New York Journal of Mathematics

New York J. Math. **5** (1999) 53–81.

# Geometric $K$-Homology and Controlled Paths

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Abstract. We show that $K$-homologous differential operators on an oriented, Riemannian manifold $M$ can be connected by a "controlled path" of operators. The analytic properties of these paths allows us to measure a winding number (in the sense of de la Harpe and Skandalis). To aid in the exposition we develop a variant of Baum's $(M, E, f)$ model for $K$-homology. Our model removes the need for $Spin^c$ structures in the description of geometric $K$-homology.

Contents



## 1. Introduction

The topological $K$-theory of a locally compact, Hausdorff space is a well understood generalized cohomology theory defined in terms of equivalence classes of stable vector bundles over the space. Its dual theory, $K$-homology, has been defined in several ways, some of which are very analytical in their flavour. Atiyah [Ati69] provided the first clue for the definition of $K$-homology in terms of elliptic pseudodifferential operators on the space—Brown, Douglas and Fillmore provided an analytic definition by developing the theory of extensions of $C^*$-algebras [BDF77]

Received May 18, 1999.
*Mathematics Subject Classification.* 19K56, 46L80.
*Key words and phrases.* $K$-homology, Dirac-type operator, Finite-propagation speed, Trace class operators.


ISSN 1076-9803/99

and Kasparov formalised Atiyah's suggestion by using equivalence classes of generalized elliptic operators to realise $K$-homology [Kas75]. These definitions essentially equate $K_*(M)$, the $K$-homology of a compact manifold $M$, to the group of extensions of the algebra $C(M)$ (of complex valued, continuous functions on $M$) by the compact operators $\mathcal{K}$ (on an infinite dimensional, complex, separable Hilbert space). For a Schatten ideal $\mathcal{L}^p$, the notion of $\mathcal{L}^p$-smooth elements in $K_*(M)$ was introduced and studied in [Dou81]. These results were extended in [Sal83, Gon90] and more recently, in [Wan92].

Our work in this paper started in an attempt to develop an $\mathcal{L}^1$-smooth model for the $K$-homology of a compact manifold $M$. Cycles for this model were to be pairs $(\mathcal{H}, \mathcal{F})$ where

(1) $\mathcal{H}$ is a Hilbert space satisfying the same conditions as in Kasparov's model (see Definition 2.1);
(2) $\mathcal{F}$ is a bounded operator on $\mathcal{H}$ satisfying the following conditions:
  (a) $\mathcal{F}^2 - 1 \in \mathcal{L}^1$ or $\mathcal{F}^* - \mathcal{F} \in \mathcal{L}^1$ (these would correspond to the groups $K_0$ or $K_1$ respectively);
  (b) $[F, f] \in \mathcal{L}^1$ for $f \in C(M)$; and
  (c) $\mathcal{F}$ satisfies the condition of polynomial growth (when represented as an integral operator, its kernel blows up at a polynomial rate as we approach the diagonal—see Definition 3.3).

The concept of a degenerate cycle was to be as in Kasparov's definition and the equivalence relation was to be by norm-continuous paths of operators which satisfy the conditions on $\mathcal{F}$ above and which also satisfy uniformly, the polynomial growth condition.

We were unable to prove a suitable technical theorem [Hig87] necessary to define the product in this model. However, what came out of our investigation was the discovery that our equivalence relation (through controlled paths of operators of the form $(\mathcal{H}, \mathcal{F})$) is a suitable substitute for the usual notion of norm-continuous paths of generalized elliptic operators [Kas75], at least for the $K$-homology of a compact Riemannian manifold (Theorem 3.8).

The key property of controlled paths is that they have a winding number in the sense of de la Harpe and Skandalis (see [Kes99, Lemma 4.1.7]). We have used this in [Kes99] (see also [Kes98]) to prove the homotopy invariance of the relative eta-invariant for manifolds whose fundamental group is torsion-free and for which the assembly map $\mu_{\max} : K_*(B\pi_1(M)) \to K_*(C^*_{\max}(\pi_1(M)))$ is an isomorphism [BCH94] (see also [Wei88]).

Higson has pointed out that another application of this result would be a more functorial proof of the Connes-Moscovici index theorem [CM90].

In proving the main theorem (3.8) we initially used Baum's $(M, E, f)$ model for $K$-homology . The presence of $Spin^c$ structures in this theory meant that to prove our result for the case of oriented manifolds we had to work with sphere bundles etc. To improve the exposition we have developed an "$(M, S, g)$" model for geometric $K$-homology in which the basic cycles are made up of manifolds which are only oriented (see Definition 2.6).

This paper is organized as follows: Section 2 defines Kasparov's analytic $K$-homology (denoted $K^a_*$), Baum's topological model (denoted $K^{top}_*$) and following Higson, the "$(M, S, g)$" model (denoted $K^h_*$). We show that they all define the same theory and then proceed to use $K_*^h$ for our model of $K$-homology in the rest of the paper. Section 3 defines the notion of a controlled path and states the main theorem. Section 4 is a technical section into which we have collected the technical results necessary to prove the main theorem and in Section 5 we give the proof.

This work along with [Kes98, Kes99] is based on the author's PhD thesis [Kes97]. We would like to thank Jonathan Rosenberg and Nigel Higson for their suggestions, guidance and moral support, without which it is difficult to imagine this work coming to fruition.

## 2. Geometric $K$-homology: $(M, S, g)$ theory

In this section we will review Kasparov and Baum's definitions of $K$-homology and present a variant of Baum's definition that removes the dependence on $Spin^c$ structures. We start with Kasparov's definition [Kas75].

Let $\mathcal{K} = \mathcal{K}(\mathcal{H})$ be the algebra of compact operators on a Hilbert space $\mathcal{H}$ (which is infinite-dimensional, complex and separable).

**Definition 2.1.** Let $A$ be a $C^*$-algebra. Consider triples $(\mathcal{H}, \phi, F)$, where $\mathcal{H}$ is a Hilbert space, $\phi : A \to \mathcal{B}(\mathcal{H})$ is a $*$-homomorphism and $F \in \mathcal{B}(\mathcal{H})$ is an operator satisfying the following conditions :

(1) $\phi(a)(F - F^*) \in \mathcal{K}$,
(2) $\phi(a)(F^2 - 1) \in \mathcal{K}$,
(3) $[\phi(a), F] \in \mathcal{K}$.

A triple $(\mathcal{H}, \phi, F)$ is called degenerate if the compact operators appearing in (1), (2) and (3) are 0. Two such triples are regarded as equivalent if there is a continuous map $[0, 1] \to \mathcal{B}(\mathcal{H}) : t \to F_t$ such that for all $t$, the triple $(\mathcal{H}, \phi, F_t)$ satisfies the conditions above. A commutative semigroup (with respect to the operation of direct sum) is constructed from these equivalence classes and $K^1(A)$ is the abelian group obtained by taking the quotient of this semigroup by the degenerates.

$K^0(A)$ is defined similarly, except we require $\mathcal{H}$ to be a $\mathbb{Z}_2$-graded Hilbert space, $\phi$ to be degree zero ($\phi(a)$ preserves the grading for every $a \in A$), and $F$ to be degree 1 (it reverses the grading of $\mathcal{H}$).

If $X$ is a locally compact topological space then the (analytic) $K$-homology of $X$, $K_*^a(X)$ is defined by

$$K_*^a(X) = K^*(C_0(X)).$$

This definition is modelled on the properties of elliptic pseudodifferential operators on manifolds (see [Ati69]). Condition (2) above codes the property of the existence of a parametrix while condition (3) codes the property of pseudolocality [Tay81]. For this reason, we shall henceforth call an operator satisfying the above conditions (1) through (3), an *abstract elliptic operator*.

Paul Baum's $(M, E, f)$ theory (as it is commonly known), provides a manifold theoretic definition of $K$-homology [BD82, BDb82].

**Definition 2.2.** A $K^{top}$-cycle on a topological space $X$ is a triple $(M, E, f)$ such that:

(1) $M$ is a compact, Riemannian, $Spin^c$ manifold without boundary. Let $\mathcal{S}_M$ be the spinor bundle of $M$. We require that there be a connection on $\mathcal{S}_M$

that is compatible with the Levi-Civita connection on $M$ in the sense that for vector fields $X, Y$ and $s \in C^\infty(\mathcal{S}_M)$,

$$\nabla_X(Ys) = (\nabla_X Y)s + Y\nabla_X s; \tag{2.1}$$

(2) $E$ is a complex Hermitian vector bundle on $M$ equipped with a connection that is compatible with the inner product on $E$ (see [Roe88, 1.27]);

(3) $f$ is a continuous map from $M$ to $X$.

$M$ is not required to be connected and its components need not have the same dimension; $E$ may have different fiber dimension on different connected components of $M$. Thus for $K^{top}$-cycles on $X$, there is the evident disjoint union operation. Denote this by $(M_1, E_1, f_1) \cup (M_2, E_2, f_2)$. Two $K^{top}$-cycles $(M, E, f)$ and $(M', E', f')$ are *isomorphic* if there exists a diffeomorphism $h$ mapping $M$ onto $M'$, preserving the Riemannian and $Spin^c$ structures, $h^*(E') \cong E$ (the connections on $E$ and $S$ are the pullbacks of the connections on $E'$ and $S'$ respectively), and the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{h} & M' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ X & = & X. \end{array}$$

Let $\Pi(X)$ be the collection of all $K^{top}$-cycles on $X$. Define an equivalence relation $\sim$ on $\Pi(X)$ generated by the following three elementary steps.

(1) *Bordism*: $(M_0, E_0, f_0) \sim (M_1, E_1, f_1)$ if there exists a compact Riemannian, $Spin^c$ manifold $W$ with boundary, a complex Hermitian vector bundle $E$ on $W$ and a continuous map $f : W \to X$ such that $(\partial W, E|_{\partial W}, f|_{\partial W})$ is isomorphic to the disjoint union $(M_0, E_0, f_0) \cup (-M_1, E_1, f_1)$. Here $-M_1$ denotes $M_1$ with the $Spin^c$ structure on $TM_1$ reversed [BDb82, Appendix]. We require that the connection on the $Spin^c$ structure on $\partial W$ is isomorphic to the connection on the $Spin^c$ structure on the disjoint union $M_0 \cup -M_1$ and that the connection on $E|_{M_i}$ is isomorphic to that on the bundles $E_i$ (for $i = 0, 1$). Also, we require that there be a collaring neighborhood of the boundary $\partial W$ over which the cycle $(W, E, f)$ is a Riemannian product, in the natural sense of the term.

(2) *Direct sum*: Suppose given $(M, E, f)$ and also given a direct sum decomposition $E = E_1 \oplus E_2$. Then,

$$(M, E_1 \oplus E_2, f) \sim (M, E_1, f) \cup (M, E_2, f).$$

(3) *Vector bundle modification*: Let $M$ be a $Spin^c$ manifold. On $M$ let $H$ be a $C^\infty$ $Spin^c$ vector bundle with even dimensional fibers. Let 1 denote the trivial real line bundle on $M$—so $1 = M \times \mathbb{R}$. Choose a smooth, positive-definite symmetric inner product on $H$ and hence on $H \oplus 1$. Let $\widehat{M} = S(H \oplus 1)$ be the unit sphere bundle of $H \oplus 1$. The $Spin^c$ structures on $TM$ and $H$ give a $Spin^c$ structure on $T\widehat{M}$ and so $\widehat{M}$ is a $Spin^c$ manifold. Let $\rho : \widehat{M} \to M$ denote the projection to the zero section.

Fix a point $p \in M$ and let $n = \dim(H)$. Since $H$ has a $Spin^c$ structure, there is a given associated bundle $\mathcal{S}_H$ of Clifford modules over $TM$ such that $Cl(H) \otimes \mathbb{C} \cong End(\mathcal{S}_H)$. There is a natural grading on $\mathcal{S}_H$ induced by Clifford multiplication by the volume element that allows us to write

$\mathcal{S}_H = \mathcal{S}_H^+ \oplus \mathcal{S}_H^-$. Let $H_0$ and $H_1$ denote the pullbacks of $\mathcal{S}_H^+$ and $\mathcal{S}_H^-$ to $H$. Then, $H$ acts on $H_0, H_1$ by Clifford multiplication and this gives a vector bundle map $\sigma : H_0 \to H_1$.

