New York Journal of Mathematics

New York J. Math. **4** (1998) 57–74.

Algebraic Non-Integrability of the Cohen Map

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Abstract. The map $\phi(x, y) = (\sqrt{1 + x^2} - y, x)$ of the plane is area preserving and has the remarkable property that in numerical studies it shows exact integrability: The plane is a union of smooth, disjoint, invariant curves of the map ϕ. However, the integral has not explicitly been known. In the current paper we will show that the map ϕ does not have an algebraic integral, i.e., there is no non-constant function $F(x, y)$ such that

1. $F \circ \phi = F$;
2. There exists a polynomial $G(x, y, z)$ of three variables with

$$G(x, y, F(x, y)) = 0.$$

Thus, the integral of ϕ, if it does exist, will have complicated singularities. We also argue that if there is an analytic integral F, then there would be a dense set of its level curves which are algebraic, and an uncountable and dense set of its level curves which are not algebraic.

Contents

Received April 24, 1997.

Mathematics Subject Classification. 58F23, 39A, 52A10.

Key words and phrases. algebraic, integrability, multi-valued map, complex dynamics.

This research has been supported in part by the National Science Foundation under grant no. DMS 9404419.

ISSN 1076-9803/98

1. Introduction

The area-preserving map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by the formula

$$\phi(x, y) = (\sqrt{1 + x^2} - y, x) \tag{1}$$

and the question of its integrability has been attributed to H. Cohen [Mos] and we learned about it from O. Knill. This map is a particular example in the family of area-preserving maps of the form $(x, y) \mapsto (f(x) - y, x)$. When $f(x) = 2x + k \sin x$, this map is equivalent to the standard map introduced in [Chi79].

When studied numerically, the map ϕ exhibits numerical integrability. Thus, up to the precision of floating point arithmetic, this system has the following properties (cf. Figure 1):

> For every initial condition (x_0, y_0) the trajectory of the map ϕ calculated according to the formula $(x_n, y_n) = \phi(x_{n-1}, y_{n-1})$ lies on a smooth, possibly analytic, Jordan curve, except for the singularity at the fixed point $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

It has been conjectured that there exists an integral $F : \mathbb{R}^2 \to \mathbb{R}$ of ϕ with good analytic properties. By an integral we mean a non-constant function satisfying the equation

$$F \circ \phi = F. \tag{2}$$

This integral is called *algebraic* iff there is a polynomial $G : \mathbb{R}^3 \to \mathbb{R}$ such that

$$G(x, y, F(x, y)) = 0. \tag{3}$$

In the current paper we show that this conjecture is false:

Theorem 1.1. *There is no algebraic integral for the map ϕ.*

Let us briefly explain the reasoning which may lead one to conjecture that the map ϕ does have an algebraic integral. Let us define the following family of mappings:

$$\phi_\epsilon(x, y) = (\sqrt{\epsilon^2 + x^2} - y, x). \tag{4}$$

Every mapping in this family, except for $\epsilon = 0$, is linearly equivalent to the map $\phi = \phi_1$ via the map $(x, y) \mapsto (\epsilon x, \epsilon y)$. On the other hand, the piecewise-linear map $\phi_0(x, y) = (|x| - y, x)$ has a piecewise linear and thus algebraic integral:

(5)
$$F_0(x, y) = y + |y - |x|| + |x - |y - |x||| + |y - |x - |y||| + |x - |y| + |y - |x - |y||||.$$

This formula is due to O. Knill. Thus, we may expect that the conjectured integral for ϕ_ϵ may arise from F_0 in the same way as ϕ_ϵ arises from ϕ_0 by replacing $|x|$ with $\sqrt{\epsilon^2 + x^2}$. Thus, this integral would be an algebraic function and obtained by successively taking quadratic algebraic extensions of the field of rational functions $\mathbb{R}(x, y)$. Theorem 1.1 shows that this is not the case.

Let us suppose that the map (1) does have an integral $F(x, y)$ which is analytic in the plane. Moreover, let us assume that the family of the level curves $\{\gamma_c\}_{c \in I}$, where $I = F(\mathbb{R}^2) \subset \mathbb{R}$ is a certain segment and $\gamma_c = F^{-1}(c)$ is a level curve of F. Let us assume that this is a real-analytic family of invariant curves diffeomorphic to the circle. In the remainder of this introduction we will study the implications of this assumption and of our theorem.

