $\psi$.

Let us rephrase the results of Sections 1 and 2 of [5] that are relevant for our question: Let $A$ be an algebra. A transition map through $A$ is a pair $(V, \psi)$ in which $V$ is a vector space and $\psi\colon V \otimes A \to A \otimes V$ is a linear map satisfying

$$(\nabla \otimes V)(A \otimes \psi)(\psi \otimes A) = \psi(V \otimes \nabla)\colon V \otimes A \otimes A \to A \otimes V$$

and $\psi(v \otimes 1) = 1 \otimes v$. Transition maps through $A$ form a monoidal category with $(V, \psi) \otimes (W, \phi) := (V \otimes W, (\psi \otimes W)(V \otimes \phi))$. Now if $A$ is finite dimensional, then there exists a bialgebra $e(A) = a(A, A)$ and an $e(A)$-comodule algebra structure $A \to A \otimes e(A)$ with the following property: A category equivalence between the category of right $e(A)$-modules and the category of transition maps through $A$ is given by assigning to an $e(A)$-module $V$ the transition map $(V, \psi)$ with $\psi(v \otimes a) = a_{(0)} \otimes v \cdot a_{(1)}$.

From the results of Tambara we have thus summed up we conclude immediately:

**Proposition 2.1.** *If the algebra $A$ has finite dimension, then every entwining structure $(A, C, \psi)$ is induced by a Doi-Koppinen datum.*

In fact we need only add the observation that an entwining structure $(A, C, \psi)$ is the same as a coalgebra in the monoidal category of transition maps through $A$, hence the same as a coalgebra in the monoidal category of $e(A)$-modules; thus $(A, C, \psi)$ is induced by a Doi-Koppinen datum $(A, C, e(A))$.

## 3. Counterexamples

There exist entwining structures $(A, C, \psi)$ that cannot be obtained from a Doi-Koppinen datum $(A, C, H)$; this can even happen for finite dimensional $C$.

The examples rely on the following easy observation.

**Lemma 3.1.** *Let $(A, C, \psi)$ be an entwining structure. Fix $c \in C$ and $\gamma \in C^*$, and define the vector space endomorphism $T_{c,\gamma}$ of $A$ by $T_{c,\gamma}(a) = (A \otimes \gamma)\psi(c \otimes a)$.*

*If the entwining structure $(A, C, \psi)$ is induced by a Doi-Koppinen datum, then every $a \in A$ is contained in a finite dimensional $T_{c,\gamma}$-invariant subspace of $A$.*

**Proof.** If $(A, C, \psi)$ is induced by the Doi-Koppinen datum $(A, C, H)$, then we find $T_{c,\gamma}(a) = a_{(0)}\gamma(c \cdot a_{(1)})$. From this formula it is obvious that every $H$-subcomodule of $A$ is a $T_{c,\gamma}$-invariant subspace. But every $a \in A$ is contained in a finite dimensional $H$-subcomodule of $A$. $\square$

**Example 3.2.** Let $C$ be the two-dimensional coalgebra $k1 \oplus kt$ with grouplike element 1 and $(1, 1)$-primitive $t$. Let $A$ be the free algebra on generators $X_i$ for $i \in \mathbb{Z}$. Define the entwining map $\psi\colon C \otimes A \to A \otimes C$ by $\psi(1 \otimes a) = a \otimes 1$ for all $a \in A$ and

$$\psi(t \otimes X_{i_1} X_{i_2} \dots X_{i_n}) = X_{i_1+1} X_{i_2+1} \dots X_{i_n+1} \otimes t.$$

Then $(A, C, \psi)$ is an entwining structure that cannot be obtained from any Doi-Koppinen datum.

**Proof.** It is easy to check directly that $\psi$ is an entwining map; see below for a more conceptual idea. Choose $\gamma \in C^*$ with $\gamma(t) = 1$. Then we have $T_{t,\gamma}(X_i) = X_{i+1}$. Thus the $T_{t,\gamma}$-invariant subspace of $A$ generated by $X_0$ is infinite dimensional. We conclude that $(A, C, \psi)$ cannot be obtained from any Doi-Koppinen datum. $\square$

While our counterexample cannot be obtained from a Doi-Koppinen datum, it can be derived from something very similar, which we will call an alternative Doi-Koppinen datum. This is by definition a triple $(A, C, H)$ consisting of a bialgebra $H$, a left $H$-module algebra $A$, and a left $H$-comodule coalgebra $C$; we will write the comodule structure of the latter as $c \mapsto c_{[-1]} \otimes c_{[0]}$. One can check that every alternative Doi-Koppinen datum induces an entwining structure in a fashion very similar to the case of a Doi-Koppinen datum: One defines $\psi\colon C \otimes A \to A \otimes C$ by $\psi(c \otimes a) = c_{[-1]} \cdot a \otimes c_{[0]}$. We omit the necessary verifications as they are very similar to the ones for Doi-Koppinen data. Our example above can be obtained from an alternative Doi-Koppinen datum $(C, A, k\mathbb{Z})$: The necessary comodule structure on $C$ corresponds to the $\mathbb{Z}$-grading of $C$ in which 1 and $t$ have degrees 0 and 1, respectively, and the module algebra structure on $A$ is that for which the generator $1 \in \mathbb{Z}$ shifts the indices in all the free generators $X_i$ by one.

**Remark 3.3.** If the coalgebra $C$ has finite dimension, then every entwining structure $(A, C, \psi)$ is induced by an alternative Doi-Koppinen datum.

We will not supply any details of the proof, which is very similar to that of Proposition 2.1; we only remark that Tambara's $a(C^*, C^*)$ can serve as the necessary bialgebra.

The question remains whether there exist entwining structures $(A, C, \psi)$ that is induced neither by a Doi-Koppinen datum, nor by an alternative Doi-Koppinen datum.

**Example 3.4.** Let $C = k1 \oplus \bigoplus_{i\in\mathbb{Z}} kt_i$ where 1 is grouplike and each $t_i$ is $(1,1)$-primitive, and we let $A = k[\tau_i \,|\, i \in \mathbb{Z}]/(\tau_i\tau_j \,|\, i, j \in \mathbb{Z})$. An entwining structure $\psi\colon C \otimes A \to A \otimes C$ can be defined by $\psi(1 \otimes \tau_j) = \tau_j \otimes 1$, $\psi(t_i \otimes 1) = 1 \otimes t_i$, and $\psi(t_i \otimes \tau_j) = \tau_{j+1} \otimes t_{i+1}$.

If $\gamma \in C^*$ satisfies $\gamma(t_1) = 1$, then we find $T_{t_0,\gamma}(\tau_j) = \tau_{j+1}$ for all $j \in \mathbb{Z}$, so that the $T_{t_0,\gamma}$-invariant subset of $A$ generated by $\tau_0$ is infinite dimensional. Thus $\psi$ is not induced by a Doi-Koppinen datum. A similar argument using the endomorphism $T'\colon C \to C$ defined by $T'(c) = (\alpha \otimes C)\psi(c \otimes \tau_0)$ with $\alpha \in A^*$ satisfying $\alpha(\tau_1) = 1$ shows that $\psi$ is not induced by an alternative Doi-Koppinen datum. We omit checking that $\psi$ is an entwining map.

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