Now $\widehat{M} = S(H \oplus 1)$ can be thought of as two copies of the unit ball bundle of $H$, $B_0(H)$ and $B_1(H)$, glued together by the identity map of $S(H)$—i.e $\widehat{M} = B_0(H) \cup_{S(H)} B_1(H)$. Form a vector bundle $\widehat{H}$ on $\widehat{M}$ by putting $H_0$ on $B_0(H)$ and $H_1$ on $B_1(H)$ and then clutching these two bundles along $S(H)$ by the map $\sigma$ [Kar78]. So, $\widehat{H}$ is constructed by gluing together the two Clifford bundles $\mathcal{S}_H^+$ and $\mathcal{S}_H^-$, one over the northern hemisphere, the other over the southern hemisphere—the gluing operation being described by Clifford multiplication.

Notice that starting with $M, H$ this clutching construction has produced $\widehat{M}, \widehat{H}, \rho$.

Suppose now given $(M, E, f)$ and a $C^\infty$ $Spin^c$ vector bundle $H$ on $M$ with even-dimensional fibers. Use the above construction to obtain $\widehat{M}, \widehat{H}, \rho$. Then the relation of vector bundle modification is given by:

$$(M, E, f) \sim (\widehat{M}, \widehat{H} \otimes \rho^*(E), f \circ \rho).$$

**Definition 2.3.** Set

$$K_*^{top}(X) = \Pi(X)/\sim.$$

$K_*^{top}(X)$ is an abelian group with respect to the operation of disjoint union. Note that for a $K^{top}$-cycle $(M, E, f)$ on $X$, the equivalence relation $\sim$ preserves the parity of the dimension of $M$. In $K_*^{top}(X)$ let $K_0^{top}(X)$ (respectively $K_1^{top}(X)$), be the subgroups given by all $(M, E, f)$ with each connected component of $M$ even dimensional (respectively odd dimensional). Then

$$K_*^{top}(X) = K_0^{top}(X) \oplus K_1^{top}(X).$$

2.1. **Isomorphism with analytic *K*-homology.** The isomorphism

$$\xi : K_*^{top}(X) \to K_*^a(X)$$

has a description in terms of the Dirac operator $D$ on a $Spin^c$-manifold $M$. Recall that this is an order 1, elliptic differential operator on $M$ defined on the space of smooth sections of the spinor bundle $\mathcal{S}_M$ on $M$ [BD82, Roe88, LM90]. Given a vector bundle $E$ over $M$ with a connection, we can form the Dirac operator on $M$ with coefficients in $E$,

$$D_{\mathcal{S}}^E : C^\infty(\mathcal{S}_M \otimes E) \to C^\infty(\mathcal{S}_M \otimes E)$$

and this is also an order 1, elliptic differential operator on $M$ [BD82, BDb82].

**Definition 2.4.** A chopping function is a function $\chi : \mathbb{R} \to \mathbb{R}$ such that

(1) $\chi$ is a continuous odd function; and
(2) $\lim_{x\to\infty} \chi(x) = 1$.

By the functional calculus for pseudodifferential operators [Tay81, Ch. 12], $\chi(D)$ and $\chi(D_{\mathcal{S}}^E)$ define abstract elliptic operators in the sense of Kasparov (see Definition 2.1).

The isomorphism between Baum's topological $K$-homology and analytic $K$-homology $\xi : K_*^{top}(X) \to K_*^a(X)$ is given by [BD82],

$$\xi(M, E, f) = [f_*(\chi(D_{\mathcal{S}}^E))].$$

2.2. **$(M, S, g)$ theory.** We now define a variant of Baum's theory. The main difference is that this theory will not use $Spin^c$ manifolds, instead the manifolds appearing here will only need to be orientable. The aim is to represent fundamental geometric operators such as the signature operator or the deRham operator on non $Spin^c$ manifolds directly as $K$-cycles. Our model is motivated by Guentner's use of $K$-homology to prove the index theorem [Gue93] and also by suggestions of Higson.

Let $Cl(TM)$ denote the bundle of Clifford algebras on a Riemannian manifold $M$—so, the fiber of $Cl(TM)$ at a point $m \in M$ is the Clifford algebra $Cl(T_m M)$ of the inner-product space $T_m M$ [Roe88].

**Definition 2.5.** [Roe88, (2.3)] Let $S$ be a bundle of Clifford modules over a Riemannian manifold $M$. $S$ is a Clifford bundle if it is equipped with a Hermitian metric and compatible connection such that

(1) The Clifford action of a vector $v \in T_m M$ on $S_m$ is skew-adjoint:

$$(vs_1, s_2) + (s_1, vs_2) = 0;$$

(2) The connection on $S$ is compatible with the Levi-Civita connection on $M$ as in (2.1).

Let $S$ be a Clifford bundle over a Riemannian manifold $M$ and let $\gamma : Cl(TM) \otimes S \to S$ be the Clifford module structure. Define a new module structure $\tilde{\gamma} : Cl(TM) \otimes S \to S$ by

$$\tilde{\gamma}(v \otimes s) = -\gamma(v \otimes s), \quad v \in T_m M, s \in S_m.$$

$S$ with the new module structure $\tilde{\gamma}$ is denoted by $-S$. $-S$ is said to have the *opposite Clifford structure* to $S$.

**Definition 2.6.** Let $X$ be a topological space. A $K^h$-cycle [1] for $X$ is a triple $(M, S, g)$ such that:

(1) $M$ is a smooth, compact, oriented Riemannian manifold of dimension $n$;
(2) $S$ is a Clifford bundle on $M$; and
(3) $g : M \to X$ is a continuous map.

As in Baum's theory, $M$ is not required to be connected and its components need not have the same dimension; $S$ must have locally constant fiber dimension. Thus there is an evident disjoint union operation between the $K^h$-cycles on $X$—denote this by $(M_1, S_1, g_1) \cup (M_2, S_2, g_2)$. The notion of isomorphism of $K^h$-cycles is as in Baum's definition.

If $M$ is an even-dimensional, oriented, Riemannian manifold then a Clifford bundle $S$ over $M$ has a $\mathbb{Z}_2$-grading given by Clifford multiplication by the volume element [LM90, II.6]. In this case we will denote the decomposition of $S$ as $S = S^+ \oplus S^-$. If $M$ is odd-dimensional then such a grading does not exist.

Let $\Sigma(X)$ be the collection of all $K^h$-cycles on $X$. Define an equivalence relation $\sim$ on $\Sigma(X)$ generated by the following three elementary steps.

(1) *Bordism*: $(M_0, S_0, g_0) \sim (M_1, S_1, g_1)$ if there exists a compact, oriented Riemannian manifold $W$ with boundary $\partial W$, a Clifford bundle $S$ on $W$ and a map $g : W \to X$ such that:

[1] The superscript h if for Higson who suggested this definition to us.

(a) If $M_0, M_1$ are even-dimensional then

$$(\partial W, S|_{\partial W}, g|_{\partial W}) = (M_0, S_0, g_0) \cup (-M_1, S_1, g_1) \text{ and;}$$

(b) If $M_0, M_1$ are odd-dimensional then

$$(\partial W, S^+|_{\partial W}, g|_{\partial W}) = (M_0, S_0, g_0) \cup (-M_1, -S_1, g_1).$$

Here $-M_1$ denotes $M_1$ with the orientation on $M_1$ reversed. We also require that the connection on $S|_{M_i}$ is isomorphic to the connection on the bundles $S_i$ (for $i = 0, 1$). Finally, we require that there be a collaring neighborhood of the boundary $\partial W$ over which $(W, S, g)$ is a Riemannian product in the natural sense of the term. This definition is motivated by Baum's definition of Bordism—see [BD82], [BDb82, Appendix].

(2) *Direct Sum*: Suppose given $(M, S, g)$ and also given a direct sum decomposition $S = S_1 \oplus S_2$. Then,

$$(M, S_1 \oplus S_2, g) \sim (M, S_1, g) \cup (M, S_2, g).$$

(3) *Vector bundle modification*: (cf. [Gue93, §4.3]) Let $(M, S, g)$ be a $K$-cycle on $X$. A sphere bundle over $M$ is a fiber bundle with fiber $S^{2n}$ and structure group $SO(2n)$—for example it is the unit sphere bundle of $H \oplus 1$ where $H$ is an even dimensional, oriented Riemannian vector bundle over $M$ and 1 is the trivial complex line bundle over $M$. Let $\widehat{M}$ be a sphere bundle over $M$ with projection $\pi : \widehat{M} \to M$. The bundle of *vertical* tangent vectors is $T_{vert}\widehat{M} = \ker \pi_* \subset T\widehat{M}$. Suppose $\widehat{M}$ is equipped with an orientation and Riemannian structure compatible with those of $M$ and $S^{2n}$, meaning that:

(a) The restriction of the metric to each fiber of $\widehat{M}$ gives the standard metric on the sphere $S^{2n}$.

(b) The projection $\pi : M \to \widehat{M}$ is a Riemannian submersion. That is, if the bundle of horizontal tangent vectors is defined by $T_{horiz}\widehat{M} = (T_{vert}\widehat{M})^{\perp} \subset T\widehat{M}$, then for $\widehat{p} \in \widehat{M}$ the restriction of $\pi_*$ to $T_{horiz}\widehat{M}_{\widehat{p}}$ is an isometry onto $TM_{\pi(\widehat{p})}$.

(c) If $T_{vert}(\widehat{M})$ and $T_{horiz}(\widehat{M})$ are equipped with orientations such that the inclusion of the fiber in (a) and submersion in (b) are orientation preserving, then $T\widehat{M}$ is oriented as the direct sum $T_{vert}(\widehat{M}) \oplus T_{horiz}(\widehat{M})$. The complexified exterior algebra $\Lambda^*_{\mathbb{C}} T_{vert}(\widehat{M})$ is a Hermitian bundle on $\widehat{M}$. There are decompositions:

$$\Lambda^*_{\mathbb{C}} T_{vert}(\widehat{M}) = \Lambda^{even}_{\mathbb{C}} T_{vert}(\widehat{M}) \oplus \Lambda^{odd}_{\mathbb{C}} T_{vert}(\widehat{M})$$

$$\Lambda^*_{\mathbb{C}} T_{vert}(\widehat{M}) = \Lambda^{+}_{\mathbb{C}} T_{vert}(\widehat{M}) \oplus \Lambda^{-}_{\mathbb{C}} T_{vert}(\widehat{M}),$$

corresponding to the deRham grading operator $\epsilon_1$ and the signature grading operator $\epsilon_2$ (Clifford multiplication by the volume element) respectively.

Let $V$ be the $+1$ eigenbundle of $\epsilon_1\epsilon_2$. Equip $V$ with a connection that is compatible with the metric and with the Levi-Civita connection on $\widehat{M}$. Let $c_V$ denote "Clifford multiplication" of $T_{vert}(\widehat{M})$ on $V$ (via internal and external multiplication [Roe88, Lemma 1.12]) and let $c_S$ denote "Clifford multiplication" of $TM$ on $S$. Let $\epsilon_V$ be the grading operator on $V$ given by the restriction of $\epsilon_1$ to $V$ and define a Clifford multiplication on $\widehat{S} = \pi^*(S) \otimes V$ by,

$$\widehat{\epsilon} = \epsilon \otimes \epsilon_V$$

$$\widehat{c}(v) = c_S(\pi_* v) \otimes 1 \oplus \epsilon \otimes c_V(v_{vert}), \quad v \in T\widehat{M},$$

where $v = v_{horiz} \oplus v_{vert} \in T_{horiz}\widehat{M} \oplus T_{vert}\widehat{M}$ is the decomposition into horizontal and vertical components. Finally equip $\widehat{S}$ with the inner product and compatible connection induced by those on $V$ and $E$—thus we make $\widehat{S}$ a Clifford bundle over $\widehat{M}$.

Letting $\widehat{g} = g \circ \pi$, the vector bundle modification relation is

$$(M, S, g) \sim (\widehat{M}, \widehat{S}, \widehat{g}).$$

As in Baum's $(M, E, f)$-theory define

$$K_*^h(X) = \Sigma(X)/\sim .$$

$K_*^h(X)$ is an abelian group with respect to the operation of disjoint union. Note that for a $K$-cycle $(M, S, g)$ on $X$, the equivalence relation $\sim$ preserves the parity of the dimension of $M$. In $K_*^h(X)$ let $K_0^h(X)$ (respectively $K_1^h(X)$), be the subgroups given by all $(M, S, g)$ with each connected component of $M$ even dimensional (respectively odd dimensional). Then

$$K_*^h(X) = K_0^h(X) \oplus K_1^h(X).$$

2.3. **Isomorphism between $K_*^{top}(X)$ and $K_*^h(X)$.** Let

$$\Phi : K_*^{top}(X) \to K_*^h(X)$$

be given by

$$\Pi(X) \ni [(M, E, f)] \to [(M, E \otimes \mathcal{S}_M, f)] \in \Sigma(X),$$

where $\mathcal{S}_M$ is the $Spin^c$ structure on the $Spin^c$ manifold $M$. Since $\mathcal{S}_M$ is naturally a Clifford bundle [LM90, II.7], $E \otimes \mathcal{S}_M$ is also a Clifford bundle [Roe88, 2.14] and so the triple $(M, E \otimes \mathcal{S}, f) \in \Sigma(X)$.