For every curve γ_c of this analytic family, the map $\phi|\gamma_c$ has a rotation number $\rho(c)$ varying continuously with c. The rotation number may not be constant. For instance, near the fixed point $P_0 = (1/\sqrt{3}, 1/\sqrt{3})$ this number must be close to $\rho_0 = \arg(\lambda)/(2\pi)$, where λ is one of the two complex eigenvalues of the derivative $D\phi(P_0)$. It is clear that $\lambda + \lambda^{-1} = 1$. From this equation we deduce that ρ_0 is irrational and diophantine (badly approximated by rationals).

On the other hand, the map ϕ near ∞ resembles the map ϕ_0. Let $\eta_c = F_0^{-1}(c)$ be a level curve of F_0. One can see that η_c is a non-convex polygon with 9 sides. The rotation number of $\phi_0|\eta_c$ is $2/9$ and does not depend on c. This is explained in Figure 3. Numerical studies indicate that the invariant curves of ϕ near infinity only roughly approximate the polygonal invariant curves of ϕ_0 (Figure 2). We note that $\phi_0^9 = id$. Thus there are many other continuous integrals of ϕ_0.

The slight non-linearity of the invariant curves persists and even near ∞ the segments of the invariant curves of ϕ cannot be approximated by straight line segments. This effect can be studied analytically [AE90]. One discovers that the limit

$$X(x,y) = \lim_{\epsilon \longrightarrow 0} \frac{1}{\epsilon^2} \left[\phi_\epsilon^9(x,y) - (x,y) \right] \tag{6}$$

exists as a vector field, and is analytic except for a set of singularities along 9 rays contained in the lines $x = 0$, $y = 0$ $x = \pm y$, $y = 1/2x$ $(x > 0)$ and $y = 2x$ $(x > 0)$. The integral curves of this vector field give the first-order approximation to the invariant curves of ϕ near infinity. We know an explicit expression for $X(x,y)$. Fortunately, the calculation of $X(x,y)$ is within the reach of symbolic computation systems, and even calculation by hand. In [AE90, pp. 219–221], one finds a slightly more general example than ϕ_ϵ and an expression for $X(x,y)$. We find that the vector field $X(x,y)$ on the quadrant $x < 0$, $y < 0$ is given by the following equations:

$$X(x,y) = \frac{1}{2}\left(\frac{2}{x+2y} + \frac{1}{x+y} + \frac{1}{2x+y}, -\left(\frac{1}{x+2y} + \frac{1}{x+y} + \frac{2}{2x+y} \right) \right). \tag{7}$$

One can even calculate the first integral of this vector field:

$$H(x,y) = \frac{1}{2} \log \left[(x+2y)(x+y)(2x+y) \right]. \tag{8}$$

One finds that $X(x,y)$ is a Hamiltonian vector field with respect to the symplectic form $dy \wedge dx$ and that H is the corresponding Hamiltonian, i.e., $X = (H_x, -H_y)$. Similar expressions in the other 8 sectors of the plane can be obtained by linear changes of variables $(x,y) \mapsto (\pm x - y, x)$ and their suitable compositions. We would like to emphasize that the form of the expression for $X(x,y)$ *will change* under these coordinate changes. It is the vector field $X(x,y)$ whose integral curves approximate the invariant curves of the map ϕ near ∞. These integral curves are the level curves of H and thus are algebraic. Moreover, in [AE90] the authors showed (using KAM theory) that indeed there are uncountably many invariant curves near ∞ approximated by the integral curves of $X(x,y)$. It is the nature of KAM theory that the rotation number on these curves is irrational and it assumes a positive measure set of values.

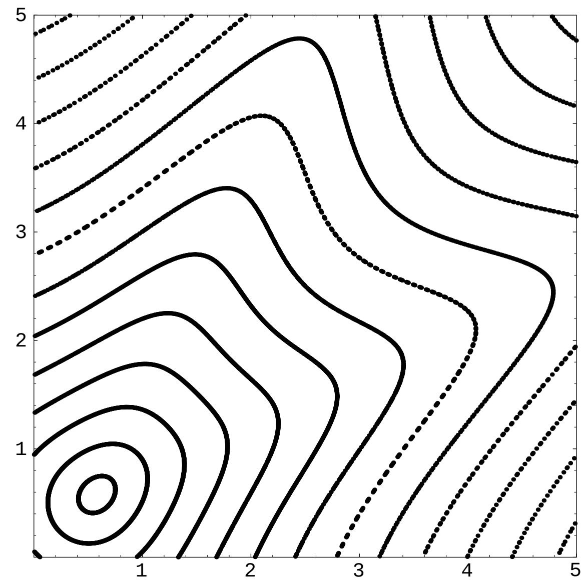


FIGURE 1. Numerically found invariant curves of the map ϕ in the square $[0, 5] \times [0, 5]$. These are simply very long orbits of ϕ.

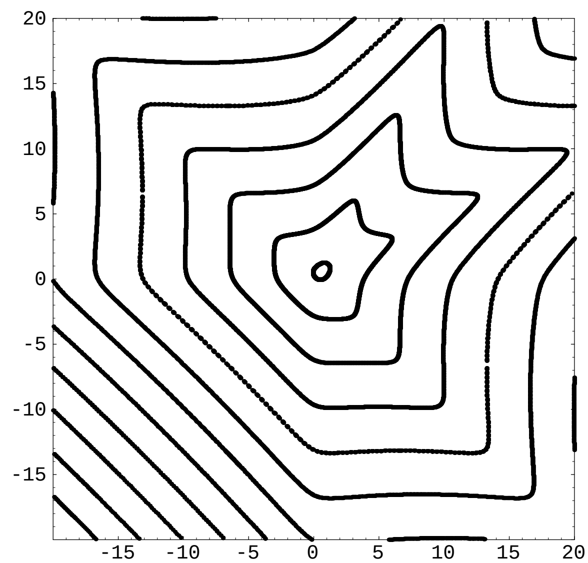


FIGURE 2. Integral curves of the map ϕ in the square $[-20, 20] \times [-20, 20]$.

It is reasonable to conjecture that the rotation number $\rho(c)$ depends analytically on c (except for the singularity at P_0). Near P_0 this function must be close to ρ_0 and near ∞ it must be close to $2/9$. It is not constant in any open subset of the plane. Thus, $\rho(c)$ takes on rational values on a dense set of values $c \in I$. The

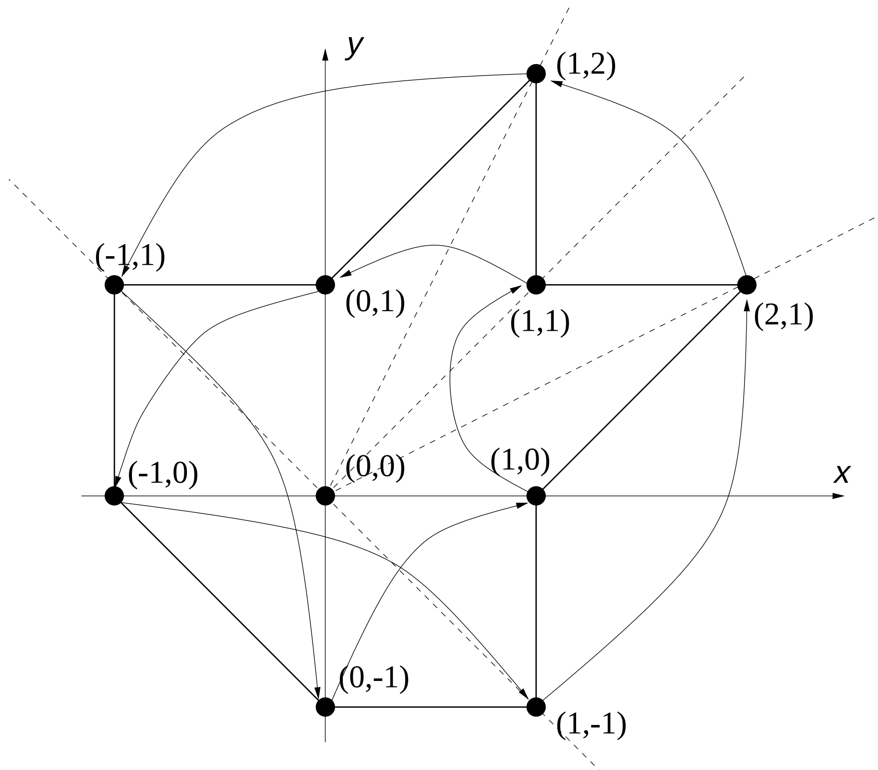


FIGURE 3. An invariant partition of the plane into 9 invariant sectors for ϕ_0. The axes and the dashed rays partition the plane into 9 sectors. Moreover, ϕ_0 is affine on each of the sectors and $\phi_0^9 = id$. The orbit of $(1,0)$ consists of 9 points marked in the figure. The 9-gon with vertices at these 9 points is invariant.