We need to check that $\Phi$ is well defined and we will show that it is an isomorphism. To do this we will first establish some crucial links between the equivalence relation on $\Pi(X)$ and $\Sigma(X)$.

**Lemma 2.7.** *If* $(M_1, E_1, f_1) \sim (M_2, E_2, f_2)$ *in* $\Pi(X)$*, then* $(M_1, E_1 \otimes \mathcal{S}_{M_1}, f_1) \sim (M_2, E_2 \otimes \mathcal{S}_{M_2}, f_2)$ *in* $\Sigma(X)$*.*

**Proof.** We treat each of the three steps of bordism, direct sum and vector bundle modification separately:

*Bordism*: Suppose given a bordism $(W, E, f)$ in $\Pi(X)$ between cycles $(M_1, E_1, f_1)$, $(M_2, E_2, f_2) \in \Pi(X)$ (so $W, M_1$ and $M_2$ are $Spin^c$ manifolds). Then by tracing through the definition of Baum's bordism equivalence relation [BD82], [BDb82, Appendix] it follows that $(W, E \otimes \mathcal{S}_W, f)$ is a bordism in $\Sigma(X)$ between the cycles $(M_1, E_1 \otimes \mathcal{S}_{M_1}, f_1), (M_2, E_2 \otimes S_{M_2}, f_2) \in \Sigma(X)$. The key point is that reversing the $Spin^c$ structure on an even dimensional $Spin^c$ manifold is tantamount to reversing the orientation of the manifold while on an odd dimensional $Spin^c$ manifold it involves reversing the orientation and using the opposite Clifford structure on the spinor bundle (see [BDb82, Appendix]).

*Direct sum*: This operation is identical in both $\Pi(X)$ and $\Sigma(X)$ and so there is nothing to show here.

*Vector bundle modification*: Let $(M, E, f)$ be a Baum $K$-cycle and let $H$ be a $2n$ dimensional $C^\infty$ $Spin^c$ vector bundle over $M$. As above let $\mathcal{S}_M$ be the $Spin^c$ structure on $M$ and $\mathcal{S}_H$ the $Spin^c$ structure on $H$. Let $(\widehat{M}, \widehat{E}, \widehat{f})$ be the result of the construction described in (3) of the equivalence relation on $\Pi(X)$. Recall that $\widehat{M} = S(H \oplus 1)$ and $\widehat{E} = \pi^*(E) \otimes \widehat{H}$ where $\widehat{H} = \mathcal{S}_H^+ \cup_\sigma \mathcal{S}_H^-$. The following claim completes the proof. □

**Claim.**

(1) The $Spin^c$ structure on $\widehat{M}$, $\mathcal{S}_{\widehat{M}}$, is isomorphic to $\pi^*(\mathcal{S}_M) \otimes \pi^*(\mathcal{S}_H)$.
(2) Let $\pi^*(\mathcal{S}_H) = \pi^*(\mathcal{S}_H)^+ \oplus \pi^*(\mathcal{S}_H)^-$ be the decomposition of $\pi^*(\mathcal{S}_H)$ given by Clifford multiplication. Then, $\widehat{H} \simeq (\pi^*(\mathcal{S}_H)^+)^*$.
(3) The $+1$ eigenbundle $V$ of the grading $\epsilon_1\epsilon_2$ (see (3) of the equivalence relation on $\Sigma(X)$) is isomorphic to $\pi^*(\mathcal{S}_H) \otimes (\pi^*(\mathcal{S}_H)^+)^*$.
(4) Let $(M, S, f) = (M, E \otimes \mathcal{S}_M, f) = \Phi(M, E, f)$ and $\widehat{M} = S(H \oplus 1)$. If $(\widehat{M}, \widehat{S}, \widehat{f})$ is obtained from $(M, S, f)$ using vector bundle modification in $\Sigma(X)$, then

$$(\widehat{M}, \widehat{S}, \widehat{g}) \simeq (\widehat{M}, \mathcal{S}_{\widehat{M}} \otimes \widehat{E}, \widehat{f}) = \Phi(\widehat{M}, \widehat{E}, \widehat{f}).$$

**Proof of Claim.** (1): Consider the diagram

$$\begin{array}{ccccc} \widehat{M} & = & S(H \oplus 1) & \xrightarrow{\subset} & H \oplus 1 \\ & & \downarrow & & \downarrow \pi \\ & & M & = & M. \end{array}$$

So,

$$T\widehat{M} \subset T(H \oplus 1)|_{\widehat{M}} = \pi^*(TM) \oplus \pi^*(H \oplus 1).$$

We will call the summands the horizontal and vertical components of $T\widehat{M}$ respectively. Thus,

$$Cl(T\widehat{M}) \subset \pi^*(Cl(TM)) \otimes \pi^*(Cl(H \oplus 1).$$

In this manner we get an action of $Cl(T\widehat{M})$ on $\pi^*(\mathcal{S}_M) \otimes \pi^*(\mathcal{S}_H)$. If $\dim(M) = m$ then the dimension of $Cl(T\widehat{M})$ is $2^{2n+m}$ and this is exactly the same as that of $End(\pi^*(\mathcal{S}_M) \otimes \pi^*(\mathcal{S}_H))$. Thus, $\pi^*(\mathcal{S}_M) \otimes \pi^*(\mathcal{S}_H)$ is a $Spin^c$ structure on $\widehat{M}$.

(2): It is a straightforward though tedious calculation to show that the clutching map for the bundle $(\pi^*(\mathcal{S}_H)^+)^*$ is exactly the same as that for $\widehat{H}$ and so as vector bundles over $\widehat{M}$, they are isomorphic.

(3): A consequence of (1) is that $\pi^*(\mathcal{S}_H)$ is a $Spin^c$ structure for $T_{vert}(\widehat{M})$. Recall that $V$ is the $+1$ eigenbundle of the grading operator $\epsilon_1\epsilon_2$ on the exterior bundle $\Lambda^*_{\mathbb{C}}(T_{vert}(\widehat{M}))$. To streamline notation, let $\Lambda^*$ denote the exterior bundle and let $\mathcal{S}$ denote $\pi^*(\mathcal{S}_H)$. Further, let $\mathcal{S}^\pm$ denote $\pi^*(\mathcal{S}_H)^\pm$. Recall (see [LM90, II.5]), that $\Lambda^* \simeq \mathcal{S} \otimes \mathcal{S}^*$. Under the de Rham grading $\epsilon_1$,

$$\begin{aligned} \mathcal{S} \otimes \mathcal{S}^* \simeq & (\mathcal{S}^+ \otimes (\mathcal{S}^+)^* \oplus \mathcal{S}^- \otimes (\mathcal{S}^-)^*) \\ & \oplus (\mathcal{S}^+ \otimes (\mathcal{S}^-)^* \oplus \mathcal{S}^- \otimes (\mathcal{S}^+)^*), \end{aligned}$$

where we have grouped together first the even-degree terms, then the odd-degree terms. Similarly, under the signature grading $\epsilon_2$,

$$\begin{aligned}\mathcal{S}\otimes\mathcal{S}^* \simeq (\mathcal{S}^+\otimes\mathcal{S}^+)^* \oplus \mathcal{S}^+\otimes(\mathcal{S}^-)^*)\\ \oplus(\mathcal{S}^-\otimes(\mathcal{S}^-)^*\oplus\mathcal{S}^-\otimes(\mathcal{S}^-)^*).\end{aligned}$$

Thus, the $+1$ eigenspace of the grading operator $\epsilon_1\epsilon_2$ is

$$(\mathcal{S}^+\oplus\mathcal{S}^-)\otimes(\mathcal{S}^-)^* = \mathcal{S}\otimes(\mathcal{S}^+)^*.$$

Reverting to the notation of the lemma,

$$V \simeq \pi^*(\mathcal{S}_H)\otimes(\pi^*(\mathcal{S}_H)^+)^*.$$

(4): By (2),

$$\widehat{E} = \pi^*(E)\otimes\widehat{H} \simeq \pi^*(E)\otimes(\pi^*(\mathcal{S}_H)^+)^*.$$

Thus, by (1) and (3),

$$\begin{aligned}\mathcal{S}_{\widehat{M}}\otimes\widehat{E} &\simeq \pi^*(\mathcal{S}_M)\otimes\pi^*(\mathcal{S}_H)\otimes\pi^*(E)\otimes(\pi^*(\mathcal{S}_H)^+)^*\\ &\simeq V\otimes\pi^*(\mathcal{S}_M\otimes E)\\ &\simeq \widehat{S}\end{aligned}$$

□

**Lemma 2.8.** $\Phi$ *is an isomorphism of abelian groups.*

**Proof.** It is clear that $\Phi$ respects the operation of disjoint union of cycles and so it is a group homomorphism.

That $\Phi$ is injective follows from methods identical to those used in the above lemma.

To see that $\Phi$ is onto, let $(N,S,g)$ be an arbitrary $K^h$-cycle (note in particular that $N$ is not necessarily a $Spin^c$ manifold). If $N$ is even dimensional, use $\widehat{N} = S(T^*N\oplus 1)$ and if $N$ is odd dimensional, use $\widehat{N} = S(T^*N)$. Then $\widehat{N}$ is a $Spin^c$ manifold [BD82, §22, 24] and by vector bundle modification $(N,S,g)\sim(\widehat{N},\widehat{S},\widehat{g})$ which is now a $K^{top}$-cycle and so is in the image of $\Phi$. □

Note that the map $K^h_*(X)\to K^a_*(X)$ is given by

$$(M,S,g)\to g_*([D_S]),$$

where $D_S$ is the Dirac operator on the Clifford bundle $S$ [Roe88, 2.4].

**Example 2.9.** The $K$-homology class of the signature operator on an oriented, Riemannian manifold $M$ is given by $(M,\Lambda^*_{\mathbb{C}}(M),id)$. Here $\Lambda^*_{\mathbb{C}}(M)$ is the exterior bundle on $M$ which is naturally a Clifford bundle [Roe88, 2.12]. Also, the Dirac operator of this bundle is the signature operator on $M$ [Roe88, 2.13].

For the rest of this paper we will use the $(M,S,g)$ model for geometric $K$-homology.

## 3. Controlled paths of unitary operators

In this section we will give a realisation of the equivalence in analytic $K$-homology through what we call "controlled paths".

**Definition 3.1.** [HR95] Let $Y$ be a metric space. An operator $F$ on a Hilbert space $\mathcal{H}$ equipped with an action of $C(Y)$ is said to have *finite propagation* if there is a constant $R > 0$ such that $\phi F \psi = 0$ whenever $\phi, \psi \in C(Y)$ have $d(Supp(\phi), Supp(\psi)) > R$ (meaning that the distance between any point in $Supp(\phi)$ and any point in $Supp(\psi)$ is greater than $R$). The smallest such constant $R$ is called the propagation of $F$.

If $F$ is an operator represented by a Schwartz kernel and acting on the $L^2$-sections of a vector bundle over a manifold $M$, then this condition is equivalent to saying that the Schwartz kernel of $F$ is supported within an $R$-neighborhood of the diagonal in $M \times M$.

**Definition 3.2.** Let $\epsilon > 0$. An $\epsilon$-compression of a bounded operator $F$ is an operator $F_\epsilon$ satisfying the following conditions:

(1) $F_\epsilon$ is a trace class perturbation of $F$;
(2) The propagation of $F_\epsilon$ is no more than $\epsilon$.

Following Higson and Roe [HR95], if $F$ satisfies $\phi F \psi \in \mathcal{L}^1$ when $Supp(\phi) \cap Supp(\psi) = \emptyset$ (which is the case for an order zero pseudodifferential operator on a smooth, closed manifold), we construct an $\epsilon$-compression $F_\epsilon$ of $F$ as follows: Let $\{U_\alpha\}$ be a cover of $M$ consisting of balls of diameter $\epsilon/2$ and let $\{\phi_\alpha\}$ be a partition of unity of $M$ subordinate to the $U_\alpha$. Define

$$F_\epsilon = \sum_{Supp(\phi_\alpha) \cap Supp(\phi_\beta) \neq \emptyset} \phi_\alpha F \phi_\beta, \tag{3.1}$$

then $F_\epsilon$ will be a trace class perturbation of $F$. Notice also that $F_\epsilon$ is an operator of propagation no more than $\epsilon$.