analyticity of the family and area preservation imply that the conditional measures of the partition of the plane into the invariant curves γ_c are absolutely continuous, and indeed have analytic densities. Thus, if $\rho(c)$ is rational then indeed γ_c consists of periodic orbits of ϕ of a fixed period. In view of the fact that the equation $\phi^n(x,y) = (x,y)$ is algebraic, we conclude that γ_c is an algebraic curve. Hence, our assumptions lead to the conclusion that algebraic curves are dense in the family $\{\gamma_c\}$. We have justified the following:

Conjecture 1.2. *The map ϕ has a non-constant, real-analytic integral $F : \mathbb{R}^2 \to \mathbb{R}$. The family of its level curves $\gamma_c = F^{-1}(c)$ contains a countable, dense in $\mathbb{R}^2$ set of algebraic curves and a dense in $\mathbb{R}^2$, non-countable set of non-algebraic curves.*

It is not obvious how to construct real-analytic or even C^∞ functions F with the property described in the above conjecture, even without the property $F \circ \phi = F$.

2. Acknowledgments

We express our gratitude to O. Knill for showing us the main example studied in this paper and for helpful discussions during our work on the manuscript.

3. Algebraic Relations

In the course of the proof of Theorem 1.1 we complexify our map. However, one immediately observes that the map $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ given by formula (1) is multi-valued. This creates some problems. However, a systematic approach to the dynamics of algebraic relations has been developed in [Ryc97, Ryc96], at least in some respects. For instance, criteria for the existence of invariant algebraic manifolds can be developed. Such criteria are key to the current result.

Let us consider the following algebraic set in $\mathbb{C}^2 \times \mathbb{C}^2$:

$$R = \{((x_1, y_1), (x_2, y_2)) : (x_2 + y_1)^2 = 1 + x_1^2,\ y_2 = x_1\} \tag{9}$$

It is not difficult to verify that this set is the Zariski closure of the union of the graphs of the local branches of the multi-valued map ϕ. This set replaces the map ϕ naturally in many dynamical considerations and allows one to carry out many standard constructions in a rigorous manner. We will often write graph(ϕ), or even ϕ in place of R, in order to clearly distinguish the algebraic relation induced by ϕ from other relations that may be discussed.

We recall that an algebraic variety W is called *pure-dimensional* if all irreducible components of W have the same dimension. If not explicitly stated, every algebraic variety under consideration is assumed to be pure-dimensional.

Definition 3.1. A pure-dimensional sub-variety $R \subset X \times Y$, where X and Y are pure-dimensional projective varieties, is called a *non-singular algebraic relation* if the following properties hold:

1. $\dim R = \dim X = \dim Y$;
2. For every irreducible component $S \subset R$ and a coordinate projection π_j, $j = 1, 2$, the set $\pi_j(S)$ is Zariski-dense in an irreducible component of X or Y ($\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$);
3. $\pi_1(R) = X$ and $\pi_2(R) = Y$. (If X or Y is irreducible then this assumption follows from the previous assumptions.)

By a *global branch* (or simply a *branch*) of the relation R we simply mean an irreducible component of R, which by itself is also a non-singular algebraic relation. When R is irreducible as a variety we will call it an *irreducible algebraic relation.* Thus an irreducible algebraic relation has only one (perhaps multi-valued) branch.

By a (non-singular) *local branch* of an algebraic relation we mean any diffeomorphism $\psi : U \to Y$ where $U \subseteq X$ is an open (in the standard, i.e., not Zariski, topology) set and such that graph(ψ) $\subseteq R$, where

$$\mathrm{graph}(\psi) = \{(P, \psi(P)) : P \in U\}.$$

It is easy to see that the number of local branches defined in an open neighborhood of a point $P \in X$ is constant on a (Zariski) open and dense subset of X. We will say that an algebraic relation is *k-valued* if generically there are k local branches of this algebraic relation.

Remark 3.2. In much of the algebraic geometry literature it is assumed that a variety is an algebraic set which is irreducible. In our paper the word *variety* is synonymous with an *algebraic set*. When we speak of an irreducible algebraic set, we will make this assumption explicit.

From the dynamical systems point of view, the most interesting case of an algebraic relation is obtained when $X = Y$ because then R can be considered as a multi-valued self-map and a dynamical system.

We note that relations can be composed and inverted in the usual way: if $R_j \subset X_{j-1} \times X_j$ is an algebraic relation for $j = 1, 2, \ldots, k$ then $R_1 \circ R_2 \circ \cdots \circ R_k$ is defined as the set:

$$\Big\{(P_0, P_k) \in X_0 \times X_k : \exists\ P_1 \in X_1, P_2 \in X_2, \ldots, P_{k-1} \in X_{k-1}\ \forall\ j \in \{1, 2, \ldots, k\}\quad (P_{j-1}, P_j) \in R_j\Big\}.$$

The inverse of R is defined as

$$R^{-1} = \{(P, Q) \in Y \times X : (Q, P) \in R\}. \tag{10}$$

We note that in general $R^{-1} \circ R \neq \Delta$, where $\Delta = \{(x, x) \in X \times X\}$ is the graph of *id*. For example, the graph of ϕ has this property, as will be shown in the proof of Lemma 5.1.

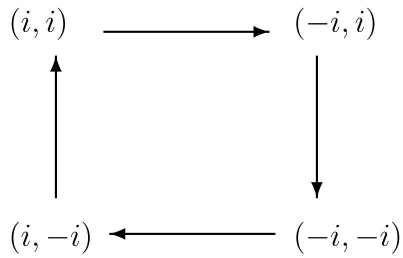
The notation R^k abbreviates $R \circ R \circ \cdots \circ R$ (k times). A composition of (non-singular) algebraic relations is again an (non-singular) algebraic relation. The inverse of an (non-singular) algebraic relation is an (non-singular) algebraic relation.

It follows from this definition that R (and R^{-1}) are unions of graphs of local diffeomorphisms, except for an algebraic subset of R of positive co-dimension. We will call this exceptional set the *singular set* of R and denote it by R^{sing}. This restriction can always be satisfied in practice because if it is violated, the dynamics can be restricted to a smaller variety and the restricted relation will satisfy the property that R^{sing} has positive co-dimension. We note that the singular set includes points due to the singularity of X, Y or the map itself. The sets $\pi_i(R^{sing})$, $i = 1, 2$, will also be called the singular sets, with little room for confusion. Strictly speaking, $\mathbb{C}^2$ is not a projective variety but an open subset of $\mathbb{P}^1(\mathbb{C})^2$. The projectivization of our problem can be performed by using homogeneous coordinates. However, this would introduce an unnecessary burden on our notation. We will avoid this inconvenience in a standard way: we will consider $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ (Riemann sphere) and in suitable places we will extend arguments for "finite" points by considering neighborhoods of $x = \infty$ and $y = \infty$ separately.

It can happen that the composition of an irreducible relation with itself is reducible. For example, $\phi^{-1} \circ \phi$ is reducible as will be shown in the proof of Lemma 5.1. However, we have the following lemma:

Lemma 3.3. *Let* $R = \operatorname{graph}(\phi)$. *For every* k *the algebraic relation* R^k *is irreducible.*

Proof. First we observe that the 4-point set $K = \{(\pm i, \pm i)\}$ is *completely invariant*, i.e., whenever $(x, y) \in K$ then every value of $\phi(x, y)$ is also in K. Moreover, for every point (x, y) in this set there is only one value of $\phi(x, y)$ and thus by analogy with the theory of functions of one complex variable the members of K are called *branch points* of the multi-valued map ϕ. The set K is mapped periodically into itself, according to the rule:



The idea of our argument is to show that the k-th image of the curve $\gamma(t) = (\pm i+t, i)$ is an irreducible curve which is branched over $t = 0$ of order 2^k. If R^k is reducible then also this image would be reducible. Thus, proving 2^k-fold branching for the test curve γ is sufficient to show our result. In order for this argument to work we need to require of the test curve γ that it is not contained in the singular set of the multi-valued map ϕ, i.e., that $x(\gamma(t))$ is not identically $\pm i$.