**Definition 3.3.** An operator $F$ is said to have *polynomial growth* if there is a polynomial $p$ such that for each $\epsilon > 0$, there is an $\epsilon$-compression of $F$, $F_\epsilon$, satisfying

$$||F - F_\epsilon||_1 < p\left(\frac{1}{\epsilon}\right).$$

If $F$ has a kernel representation via $k(x, y)$ then this condition basically says that the speed with which $k(x, y)$ becomes singular as we approach the diagonal is polynomial. If $k(x, y)$ is locally integrable off the diagonal and if $k(x, y) \leq C \cdot d(x, y)^{-n}$ for a constant $C$, then the operator has polynomial growth.

The proof of Lemma 2.2, chapter II of [Tay81] shows that pseudodifferential operators on $\mathbb{R}^n$ of order 0 satisfy the following property: for any compact $K \subset \mathbb{R}^n$, there is a constant $C > 0$ such that for $x, y \in K$,

$$|k(x, y)| \leq C|x - y|^{-n}.$$

Thus, compactly supported pseudodifferential operators are examples of operators having polynomial growth.

**Lemma 3.4.** *Let $X, Y$ be a compact metric spaces and suppose that there is a Lipschitz map $f : X \to Y$. Let $\mathcal{H}$ be a Hilbert space equipped with an action of*

*$C(X)$ and let $F$ be an operator of polynomial growth on $\mathcal{H}$. Define $f_*\mathcal{H}$ to be the Hilbert space $\mathcal{H}$ with an action of $C(Y)$ obtained by pulling back functions on $Y$ to functions on $X$ via $f$, and then using the action of $C(X)$. Then, $F$ is also an operator of polynomial growth on $f_*\mathcal{H}$.*

**Remark 1.** The condition that $f$ be Lipschitz is not optimal—having $f$ be $Lip_\alpha$ (for any $\alpha$) would suffice.

**Proof of Lemma 3.4.** The Lipschitz condition provides the control necessary to relate distances of supports of functions on $Y$ with the distances of supports of their pullbacks to $X$, so that if $K$ is the Lipschitz constant for $f$ then

$$d(Supp(\phi), Supp(\psi)) = R \quad \Longrightarrow \quad d(Supp(f_*\phi), Supp(f_*\psi)) \leq KR.$$

The lemma is an immediate consequence of this. □

**Definition 3.5.** Let $Y$ be a metric space. If $F_t$ is a path of bounded operators on a Hilbert space $\mathcal{H}$ equipped with an action of $C(Y)$ then we say that $F_t$ has polynomial growth if there is a polynomial $p$ such that given $\epsilon > 0$, for every $t$ there is an $\epsilon$-compression of $F_t$, $F_{t,\epsilon}$, satisfying,

$$\|F_t - F_{t,\epsilon}\|_1 \leq p\left(\frac{1}{\epsilon}\right).$$

We also require that the path $F_{t,\epsilon}$ have the same continuity and differentiability conditions as the path $F_t$.

**Definition 3.6.** Let $Y$ be a metric space. A path $F_t$ of bounded operators on a Hilbert space $\mathcal{H}$ equipped with an action of $C(Y)$ is called a controlled path provided the following are true:

(1) The path $F_t$ has polynomial growth in the sense of Definition 3.5;
(2) The paths $F_t^2 - 1$ and $F_t(F_t^2 - 1)$ are paths made up of trace class operators and are trace-norm continuous and piecewise continuously differentiable in the trace norm.

**Remark 2.** It is *not* required that one-sided derivatives exist at the "breaks" in the piecewise differentiable paths. The usefulness of controlled paths is illustrated by the following lemma:

**Lemma 3.7.** *If $F_t$, $t \in [a,b]$ is a controlled path of self-adjoint operators on a Hilbert space $\mathcal{H}$, then $[-\exp(i\pi F_t)]$ is a path of unitary operators on $\mathcal{H}$ such that:*

(1) *The path $[-\exp(i\pi F_t)]$ is piecewise continuously differentiable in the trace norm;*
(2) *The path $[-\exp(i\pi F_t)]$ has a well defined winding number (in the sense of de la Harpe and Skandalis [HS84]).*

**Remark 3.** (1) The proof of this lemma is found in [Kes97, Lemma 5.1.7], [Kes99, Lemma 4.1.7].
(2) The polynomial growth condition on a controlled path is in place so that we can make estimates on the winding number of $(-\exp(i\pi F_t))$. This is the key property that provides the control necessary to make estimates on the winding number of the small time path that arises in our proof of the homotopy invariance of relative $\eta$-invariants [Kes99, Theorem 4.2.1].

The following is the main theorem of this paper:

**Theorem 3.8.** *Let $Y$ be a compact Riemannian manifold with boundary. Let $(M,S,g)$ and $(M',S',g')$ be two equivalent $K$-cycles on $Y$ (in the sense of Definition 2.6) and suppose that $g : M \to Y$ and $g' : M' \to Y$ are Lipschitz maps. Let $\chi(x)$ be a chopping function such that:*

1. *The derivative of $\chi$ is Schwartz class.*
2. *The Fourier transform of $\chi$ is smooth and is supported in $[-1,1]$.*
3. *The functions $\chi^2 - 1$ and $\chi(\chi^2 - 1)$ are Schwartz class and their Fourier transforms are supported in $[-1,1]$.*

*Let $D_S, D_{S'}$ be the Dirac operators of the Clifford bundles $S$ and $S'$ respectively. Then there are degenerate operators $A, A'$ such that the following properties hold.*

1. *$\chi(D_S) \oplus A$ and $A' \oplus \chi(D_{S'})$ are defined on the same Hilbert space $\mathcal{H}$.*
2. *The Hilbert space $\mathcal{H}$ has an action of $C(Y)$.*
3. *$\chi(D_S) \oplus A$ is connected to $A' \oplus \chi(D_{S'})$ by a controlled path (in the sense of Definition 3.6).*

**Remark 4.** The condition that $Y$ be Riemannian is in place so that we can approximate certain continuous maps by Lipschitz maps. The boundary of $Y$ can be empty although the application of this theorem in [Kes99] uses the case when $\partial Y \neq \emptyset$. Also, note that this theorem is essentially a generalisation of Proposition 18.2 of [BD82].

We will devote Section 5 to the proof of this theorem. In Section 4 we prove some technical results which are used in the proof.

## 4. Some technical results

**Definition 4.1.** [RS80, VIII.7] Let $A_n$, $n = 1, 2, \ldots$ and $A$ be self-adjoint operators. Then, $A_n$ is said to converge to $A$ in the norm resolvent sense if $(A_n + i)^{-1}$ converges to $(A + i)^{-1}$ in norm.

The following is a technical lemma that will be used several times in establishing convergence in the norm resolvent sense.

**Lemma 4.2.** *Let $D, X$ be unbounded self-adjoint operators on a Hilbert space $\mathcal{H}$. Let $\mathcal{C}$ be a dense subset of $\mathcal{H}$ such that $\mathcal{C} \subseteq dom(D) \cap dom(X)$. Suppose further that*

1. *$D$ and $X$ map $\mathcal{C}$ into itself.*
2. *There is a bounded, self-adjoint operator $B$ such that $(DX + XD)v = Bv$ for any $v \in \mathcal{C}$.*
3. *$(D + t^{-1}X)$ is essentially self-adjoint.*
4. *$X$ is bounded below on $\mathcal{C}$—i.e for some $\epsilon > 0$, $\|Xv\|^2 \geq \epsilon \|v\|^2$.*

*Then:*

1. *$\|X(D + t^{-1}X + i)^{-1}w\| \leq \sqrt{t^2 + t\|B\|}\,\|w\|$ for any $w \in \mathcal{C}$ and the path $D_s = D + s^{-1}X$ is continuous in the norm resolvent sense at any $s \neq 0$.*
2. *$\|(D + t^{-1}X + i)^{-1}\| \to 0$ as $t \to 0$.*
3. *For any continuous function $f$ on $\mathbb{R}$ that vanishes at $\infty$,*

$$\|f(D + t^{-1}X)\| \to 0 \qquad \text{as } t \to 0.$$

**Proof.** By the hypotheses of the lemma, on $\mathcal{C}$,

$$(D+t^{-1}X)^2 = D^2 + t^{-2}X^2 + t^{-1}B.$$

So, for $v \in \mathcal{C}$,

$$\|(D+t^{-1}X+i)v\|^2 \geq t^{-2}\|Xv\|^2 + t^{-1}\langle Bv, v\rangle. \tag{4.1}$$

Thus,

$$\begin{aligned}\|Xv\|^2 &\leq t^2(\|(D+t^{-1}X+i)v\|^2 - t\langle Bv, v\rangle \\ &\leq t^2\|(D+t^{-1}X+i)v\|^2 + t|\langle Bv, v\rangle|.\end{aligned}$$

Now by the Cauchy-Schwarz inequality and the assumption that $B$ is bounded and self-adjoint,

$$\|Xv\|^2 \leq t^2\|(D+t^{-1}X+i)v\|^2 + t\|B\|\|v\|^2.$$

Let $w \in (D+t^{-1}X+i)\mathcal{C}$ and let $v = (i+D+t^{-1}X)^{-1}w \in \mathcal{C}$ . By the spectral radius formula, $\|(i+D+t^{-1}X)^{-1}\| \leq 1$, so $\|v\| \leq \|w\|$. Thus,

$$\begin{aligned}\|X(i+D+t^{-1}X)^{-1}w\|^2 &= \|Xv\|^2 \\ &\leq (\sqrt{t^2+t\|B\|}\|w\|)^2.\end{aligned}$$

Thus, $\|X(i+D+t^{-1}X)^{-1}w\| \leq \sqrt{t^2+t\|B\|}\|w\|$. To establish the norm continuity of the resolvents of the $D_t$, notice that for $w \in (D+t^{-1}X+i)\mathcal{C}$,

$$\begin{aligned}\|((D_s+i)^{-1} - (D_t+i)^{-1})w\| &= \|(D_s+i)^{-1}(D_t-D_s)(D_t+i)^{-1}w\| \\ &\leq \|(D_s+i)^{-1}\| \cdot |t^{-1}-s^{-1}|\sqrt{t^2+t\|B\|}\|w\|. \qquad (4.2)\end{aligned}$$

By the spectral radius formula, $\|(D_s+i)^{-1}\| \leq 1$. Therefore, by (4.2),

$$\|((D_s+i)^{-1} - (D_t+i)^{-1})w\| \leq |t^{-1}-s^{-1}|\sqrt{t^2+t\|B\|}\|w\|$$

for $w \in (D+t^{-1}X+i)\mathcal{C}$. Since $D+t^{-1}X$ is essentially self-adjoint, $(D+t^{-1}X+i)\mathcal{C}$ is dense in $\mathcal{H}$. Using this and the fact that the resolvents $(D_s+i)^{-1}$ are bounded it follows that this inequality extends to all $w \in \mathcal{H}$. Thus, as $t \to s$, $\|(D_s+i)^{-1} - (D_t+i)^{-1}\| \to 0$. So, for $s \neq 0$, the path $D_s$ is continuous in the norm resolvent sense at $s$, thus establishing (1).

For (2) we recall that from (4.1), for $v \in \mathcal{C}$,

$$\|(D+t^{-1}X+i)v\|^2 \geq \|v\|^2 + \epsilon t^{-2}\|v\|^2 + t^{-1}\langle Bv, v\rangle$$

which, by the assumptions on $X$ and $B$

$$\geq (1+\epsilon t^{-2} - t^{-1}\|B\|)\|v\|^2.$$

Thus, for $w \in \mathcal{C}$, if we set $v = (i+D+t^{-1}X)^{-1}w$, then $v \in \mathcal{C}$ also and so,

$$\begin{aligned}\|w\|^2 &= \|(i+D+t^{-1}X)v\|^2 \\ &\geq (1+\epsilon t^{-2} - t^{-1}\|B\|)\|v\|^2.\end{aligned}$$

So,

$$\|(i+D+t^{-1}X)^{-1}w\|^2 \leq \frac{1}{(1+\epsilon t^{-2} - t^{-1}\|B\|)}\|w\|^2.$$

Thus since $(i+D+t^{-1}X)^{-1}$ is bounded and $\mathcal{C}$ is dense in $\mathcal{H}$,

$$\|(i+D+t^{-1}X)^{-1}\| \leq \frac{1}{\sqrt{1+\epsilon t^{-2} - t^{-1}\|B\|}} \to 0$$

as $t \to 0$.

Finally for (3), we use the essence of the proof of Theorem VIII.20, [RS80]: By the Stone-Weierstrass theorem, polynomials in $(x+i)^{-1}$ and $(x-i)^{-1}$ are dense in $C_0(\mathbb{R})$, the continuous functions vanishing at $\infty$. Note that the methods used in proving (2) apply to prove that

$$\|(D + t^{-1}X - i)^{-1}\| \to 0 \quad \text{as } t \to 0.$$

Thus, (3) follows from (2). □

**Definition 4.3.** Let $\mathcal{R}(\mathbb{R})$ denote the space of rapidly decreasing functions on $\mathbb{R}$; thus a function $f : \mathbb{R} \to \mathbb{C}$ belongs to $\mathcal{R}(\mathbb{R})$ if it is continuous and if for each $N \geq 0$ there is a constant $C_N$ such that

$$|f(x)| \leq C_N(1 + |x|)^{-N}$$

for all $x \in \mathbb{R}$.

**Lemma 4.4.** *Let $f \in \mathcal{R}(\mathbb{R})$. Suppose $D$ is a self-adjoint operator whose $j^{th}$ eigenvalue $\lambda_j$ satisfies $|\lambda_j| \geq p(j)$ for some non-constant polynomial $p$. Then, there exists a polynomial $q$ such that for all $\epsilon > 0$,*

$$\|f(\epsilon D)\|_1 \leq q\left(\frac{1}{\epsilon}\right).$$

**Proof.** Note that

$$\|f(\epsilon D)\|_1 = \sum_j |f(\epsilon \lambda_j)|.$$

Let the degree of the polynomial $p$ be $N \geq 1$. Since $f \in \mathcal{R}(\mathbb{R})$, there is a constant $C$ such that $|f(x)| \leq C(1 + |x|)^{-2}$. Thus,

$$\begin{aligned} \|f(\epsilon D)\|_1 &\leq \sum_j \frac{C}{(1 + |\epsilon \lambda_j|)^2} \\ &\leq \sum_j \frac{C'}{(1 + \epsilon j^N)^2} \\ &\leq \frac{1}{\epsilon^2} K \end{aligned}$$

for constants $C'$ and $K$. □

**Lemma 4.5.** *Let $\tilde{\chi}$ be a chopping function defined as follows:*

$$\tilde{\chi}(x) = \begin{cases} sign(x), & |x| > 1 \\ x, & |x| \leq 1. \end{cases}$$

*If $D$ is a self-adjoint operator whose $j^{th}$ eigenvalue $\lambda_j$ satisfies $|\lambda_j| \geq p(j)$ for some non-constant polynomial $p$, then there is a polynomial $q$ such that for all $\epsilon > 0$,*

$$\|\tilde{\chi}(\epsilon D) - \tilde{\chi}(D)\|_1 \leq q\left(\frac{1}{\epsilon}\right).$$

**Proof.** Note that the function $\tilde{\chi}(\epsilon x)$ is given by

$$\tilde{\chi}(\epsilon x) = \begin{cases} sign(x), & |x| > 1/\epsilon \\ \epsilon x, & |x| \leq 1/\epsilon. \end{cases}$$

So,

$$|\tilde{\chi}(\epsilon x) - \tilde{\chi}(x)| = \begin{cases} |\epsilon x - x| & |x| < 1 \\ |\epsilon x \pm 1| & 1 \leq |x| \leq 1/\epsilon \\ 0 & 1/\epsilon \leq |x|. \end{cases}$$

Thus, assuming for simplicity that the lowest eigenvalue of $D$ is at least 1 in absolute value,

$$\begin{aligned} \|\tilde{\chi}(\epsilon D) - \tilde{\chi}(D)\|_1 &= \sum_j |\tilde{\chi}(\epsilon \lambda_j) - \tilde{\chi}(\lambda_j)| \\ &= \sum_{j=1}^{N} 1 - \epsilon \lambda_j, \end{aligned}$$

where the sum is over the eigenvalues $\lambda_j$ of absolute value less than or equal to $1/\epsilon$. Since each term is bounded by 1, and since $N$ is bounded by a polynomial, we are done. □

**Lemma 4.6.** *Let $Y$ be a compact metric space of finite dimension (i.e., $Y$ can be embedded as a subset of $\mathbb{R}^n$ for some $n > 0$). Let $F$ be a bounded operator on a Hilbert space $\mathcal{H}$ equipped with an action of $C(Y)$. Suppose there is a polynomial $q$ such that for any $\epsilon > 0$, if $\phi, \psi \in C(Y)$ are such that $d(Supp(\phi), Supp(\psi)) > \epsilon$, then*

$$\|\phi F \psi\|_1 \leq q\left(\frac{1}{\epsilon}\right).$$

*Then, for the $\epsilon$-compression $F_\epsilon$ of $F$ defined in (2.1), there is a polynomial $p$ such that for any $\epsilon > 0$,*

$$\|F - F_\epsilon\|_1 \leq p\left(\frac{1}{\epsilon}\right).$$

**Proof.** Fix $\epsilon > 0$. Recall the definition of $F_\epsilon$ from (2.1)—Let $\{U_\alpha\}$ be a cover of $M$ consisting of balls of diameter $\epsilon/2$ and let $\varphi_\alpha$ be a partition of unity subordinate to the $U_\alpha$. Define

$$F_\epsilon = \sum_{Supp(\varphi_\alpha) \cap Supp(\varphi_\beta) \neq \emptyset} \varphi_\alpha F \varphi_\beta.$$

Thus,

$$F - F_\epsilon = \sum_{Supp(\varphi_\alpha) \cap Supp(\varphi_\beta) = \emptyset} \varphi_\alpha F \varphi_\beta.$$

Since we can cover the unit cube in $\mathbb{R}^n$ by $const.(2/\epsilon)^n$ balls of diameter $\epsilon/2$, for some constant $C$,

$$\|F - F_\epsilon\|_1 \leq C\left(\frac{2}{\epsilon}\right)^n q\left(\frac{2}{\epsilon}\right).$$

Note that the right hand side of the above inequality is a polynomial $p$ in $1/\epsilon$. □

**Lemma 4.7.** *Let $\chi$ be a chopping function which differs from $\tilde{\chi}$ of Lemma 4.5 by a function $f \in \mathcal{R}(\mathbb{R})$ and suppose the Fourier transform of $\chi$ is compactly supported within $[-1, 1]$. Let $Y$ be a compact metric space. Let $D_t$, $0 \leq t \leq 1$ be a path of self-adjoint elliptic, first order differential operators on a complete Riemannian manifold $Z$ such that there is a Lipschitz map $Z \to Y$ and the $j^{th}$ eigenvalue $\lambda_{t,j}$ of $D_t$ satisfies either of the following conditions:*

(1) $\lambda_{t,j} \geq t^{-1}p(j)$, *or*
(2) $\lambda_{t,j} \geq p(j) + t^{-1}C$

*for some constant $C$ and polynomial $p$. Then $\chi(D_t)$ is a path of operators of polynomial growth on a Hilbert space equipped with an action of $C(Y)$.*

**Proof.** Fix $\epsilon > 0$. Let $F_t = \chi(D_t)$ and let $F_t^\epsilon = \chi(\epsilon D_t)$. Note that from our assumption on $\chi$, $F_t$ will have propagation 1 and $F_t^\epsilon$ will have propagation $\epsilon$, (see [Roe89, §2]).

Let $\psi, \phi \in C(Y)$ so that $|\phi(y)| \leq 1$ and $|\psi(y)| \leq 1$ and so that $d(Supp(\phi), Supp(\psi)) > \epsilon$. For technical convenience we will assume that the Lipschitz constant for the map $Z \to Y$ is 1. Then, by finite propagation speed considerations, $\phi F_t^\epsilon \psi = 0$. Thus,

$$\begin{aligned} \|\phi F_t \psi\|_1 &= \|\phi(F_t - F_t^\epsilon)\psi\|_1 \\ &\leq \|F_t - F_t^\epsilon\|_1. \end{aligned} \tag{4.3}$$

Let $\tilde{\chi}$ be as in Lemma 4.5 and let

$$\gamma_\epsilon(D_t) = \chi(D_t) - \chi(\epsilon D_t), \qquad \tilde{\gamma}_\epsilon(D_t) = \tilde{\chi}(D_t) - \tilde{\chi}(\epsilon D_t).$$

Then, letting $f = \chi - \tilde{\chi}$,

$$\gamma_\epsilon(D_t) - \tilde{\gamma}_\epsilon(D_t) = f(D_t) - f(\epsilon D_t). \tag{4.4}$$

By an argument similar to Lemma 4.4, $\|f(D_t) - f(\epsilon D_t)\|_1$ is uniformly bounded by a polynomial in $1/\epsilon$ and by Lemma 4.5, $\|\tilde{\gamma}_\epsilon(D_t)\|_1$ is also bounded by a polynomial in $1/\epsilon$; (examining the proof of the lemma, as $t \to 0$ we see that $\|\tilde{\gamma}_\epsilon(D_t)\|_1 \to 0$ and so this estimate is uniform). Thus, from (4.4) we may conclude that $\|\gamma_\epsilon(D_t)\|_1$ is uniformly bounded by a polynomial in $1/\epsilon$ and so by (4.3) and Lemma 4.6 we are done. □

## 5. Proof of the main theorem

We will first show that a controlled path can be made to implement each of the equivalence relations of bordism, direct sum and vector bundle modification. The techniques used here are motivated by the works of Higson and Roe ([Hig91], [Hig90] and [Roe89]).

5.1. **Bordism.** Recall the definition of the step of bordism in the equivalence relation on $\Pi(X)$.

**Theorem 5.1.0.** *Suppose that $(M, S, g)$ is a $K$-cycle over a compact, Riemannian manifold $Y$ such that $g : M \to Y$ is Lipschitz. Suppose further that $(M, S, g)$ is null bordant via a triple $(Z', F', \omega)$ and that $\chi$ is a chopping function that satisfies the conditions on the chopping function in Theorem 3.8. Then, there is a Hilbert space $\mathcal{H}$ equipped with an action of $C(Y)$ and degenerate operators $A, A'$ such that there is a path of controlled operators on $\mathcal{H}$ connecting $\chi(D_S) \oplus A$ to $A'$.*

**Remark 5.** Note that it is implicit that the Hilbert space $\mathcal{H}$ contains $L^2(M, S)$ as a direct summand.

**Scheme of Proof.** We proceed in the spirit of the argument from Higson's work on the cobordism invariance of the index [Hig91]. Notice first that since $Y$ is Riemannian and $g$ is Lipschitz, we can perturb $\omega : Z' \to Y$ to a Lipschitz map.

Modify $Z'$ by attaching the cylinder $M \times [-10, \infty)$ to the boundary of $Z'$ and call the resulting complete, non-compact manifold $Z$. There is a Lipschitz map $Z \to Y$ obtained by taking $\omega$ on $Z'$ and $g$ composed with projection onto the first factor on the product part. Let $F = F'$ on $Z'$ extended by $S \oplus 1$ on the product part. Let $W = M \times \mathbb{R}$ and $S' = S \oplus 1$. We will construct a controlled path that connects the operator $\chi(D_S)$ to a degenerate as follows:

(I) Connect ($\chi(D_S) \oplus$ degenerate) to an operator $\chi(D_{S',x})$ .

(II) Connect $\begin{pmatrix} \chi(D_{S',x}) & 0 \\ 0 & \chi(-D_{F,x}) \end{pmatrix}$ to a degenerate.

(III) Notice that from (II),

$$\begin{pmatrix} \chi(D_{S',x}) & 0 & 0 \\ 0 & \chi(-D_{F,x}) & 0 \\ 0 & 0 & \chi(D_{F,x}) \end{pmatrix} \sim \text{degenerate} \oplus \chi(D_{F,x}). \tag{5.1.1}$$

We will show that $\begin{pmatrix} \chi(-D_{F,x}) & 0 \\ 0 & \chi(D_{F,x}) \end{pmatrix}$ is connected to a degenerate and so the left hand side of (5.1.1) is equal to ($\chi(D_{S',x}) \oplus$ degenerate). Thus, the path just constructed and the path from (II) implements an equivalence between $\chi(D_{F,x})$ and $\chi(D_{S',x})$.

(IV) Connect $\chi(D_{F,x})$ to a degenerate.

5.1.1. Step I (The case of a product). If $M$ is odd dimensional, then $W = \mathbb{R} \times M$ is even dimensional and the Dirac operator of $S'$ can be written as

$$D_{S'} = \begin{pmatrix} 0 & D_S + \frac{d}{dx} \\ D_S - \frac{d}{dx} & 0 \end{pmatrix}.$$

If $M$ is even dimensional then $D_S$ acts on the Clifford bundle $S$ which has a $\mathbb{Z}_2$-grading $S = S^+ \oplus S^-$ given by Clifford multiplication. Pull back $S$ and its connection to $W$ and extend the Clifford action of $TM$ to a Clifford action of $TW$ by letting the unit tangent vector $e_0$ for $\mathbb{R}$ act as $-i \cdot vol$ where $vol$ is the volume element of $M$. The connection is compatible with the larger Clifford action and we call $D_{S'}$ the Dirac operator of this bundle. It can be checked that $D_{S'}$ is described precisely by the same formula as in the odd case above.