We will show that if $\gamma(t) = (\pm i + ct + O(t^2), \pm i + O(t^2))$ is a germ of an analytic curve at $t = 0$ and $c \neq 0$ then there is a germ of an analytic curve η of the same form as γ and such that

$$\phi \circ \gamma(s^2) = \eta(s). \tag{11}$$

Moreover, for every $t \neq 0$ sufficiently close to 0 and for every two values of the composition $\phi \circ \gamma(t)$ there is exactly one root of the equation $s^2 = t$ such that $\phi \circ \gamma(t) = \eta(s)$, i.e., the new parameterization is locally 1:1 near 0. The following calculation proves this fact:

$$\begin{aligned}
\phi(\gamma(t)) &= (\sqrt{1 + (\pm i + ct + O(t^2))^2} \mp i - O(t^2), \pm i + ct + O(t^2)) \\
&= (\mp i + \sqrt{2ict} + O(t^{3/2}), \pm i + ct + O(t^2)) \\
&= (\mp i + \sqrt{2ic}s + O(s^3), \pm i + O(s^2))
\end{aligned}$$

and the last expression is clearly analytic in s. Now by induction we show that for every k the curve $\phi^k \circ \gamma$ can be parameterized locally 1:1 by an analytic function of s, where $s^{2^k} = t$. □

Remark 3.4. We have shown that the monodromy group of the curve $R^k \circ \gamma(t)$ at 0 is a cyclic group of order 2^k.

4. Invariant Varieties

The local theory of invariant manifolds for algebraic relations is identical to the standard local theory of dynamical systems, provided that we stay away from the singular set. However, the global theory is richer than that of single-valued diffeomorphisms. For instance, the global invariant curve of a hyperbolic fixed point can have genus ≥ 1. In particular, it does not have to be an embedded copy of $\mathbb{C}$ (cf. [Ryc97]).

We adopt the following definition:

Definition 4.1. Let $R \subset X \times X$ be a non-singular algebraic relation. A sub-variety $V \subset X$ is called *invariant* if there is a non-singular algebraic relation S on V such that $S \subseteq R$ under the natural inclusion $V \times V \subseteq X \times X$.

In practical terms, this definition means that a sub-variety $V \subseteq \mathbb{C}^2$ is invariant under the multi-valued map ϕ if for every point $(x, y) \in V$ at least one of the two values of $\phi(x, y) = (\pm\sqrt{1 + x^2} - y, x)$ belongs to V. This condition can be relaxed to hold on V except for a sub-variety $V_1 \subset V$ of positive co-dimension. If V is of dimension 1 (the most interesting case) then V_1 is a finite set.

We will use the terms "multi-valued map" and "non-singular algebraic relation" interchangeably. Arguments using the notion of a "branch of a multi-valued map" tend to be somewhat non-rigorous but very intuitive. Typically such arguments can be easily formalized. The usual procedure introduces a non-singular algebraic relation which is the Zariski closure of the union of the graphs of the local branches of the multi-valued map.

5. The Existence of a Single-Valued Branch

Let us suppose that V is a pure-dimensional invariant variety of dimension 1 for the multi-valued map ϕ. We will simply refer to such a variety as *curve*. *A priori* it cannot be assumed that V is irreducible. The condition that *locally* at least one branch of ϕ maps V to V allows us to show the following:

Lemma 5.1. *Let V be a ϕ-invariant curve which does not contain a horizontal line $\mathbb{C} \times \{y_0\}$ for any $y_0 \in \mathbb{C}$ (equivalently, it does not contain a vertical line $\{x_0\} \times \mathbb{C}$). Let $S \subset \mathrm{graph}(\phi) \cap (V \times V)$ be a non-singular sub-relation of ϕ on V such that V is still invariant under S.*

The following objects exist:

1. *An invariant curve $V' \subset V$ and a decomposition*

$$V' = \bigcup_{j=0}^{r-1} V_j \tag{12}$$

of V' into irreducible components;