Let $\partial$ denote the operator $d/dx$ and let,

$$D_{S',x} = \begin{pmatrix} x & D_S + \partial \\ D_S - \partial & -x \end{pmatrix}.$$

From Chapter 10, Section C of [Roe88],

$$\mathcal{U} = \ker \begin{pmatrix} x & \partial \\ -\partial & -x \end{pmatrix} = \begin{pmatrix} e^{-x^2/2} \\ e^{-x^2/2} \end{pmatrix}.$$

By definition, on $V$ the operator $D_{S',x} = \begin{pmatrix} 0 & D_S \\ D_S & 0 \end{pmatrix}$. We will show that on $\mathcal{U}^\perp$, $D_{S',x}$ is connected to a degenerate via a path $B_s$ defined by

$$B_s = \begin{pmatrix} 0 & D_S \\ D_S & 0 \end{pmatrix} + s^{-1} \begin{pmatrix} x & \partial \\ -\partial & x \end{pmatrix}, \qquad 0 < s \le 1.$$

The main idea here is that on $\mathcal{U}^\perp$ the operator $\begin{pmatrix} x & \partial \\ -\partial & x \end{pmatrix}$ is bounded below and so by taking $s$ to be small, we can make $B_s$ "close to being invertible". This is made more formal by the following lemma and the technical tools developed in Section 4.

**Lemma 5.1.1.** *$B_s$ is continuous in the norm resolvent sense and for any function $f \in C_0(\mathbb{R})$, $f(B_s)$ converges in norm to 0.*

**Proof.** Let $X = \begin{pmatrix} x & \partial \\ -\partial & -x \end{pmatrix}$. Then, $B_t = D_S + t^{-1}X$. Let $\mathcal{C} = C_c^\infty(W, S') \cap \mathcal{U}^\perp$ (the compactly supported smooth sections of $S'$ off the kernel of $X$). Notice that $\mathcal{C}$ is dense in $\mathcal{H} = L^2(W, S') \cap \mathcal{U}^\perp$ and $D$ and $X$ are unbounded, self-adjoint operators on $\mathcal{H}$ such that the following properties hold.

(1) $\mathcal{C} \subseteq \mathrm{dom}(D_S) \cap \mathrm{dom}(X)$.
(2) Since $D_S$ commutes with $\partial$ and with $x$, $(D_S X + X D_S)v = 0$ for any $v \in \mathcal{C}$.
(3) Since $D_{S'}$ and $X$ are essentially self-adjoint operators, $D_{S'} + t^{-1}X$ is also essentially self-adjoint.
(4) On $V^\perp$, the operator $X$ is bounded below. This is because $X$ is unitarily equivalent to the harmonic oscillator $H = \begin{pmatrix} 0 & \partial + x \\ -\partial + x & 0 \end{pmatrix}$ and as in Section C, Chapter 10 of [Roe88], $H$ has a minimum non-zero eigenvalue of 1 and so is bounded below off its kernel.

Thus, applying Lemma 4.2, our conclusion follows. □

**Lemma 5.1.2.** *Let $f \in \mathcal{R}(\mathbb{R})$ and $f(x) > 0$ for $x > 0$. Then $f(B_s)$ is continuously differentiable in $\mathcal{L}^1$ and $f(B_s) \in \mathcal{L}^1$ for all $0 < s \leq 1$.*

**Sketch of Proof.** Adopting the notation of the previous lemma, our path

$$B_s = D_S + s^{-1}X.$$

$B_s^2$ is a positive operator and

$$B_s^2 = (D_S)^2 + s^{-2}X^2 + s^{-1}A,$$

where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a bounded operator. Since $X$ is unitarily equivalent to the harmonic oscillator, by Section C, Chapter 10 of [Roe88] the eigenvalues of $s^{-2}X^2$ are $s^{-2}(2k+1)$, for $k \geq 0$. Let the corresponding (normalized) eigenvectors of $s^{-2}X^2$ be $\{\psi_k\}$. Then, by Section C, Chapter 10 of [Roe88], the $\psi_k$ form an orthonormal basis for $L^2(\mathbb{R})$. Similarly, let the eigenvalues of $(D_S)^2$ be $\lambda_j$. Then, by Theorem 7.3 of [Roe88], $\lambda_j \sim j^{2/n}$ ($n = dim(M)$), and if $\{\phi_j\}$ are the corresponding (normalized) eigenvectors, then they form a basis of $L^2(M, S)$. Let $\mathcal{H} = L^2(W, S') \cap V^\perp$. Since $L^2(M \times \mathbb{R}) \simeq L^2(M) \otimes L^2(\mathbb{R})$, the vectors $\{\phi_j \otimes \psi_k\}$ form a basis for $\mathcal{H}$ and are eigenvectors of $B_s = D_S \otimes I + I \otimes s^{-1}X$. The corresponding eigenvalues of $B_s^2$ are

$$\mu_{j,k} = \lambda_j + s^{-2}(2k+1) + s^{-1}c,$$

where $|c| \leq \|A\| = 1$.

Now use the rapid decay of $f$ to show by an explicit calculation that $\|f(B_s)\|_1 \to 0$ as $s \to 0$ and that $\|(f(B_s) - f(B_t))/(s-t)\|_1$ exists as $t \to s$ and in fact goes to 0 as $s \to 0$. □

**Lemma 5.1.3.** *Assuming the hypotheses of Theorem 5.1.0, $\chi(B_s)$ is a controlled path on the Hilbert space $L^2(W, S')$, connecting $\chi(D_{S'})$ to $(\chi(D_S) \oplus$ degenerate$)$.*

**Proof.** $W$ is the product manifold $M \times \mathbb{R}$. The map $W \to Y$ obtained by composing $g$ with projection onto the first factor is Lipschitz since it is the composition of Lipschitz maps.

As noted in the proof of Lemma 5.1.2, the eigenvalues of $B_s^2$ are

$$\mu_{j,k} = \lambda_j + s^{-2}(2k+1) \pm s^{-1}c.$$

Since $\lambda_j \sim j^{2/n}$, the $\mu_{j,k}$ grow at a rate bounded by a polynomial in $k$ and $j$. Applying Lemma 4.7 we see that $\chi(B_s)$ has polynomial growth.

Now let $f = \chi^2 - 1$ and $g = \chi(\chi^2 - 1)$. Then $f, g \in \mathcal{R}(\mathbb{R})$ by assumption and also $f(x)$ and $g(x)$ are positive for $x$ positive, so by Lemma 4.2 we may conclude that $f(B_s)$ and $g(B_s)$ are smooth in the trace norm and consist of trace class operators.

We have verified the conditions of Definition 3.6 and so we conclude that $\chi(B_s)$ is a controlled path. □

5.1.2. Step II. Recall that $(M, S, g)$ is null bordant via $(Z', F', \omega)$, and we construct $Z$ from $Z'$ by attaching the cylinder $M \times [-10, \infty)$ to the boundary of $Z'$. Accordingly, $F'$ extends to a bundle $F$ over $Z$ by allowing the bundle over the cylinder to be $S \oplus 1$. If $M$ is odd dimensional so that $Z$ is even dimensional, let $D_F = \begin{pmatrix} 0 & D_F^- \\ D_F^+ & 0 \end{pmatrix}$ be the Dirac operator of the Clifford bundle $F$. If $M$ is even dimensional so that $Z$ is odd dimensional then there is no natural grading on $F$ and so we abuse notation to call $D_F$ the operator $\begin{pmatrix} 0 & D_F \\ D_F & 0 \end{pmatrix}$. Let $D_{F,x}$ be defined by

$$D_{F,x} = D_F + \gamma \cdot x,$$

where $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the grading operator. For clarity we will drop the grading operator and write $D_{F,x} = D_F + x$.

Let $\sigma$ be a smooth bump function on $Z$ such that $\sigma = 1$ on $M \times (-1/2, 1/2)$ and $\sigma = 0$ off $M \times (-1, 1)$. Let $A = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$. Let

$${}_1P_t = \begin{pmatrix} D_{S',x} & tA \\ tA & -D_{F,x} \end{pmatrix} \quad 0 \le t \le 1.$$

Note that ${}_1P_0 = \begin{pmatrix} D_{S',x} & 0 \\ 0 & -D_{F,x} \end{pmatrix}$ and ${}_1P_1 = \begin{pmatrix} D_{S',x} & A \\ A & -D_{F,x} \end{pmatrix}$. Let

$${}_2P_s = \begin{pmatrix} D_{S'} + s^{-1}x & s^{-1}A \\ s^{-1}A & -D_F - s^{-1}x \end{pmatrix} \quad 0 \le s \le 1.$$

Note that ${}_1P_1 = {}_2P_1$—set

$$P_t = \begin{cases} {}_1P_t, & 0 \le t \le 1 \\ {}_2P_{(2-t)}, & 1 \le t < 2. \end{cases}$$

**Lemma 5.1.4.** *$P_t$ is continuous in the norm resolvent sense and for any function $f \in C_0(\mathbb{R})$, $f(P_t)$ converges in norm to $0$ as $t \to 2$.*

**Proof.** It suffices to prove continuity in the norm resolvent sense for the two paths ${}_1P_t$ and ${}_2P_t$ separately. Note that for any $t, s \in [0,1]$,

$$\begin{aligned}\|({}_1P_t + i)^{-1} - ({}_1P_s + i)^{-1}\| &= \|({}_1P_t + i)^{-1}({}_1P_s - {}_1P_t)({}_1P_s + i)^{-1}\| \\ &= \|({}_1P_t + i)^{-1}(s-t)A({}_1P_s + i)^{-1}\| \\ &\leq |s-t|\|A\| \to 0 \qquad \text{as } s \to t.\end{aligned}$$

Thus ${}_1P_t$ is continuous in the norm resolvent sense.

Let

$$X = \begin{pmatrix} x & A \\ A & -x \end{pmatrix} \quad D = \begin{pmatrix} D_W & 0 \\ 0 & -D_Z \end{pmatrix}.$$

Then, ${}_2P_s = D + s^{-1}X$, for $0 \leq s \leq 1$. $D$ and $X$ are unbounded, self-adjoint operators on the Hilbert space $\mathcal{H} = L^2(W, S') \oplus L^2(Z, F)$. Let $\mathcal{C} = C_c^\infty(W, S') \oplus C_c^\infty(Z, F)$. Then $\mathcal{C}$ is dense in $\mathcal{H}$ and

(1) $\mathcal{C} \subseteq \mathrm{dom}(D) \cap \mathrm{dom}(X)$;

(2) Let $Bv = (DX + XD)v$ for $v \in \mathcal{C}$. Then,

$$B = \begin{pmatrix} xD_W + D_W x & AD_Z - D_W A \\ AD_W - D_Z A & xD_{Z'} + D_{Z'} x \end{pmatrix}.$$

We claim that $B$ is a bounded operator:

That the diagonal entries of $B$ are bounded follows from the grading and the fact that the commutators of differential operators with differentiable functions are bounded. The key point to the off diagonal entries is that differential operators are local and so, on the support of $A$, the differential operators $D_Z$ and $D_W$ are the same. Thus, the off diagonal entries are essentially the commutators of $D_W$ with $A$ and $D_Z$ with $A$—since $A$ is made up from the differentiable, bounded function $\sigma$, these are bounded.

(3) The operators $D + t^{-1}X$ are essentially self-adjoint.

(4) $X^2 = \begin{pmatrix} x^2 + A^2 & 0 \\ 0 & x^2 + A^2 \end{pmatrix}$ and so is bounded below on $\mathcal{H}$.

By Lemma 4.2, it follows that ${}_2P_s$ is continuous in the norm resolvent sense and for any $f \in C_0(\mathbb{R})$, $f({}_2P_s) \to 0$ in norm as $s \to 0$. Note that this is equivalent to $P_t$ being continuous in the norm resolvent sense and for any $f \in C_0(\mathbb{R})$, $f(P_t) \to 0$ in norm as $t \to 2$. □

We will need the following technical results to establish condition (2) of the Definition 3.6 for the controlled path we will construct from $P_t$.

**Theorem 5.1.5.** *Let $f$ be a function on $\mathbb{R}$ whose Fourier transform is smooth and compactly supported. Let $D_t$ be a family of first order differential operators on a compact Riemannian manifold $M$ such that locally,*

$$D_t = \sum a_t \frac{\partial}{\partial x_i} + b_t,$$

*where $a_t, b_t$ are smooth in $t$. Then, $f(D_t)$ is a family of trace class operators whose Schwartz kernels vary smoothly in $t$.*

In [Roe87, Thm. 2.1], Roe proves a more general version of this theorem—his proof covers a leafwise Dirac operator for a foliation on a compact manifold. $D_t$

can be regarded as such a leafwise Dirac operator if we let the parameter space be the leaf space and take the leaves to all be $M$.

**Lemma 5.1.6.** *If $f$ is a Schwartz class function whose Fourier transform is smooth and supported within $[-1,1]$, then $f(P_t)$ is smooth in $\mathcal{L}^1$ and $f(P_t) \in \mathcal{L}^1$ for all $0 \le t < 2$.*

**Proof.** Notice first that the operators $f(P_t)$ have finite propagation (equal to 1) [Roe89, Prop. 2.2].

The operators $P_t$ are defined on the Hilbert space $L^2(Z,F) \oplus L^2(W,S')$. Partition $Z \cup W$ into a compact piece $C$ and a non compact piece $NC$.

Let $V_1$ be projection onto the sections supported on $C$ and $V_2$ be projection onto the sections supported on $NC$. Then,

$$f(P_t) = f(P_t)V_1 + f(P_t)V_2.$$

The range of the projection $V_2$ is isomorphic to the space of sections supported on the product manifold $W$ and since the propagation of $f(P_t)$ is finite, the operators $f(P_t)V_2$ are unitarily equivalent to the operators $f(B_s)$ on the product manifold $W$. Thus, $f(P_t)V_2$ have the same spectral theory as the operators $f(B_s)$ considered in Lemma 5.1.2 and so the proof of the lemma can be adapted to prove that the operators $f(P_t)V_2$ are smooth in the trace norm and are each of trace class.

On the range of $V_1$, the operators $P_t$ are unitarily equivalent to first order elliptic operators on a compact manifold and so by the functional calculus of pseudodifferential operators [Tay81, Ch. 12], the operators $f(P_t)V_1$ are trace class.

As noted earlier, on the range of $V_1$ the $P_t$ are unitarily equivalent to a family of first order differential operators on a compact manifold, (basically $C$ doubled) and so Theorem 5.1.5 applies to give us smoothness in the trace norm of $f(P_t)V_1$.

It remains to show that in fact, the operators $f(P_t)$ go to 0 in the trace norm as $t \to 2$. From the definition of the path $P_t$, we are required to show that $f({}_2P_s) \to 0$ in the trace norm, as $s \to 0$. Using the notation of the proof of Lemma 5.1.2, the path ${}_2P_s = D + s^{-1}X$. Let $R_s = ({}_2P_s)^2$. Then,

$$R_s = D^2 + s^{-2}X^2 + s^{-1}B,$$

where $B$ is a bounded operator and $X^2$ is bounded below. Thus, for small $s$, $s^{-2}X^2 + s^{-1}B \ge I$ and so for small enough $s$, and for $g(x) = 1 + x^2$,

$$g(D + s^{-1}X) = R_s + 1 \ge D^2 + I.$$

Let $h(x) = 1/g(x)$. So, $h \in C_0(\mathbb{R})$ and

$$h(D + s^{-1}X) \le (D^2 + I)^{-1}.$$

Further, by Lemma 5.1.2, $\|h(D + s^{-1}X)\| \to 0$ as $s \to 0$. By the eigenvalue estimates for $D$ in [Roe88, Thm. 7.3], we know that $(D^2 + I)^{-1}$ is in the Schatten class $\mathcal{L}^p$ for $p > dim(M)/2$. Thus, by the dominated convergence theorem [Sim79, Thm. 2.16], for $p = [dim(M)/2 + 1]$,

$$\|h(D + s^{-1}X)\|_p \to 0.$$

By the Hölder inequality for Schatten classes [Sim79, Thm. 2.8], this implies that

$$\|(h(D + s^{-1}X))^p\|_1 \to 0. \tag{5.1.2}$$

Notice that since $g$ is a polynomial, the function $z(x) = (g(x))^p f(x)$ is also a Schwartz function. Thus, $f(D + s^{-1}X) = h^p(D + s^{-1}X).z(D + s^{-1}X)$. So, by (5.1.2) and the inequality $\|AB\|_1 \leq \|A\|_1\|B\|_\infty$, for $A \in \mathcal{L}^1, B \in \mathcal{B}(\mathcal{H})$, [Con90, IX.2],

$$\begin{aligned}\|f(D + s^{-1}X)\|_1 &\leq \|(h(D + s^{-1}X))^p\|_1\|z(D + s^{-1}X)\|_\infty \\ &\to 0 \qquad \text{as } s \to 0.\end{aligned}$$

□

**Lemma 5.1.7.** *Assuming the hypotheses of Theorem 5.1.0, $\chi(P_t)$ is a controlled path of operators on the Hilbert space $L^2(Z, F) \oplus L^2(W, S')$, connecting*

$$\begin{pmatrix}\chi(D_{S',x}) & 0 \\ 0 & \chi(-D_{F,x})\end{pmatrix}$$

*to a degenerate operator.*

**Proof.** We proceed in a similar fashion to Lemma 5.1.6. Partition $Z \cup W$ into a compact part $C$ and a non-compact part $NC$. Since $\chi$ has compactly supported Fourier transform, the operators $\chi(P_t)$ have finite propagation (equal to 1) [Roe89, Prop. 2.2].

Let $V_1$ be projection onto the sections supported on $C$ and $V_2$ be projection onto the sections supported on $NC$. Then,

$$\chi(P_t) = \chi(P_t)V_1 + \chi(P_t)V_2.$$

The range of the projection $V_2$ is isomorphic to the space of sections supported on the product manifold $W$ and since the propagation of $\chi(P_t)$ is finite, the operators $\chi(P_t)V_2$ are unitarily equivalent to the operators $\chi(B_s)$ on the product manifold $W$. Thus, $\chi(P_t)V_2$ have the same spectral theory as the operators $\chi(B_s)$ considered in Lemma 5.1.3. Thus by Lemma 4.7 $\chi(P_t)V_2$ has polynomial growth.

On the range of $V_1$ the operators $P_t$ are unitarily equivalent to elliptic differential operators on a compact manifold (basically $C$ doubled). By [LM90, Rmk. 5.11], these are Dirac type operators and thus they have the same spectral theory as Dirac operators on a compact manifold. Thus [Roe88, Thm. 7.3], the spectrum of the $\chi(P_t)V_1$ grow at a polynomial rate and so Lemma 4.7 applies to allow us to conclude that $\chi(P_t)V_1$ has polynomial growth.

Since $\chi^2 - 1$ and $\chi(\chi^2 - 1)$ are Schwartz class functions with finitely supported, smooth Fourier transforms, by Lemma 5.1.6 we see that $\chi^2(P_t) - 1$ and $\chi(P_t)(\chi^2(P_t) - 1)$ are paths of trace class operators that are differentiable in the trace norm.

Finally, since the maps $Z \to Y$ and $W \to Y$ are Lipschitz, we may conclude that the path $\chi(P_t)$ is a controlled path on the Hilbert space $L^2(Z, F) \oplus L^2(W, S')$ (which has an action of $C(Y)$). □

5.1.3. Step III. Recall from the introduction to this section that in this step we need to establish that $\begin{pmatrix}\chi(-D_{F,x}) & 0 \\ 0 & \chi(D_{F,x})\end{pmatrix}$ is connected to a degenerate. We proceed as follows:

Let

$${}_1Q_t = \begin{pmatrix}-D_{F,x} & tI \\ tI & D_{F,x}\end{pmatrix}, \qquad 0 \leq t \leq 1$$

and

$$ {}_2Q_s = \begin{pmatrix} -D_{F,x} & s^{-1}I \\ s^{-1}I & D_{F,x} \end{pmatrix}, \qquad 0 < s \leq 1. $$

Notice that ${}_1Q_0 = \begin{pmatrix} -D_{F,x} & 0 \\ 0 & D_{F,x} \end{pmatrix}$ and ${}_1Q_1 = {}_2Q_1 = \begin{pmatrix} -D_{F,x} & I \\ I & D_{F,x} \end{pmatrix}$. Define

$$ Q_t = \begin{cases} {}_1Q_t, & 0 \leq t \leq 1 \\ {}_2Q_{(2-t)}, & 1 \leq t < 2. \end{cases} $$

**Lemma 5.1.8.** *The path $Q_t$ is continuous in the norm resolvent sense and for any $f \in C_0(\mathbb{R})$, $f(Q_t) \to 0$ in norm as $t \to 2$.*

**Proof.** As with Lemma 5.1.2, we proceed by establishing that both ${}_1Q_t$ and ${}_2Q_s$ are continuous in the norm resolvent sense and that for any $f \in C_0(\mathbb{R})$, $f({}_2Q_s) \to 0$ as $s \to 0$. Let $X = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ (in this case, $X$ turns out to be a bounded operator).

Note that for any $t, s \in [0,1]$,

$$ \begin{aligned} \|({}_1Q_t + i)^{-1} - ({}_1Q_s + i)^{-1}\| &= \|({}_1Q_t + i)^{-1}({}_1Q_s - {}_1Q_t)({}_1Q_s + i)^{-1}\| \\ &= \|({}_1Q_t + i)^{-1}(s-t)X({}_1Q_s + i)^{-1}\| \\ &\leq |s-t|\|X\| \to 0 \qquad \text{as } s \to t. \end{aligned} $$

So, ${}_1Q_t$ is continuous in the norm resolvent sense.

Let $D = \begin{pmatrix} -D_{F,x} & 0 \\ 0 & D_{F,x} \end{pmatrix}$. Then,

$$ {}_2Q_s = D + s^{-1}X, \qquad 0 < s \leq 1. $$

Now, $D$ and $X$ are self-adjoint operators on the Hilbert space $\mathcal{H} = L^2(Z,F) \oplus L^2(Z,F)$ and let $\mathcal{C} = C_c^\infty(Z,F) \oplus C_c^\infty(Z,F)$. Then $\mathcal{C}$ is dense in $\mathcal{H}$ and noting that $\operatorname{dom}(X) = \mathcal{H}$, we have:

(1) $\mathcal{C} = \operatorname{dom}(D) \cap \operatorname{dom}(X)$.
(2) $(DX + XD)v = 0$ for any $v \in \mathcal{C}$.
(3) The operators $(D + t^{-1}X)$ are essentially self-adjoint.
(4) $X$ is bounded below.

Thus, by Lemma 4.2 we may conclude that ${}_2Q_s$ is continuous in the norm resolvent sense and that for any $f \in C_0(\mathbb{R})$, $f({}_2Q_s) \to 0$ in norm as $s \to 0$. □

**Lemma 5.1.9.** *Assuming the hypotheses of Theorem 5.1.0, $\chi(Q_t)$ is a controlled path of operators on the Hilbert space $L^2(Z,F) \oplus L^2(Z,F)$, connecting*

$$ \begin{pmatrix} \chi(-D_{F,x}) & 0 \\ 0 & \chi(D_{F,x}) \end{pmatrix} $$

*to a degenerate.*

**Proof.** Partition the manifold $Z$ into a compact part $C$ and a non compact part $NC$. Let $V_1$ be projection onto the sections supported on $C$ and $V_2$ be projection onto the sections supported on $NC$. Then,

$$ \chi(Q_t) = \chi(Q_t)V_1 + \chi(Q_t)V_2. $$

The proof now proceeds in a similar manner to that of Lemma 5.1.7. □

5.1.4. Step IV. Recall that now we have to connect $\chi(D_{F,x})$ to a degenerate.

We do this as follows: Let ${}_1R_s = D_Z + x + s$ for $s \geq 0$. Notice that this has the effect of sliding the function $x$ by $s$ units to the left. Since the manifold $Z'$ (from which $Z$ is obtained by attaching the cylinder $M \times [-10, \infty)$ to the boundary $M$ of $Z'$) is compact, for some $s_0$ the function $x + s_0$ is non zero and bounded below on the entire manifold $Z$.

Let ${}_2R_s = D_Z + s^{-1}(x+s_0)$ for $0 < s \leq 1$. Note that ${}_1R_{s_0} = {}_2R_1 = D_Z + x + s_0$. Let

$$R_s = \begin{cases} {}_1R_s, & 0 \leq s \leq s_0 \\ {}_2Q_{(s_0+1-s)}, & s_0 \leq s < s_0 + 1. \end{cases}$$

**Lemma 5.1.10.** *The path $R_s$ is continuous in the norm resolvent sense and for any $f \in C_0(\mathbb{R})$, $f(R_s) \to 0$ in norm as $s \to s_0 + 1$.*

**Proof.** Proceeding as for Lemma 5.1.8, we establish continuity in the norm resolvent sense for the paths ${}_1R_s$ and ${}_2R_s$ and show that for any $f \in C_0(\mathbb{R})$, $f({}_2R_s) \to 0$ in norm as $s \to 0$. Note that for any $t, s \in [0, s_0]$,

$$\begin{aligned} \|({}_1R_t + i)^{-1} - ({}_1R_s + i)^{-1}\| &= \|({}_1R_t + i)^{-1}({}_1R_s - {}_1R_t)({}_1R_s + i)^{-1}\| \\ &= \|({}_1R_t + i)^{-1}(s-t)I({}_1R_s + i)^{-1}\| \\ &\leq |s - t| \to 0 \qquad \text{as } s \to t. \end{aligned}$$

So, ${}_1R_t$ is continuous in the norm resolvent sense.

Let $D = D_Z$ and $X = (x + s_0)$. Note that ${}_2R_s = D + s^{-1}X$. Now $D$ and $X$ are unbounded, self-adjoint operators on the Hilbert space $\mathcal{H} = L^2(Z, F)$. Let $\mathcal{C} = C_c^\infty(Z, F)$ be a dense subset of $\mathcal{H}$. Then,

(1) $\mathcal{C} \subseteq \mathrm{dom}(D) \cap \mathrm{dom}(X)$.
(2) Let $Bv = (DX + XD)v$ for $v \in \mathcal{C}$. Then, $B$ is a bounded operator since $B$ is made up out of the commutator of the differential operator $D_Z$ with the function smooth function $x$.
(3) The operators $(D + t^{-1}X)$ are essentially self-adjoint.
(4) Recall that $s_0$ was chosen so that $X = x + s_0$ will be bounded below on $\mathcal{H}$.

Thus, by applying Lemma 4.2 we may conclude that ${}_2R_s$ is continuous in the norm resolvent sense and that for any $f \in C_0(\mathbb{R})$, $f({}_2R_s) \to 0$ in norm as $s \to 0$. □

**Lemma 5.1.11.** *Assuming the hypotheses of Theorem 5.1.0, $\chi(R_s)$ is a controlled path of operators on the Hilbert space $L^2(Z, F)$, connecting $\chi(D_{F,x})$ to a degenerate.*

**Proof.** As for Lemma 5.1.9. □

5.2. **Vector bundle modification.** Suppose $(\widehat{M}, \widehat{S}, \widehat{g})$ is obtained from $(M, S, g)$ by vector bundle modification. Using techniques from Higson's work on $\mathbb{Z}/k$ index theory [Hig90], we will construct a controlled path connecting ($\chi(D_S)\oplus$ degenerate) with $\chi(D_{\widehat{S}})$.

Recall that $\widehat{M}$ is a sphere bundle over $M$ (with fibers spheres of dimension say $2n$) with projection $\pi : \widehat{M} \to M$ and $\widehat{S} = V \otimes \pi^*(S)$ where $V$ is the $+1$ eigenspace of the operator $\epsilon_1\epsilon_2$ on the complexified exterior bundle of $T_{vert}\widehat{M}$.

We work with the following decomposition of the tangent bundle of $\widehat{M}$,

$$T\widehat{M} = \pi^*(TM) \oplus T_{vert}(\widehat{M}).$$

Locally, $\widehat{M} = M \times N$ where $N$ is the sphere of dimension $2n$ and locally,

$$D_{\widehat{S}} = D_S \otimes 1 \oplus 1 \otimes D_V. \tag{5.2.1}$$

**Lemma 5.2.1.** *Let vol denote the volume form on $N$. The kernel of $D_V$ is one dimensional and is generated by $1 + vol$. Also, $D_V$ is an $SO(2n)$ equivariant operator.*

**Proof.** The kernel of the de Rham operator on $N$ is two dimensional and generated by 1 and *vol*. $1 + vol$ is in the $+1$ eigenspace of $\epsilon_1\epsilon_1$ while $1 - vol$ is in the -1 eigenspace. Thus, ker $(D_V)$ is generated by $1 + vol$.

Since $V$ is "half the complexified exterior bundle of $T_{vert}\widehat{M}$", $D_V$ can be thought of as "half the de Rham operator" in the sense that it is the de Rham operator restricted to half of its domain. Since the de Rham operator is an $SO(2n)$ equivariant operator and the splitting of $\Lambda^*$ under $\epsilon_1\epsilon_2$ is $SO(2n)$ equivariant, the operator $D_V$ is an $SO(2n)$ equivariant operator. □

Let $\{U_\alpha\}$ be a locally finite cover of $\widehat{M}$ consisting of contractible open sets and let $\{\phi_\alpha^2\}$ be a smooth partition of unity subordinate to the $U_\alpha$. Let $\overline{\phi}_\alpha$ denote the pullbacks of the $\phi_\alpha$ to $\widehat{H}$ – this is also a partition of unity. Use this to splice together the local picture (5.2.1) to write

$$D_{\widehat{S}} = \sum_\alpha \overline{\phi}_\alpha (D_S \otimes 1 + 1 \otimes D_V)\overline{\phi}_\alpha + Z_0,$$

where $Z_0$ is an order 0 differential operator arising from the fact that the symbols of both sides of (5.2.1) are the same. Note that by the $SO(2n)$ equivariance of $D_V$, $\sum \overline{\phi}_\alpha(1 \otimes D_V)\overline{\phi}_\alpha$ gives a canonical, well defined global operator on the vertical vectors of $\widehat{M}$. We will abuse notation slightly and call this global operator $1 \otimes D_V$. Thus,

$$D_{\widehat{S}} = \sum_\alpha \overline{\phi}_\alpha (D_S \otimes 1)\overline{\phi}_\alpha + 1 \otimes D_V + Z_0.$$

Our first homotopy will be to shrink off the order 0 term:

$$B_s = \sum_\alpha \overline{\phi}_\alpha (D_S \otimes 1)\overline{\phi}_\alpha + 1 \otimes D_V + (1 - s)Z_0, \qquad 0 \le s \le 1.$$

So, $B_0 = D_{\widehat{S}}$ and $B_1 = \sum_\alpha \overline{\phi}_\alpha (D_S \otimes 1)\overline{\phi}_\alpha + 1 \otimes D_V$. Let $\mathcal{U} = \ker\,(1 \otimes D_V)$. By Lemma 5.2.1, $\mathcal{U} \simeq L^2(S) \otimes \ker\,(D_V)$ which can be identified with $L^2(S)$ since ker $(D_V)$ is one dimensional. On $\mathcal{U}$,

$$B_1 = \sum_\alpha \overline{\phi}_\alpha (D_S \otimes 1)\overline{\phi}_\alpha,$$

and by a symbol calculation similar to the one above,

$$\sum_\alpha \overline{\phi}_\alpha (D_S \otimes 1)\overline{\phi}_\alpha + Z_1 = D_S,$$

for an order 0 operator $Z_1$. Thus, we may connect $D_S$ to $B_1$ by

$$B_s = \sum_\alpha \overline{\phi}_\alpha (D_S \otimes 1)\overline{\phi}_\alpha + (s - 1)Z_1, \qquad 1 \le s \le 2.$$

To complete the construction of our path we will connect $B_1$ on $\mathcal{U}^\perp$ to a degenerate using the same techniques as in stage I of the cobordism argument. On $\mathcal{U}^\perp$, the operator $1 \otimes D_V^2$ is bounded below (in the sense that it has a minimum, non zero eigenvalue) and thus

$$D_t = \sum_\alpha \overline{\phi}_\alpha (D_S \otimes 1)\overline{\phi}_\alpha \oplus t^{-1}(1 \otimes D_V) \quad 0 < t \leq 1,$$

will connect $B_1$ on $\mathcal{U}^\perp$ to a degenerate.

**Lemma 5.2.2.** *The paths $B_s$, $D_t$ are continuous in the norm resolvent sense and for any function $f \in C_0(\mathbb{R})$, $f(D_t)$ converges in norm to $0$ as $t \to 0$.*

**Proof.** For the paths $B_s$ this is straightforward since $Z_k$ is a bounded operator for $k = 0, 1$ and so

$$\begin{aligned} \|(B_s + i)^{-1} - (B_t + i)^{-1}\| &= \|(B_s + i)^{-1}(t - s)Z_k(B_t + i)^{-1}\| \\ &\leq |t - s|\|Z_k\| \to 0 \quad \text{as } t \to s. \end{aligned}$$

The path $D_t$ is of the form $D + t^{-1}X$ where $D = \sum_\alpha \overline{\phi}_\alpha(D_S \otimes 1)\overline{\phi}_\alpha$ and $X = 1 \otimes D_V$ on the space $U^\perp$. Thus $X$ is bounded below and $D$ and $X$ anticommute. Let $\mathcal{C} = C^\infty(\widehat{M})$ be the space of smooth sections of the bundle $\widehat{S}$ over $\widehat{M}$. This is dense in the Hilbert space $L^2(\widehat{M}, \widehat{S})$ and is also in the domains of $D$ and $X$. Further, since $D$ and $X$ are local operators, $(i + D + t^{-1}X)^{-1}\mathcal{C} \subset \mathcal{C}$. Thus, by Lemma 4.2 we conclude that $D_t$ is continuous in the norm resolvent sense and for any $f \in C_0(\mathbb{R})$, $f(D_t) \to 0$ as $t \to 0$. □

**Lemma 5.2.3.** *If $f$ is a Schwartz class function whose Fourier transform is smooth and compactly supported within $[-1, 1]$, then $f(B_s)$ and $f(D_t)$ are smooth in the trace norm and consist of trace class operators.*

**Proof.** Since $\widehat{M}$ is a compact manifold we may use the functional calculus for pseudodifferential operators [Tay81, Ch. 12] to conclude that the operators $f(B_s)$ and $f(D_t)$ are trace class. For the smoothness in the trace norm we use Theorem 5.1.5. □

**Lemma 5.2.4.** *Let $Y$ be a compact Riemannian manifold and let $(M, S, g)$ be a $K$-cycle for $Y$ with the property that $g : M \to Y$ is Lipschitz. Suppose $(\widehat{M}, \widehat{S}, \widehat{g})$ is obtained from $(M, S, g)$ by vector bundle modification. Let $\chi$ be a chopping function such that*

1. *The Fourier transform of $\chi$ is smooth and compactly supported;*
2. *The functions $\chi^2 - 1$ and $\chi(\chi^2 - 1)$ are in the Schwartz class and their Fourier transforms are smooth and supported in $[-1, 1]$.*

*Then $\chi(B_s), \chi(D_t)$ are controlled paths defined on the Hilbert space $L^2(\widehat{M}, \widehat{S})$. Their concatenation connects $\chi(D_S) \oplus$ degenerate to $\chi(D_{\widehat{S}})$.*

**Proof.** The operators $B_s$ are bounded perturbations of Dirac operators on compact manifolds and $D_t$ are of the form $D + t^{-1}D'$ where $D$ and $D'$ are both Dirac operators on compact manifolds. Thus, the spectra of $B_s$ and $D_t$ grow at a polynomial rate [Roe88, Thm. 7.3]. Since the map $\widehat{g} : \widehat{M} \to Y$ is the composition of Lipschitz maps, it is itself Lipschitz and so, by Lemma 4.7 we may conclude that $\chi(B_s), \chi(D_t)$ have polynomial growth.

Lemma 5.2.3 settles (2) of the Definition 3.6 and so we may conclude that $\chi(B_s)$ and $\chi(D_t)$ are controlled paths. □

We gather together the work of this section to prove its main theorem:

**Proof of Theorem 3.8.** It suffices to prove the theorem for the situation in which $(M', S', g')$ is obtained from $(M, S, g)$ by *one* move of either bordism, vector bundle modification or direct sum. Theorem 5.1.0 settles the case for bordism and Lemma 5.2.4 settles vector bundle modification. Recall that the operation of direct sum simply states that if $S = S_1 \oplus S_2$ is a direct sum decomposition of a complex, Hermitian vector bundle $E$ over an oriented manifold $M$, then

$$(M, S, g) \sim (M, S_1, g) \cup (M, S_2, g).$$

Since the Hilbert space $L^2(M, S)$ of sections of $S$ over $M$ will split as $L^2(M, S_1)$ $\oplus L^2(M, S_2)$, it follows that the Dirac operator on $M$ with coefficients in $S$, $D_S$, can be written as

$$D_S = D_{S_1} \oplus D_{S_2}.$$

Thus, this operation is trivially realised. □

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