2. *For every $j \in \{0, 1, \ldots, r-1\}$ there is a choice of a bi-rational map $\phi_j : V_j \to V_{j+1}$ such that $\mathrm{graph}(\phi_j) \subset S$ and ϕ_j is a single-valued branch of $\phi|V_j$. We assume that $j + 1 = 0$ for $j = r - 1$, i.e., the index arithmetic is modulo r.*

Proof. We note that if we replace ϕ_j with a relation S_j then the theorem is true. In other words, the lemma is obvious if we allow ϕ_j to be multi-valued. Thus, we need to see that ϕ_j is indeed single-valued except for a finite set of removable singularities. It is known that any such map is bi-rational [Sha94], Volume II, p. 179, Theorem 1.

First, we observe that for every variety W the map $\phi|W$ has either one 2-valued branch or 2 single-valued branches. This holds iff the algebraic function $\sqrt{1 + x^2}$ is 2-valued on W or it splits into two single-valued branches, respectively. This statement can be made even more explicit. Let $Q = \{P_1, P_2, \ldots, P_s\}$ be the intersection of the variety W with the set $1 + x^2 = 0$, i.e., with the union of the lines $x = \pm i$. Let $P_j = (\pm i, y_j)$ be one of these points. There is a punctured neighborhood of 0 in $\mathbb{C}$ and a 1:1 meromorphic function $\psi_j : U \to V \subset \mathbb{C}^2$ which parameterizes a neighborhood of P_j in V. Let $\psi_j = (\psi_j^1, \psi_j^2)$. There is an expression of $1 + x^2$ in terms of the local uniformizing parameter

$$1 + (\psi_j^1(\zeta))^2 = \sum_{k=\nu_j}^{\infty} a_k \zeta^k = \zeta^{\nu_j} h_j(\zeta) \tag{13}$$

where $a_{\nu_j} \neq 0$ and h_j is analytic at 0 and $h_j(0) \neq 0$. We note that the left-hand side is not identically 0. It is easy to see that locally $\sqrt{1+x^2}$ is 2-valued iff ν is odd. Otherwise, it has two single-valued branches. It is clear that the function $\sqrt{1+x^2}$ has two single-valued branches on W iff it has a single-valued branch at every point $P_j \in Q$, i.e., for all j the number ν_j is even.

Let us assume that the map ψ_j is 2-valued for some j. We may assume without loss of generality that $\psi_0 : V_0 \to V_1$ is 2-valued. Let us consider the inverse (multi-valued map) $\phi^{-1}(x,y) = (y, \sqrt{1+y^2} - x)$ on V_1. A formal calculation shows that

$$\phi \circ \phi^{-1}(x,y) = (x \pm \sqrt{1+y^2} \pm \sqrt{1+y^2}, y). \tag{14}$$

Any combination of signs is permitted. We note that the right-hand side represents a *3-valued* map on $\mathbb{C}^2$. This is seemingly contradictory, as we should obtain a 4-valued map by composing two 2-valued maps. However, this contradiction can be removed by considering the following generalization of Definition 3.1 including multiplicities:

Definition 5.2. A *system of non-singular algebraic relations* is a formal finite linear combination $\sum_j n_j R_j$, where R_j is a non-singular algebraic relation contained in $X \times Y$ and n_j is a positive integer coefficient . Thus R_j is a non-singular algebraic relation. Let r_j denote the number of branches of R_j. We will say that $\mathcal{R}$ is r-valued, where $r = \sum_j n_j r_j$.

An ordinary non-singular algebraic relation R in the sense of Definition 3.1 is represented as $1 \cdot R$ or by an ambiguous but convenient abbreviation R. If $\mathcal{R} = \sum_j n_j R_j$ is a system of non-singular algebraic relations in $X \times Y$ and $\mathcal{S} = \sum_l m_l S_l$ is another such system in $Y \times Z$ then the composition $\mathcal{S} \circ \mathcal{R}$ is well defined as a system of non-singular algebraic relations in $X \times Z$. Formally,

$$\mathcal{S} \circ \mathcal{R} = \sum_{j,l} m_l n_j \ (S_l \circ R_j) \tag{15}$$

with the understanding that a pair of varieties $S_l \circ R_j$ and $S_{l'} \circ R_{j'}$ in $X \times Z$ could coincide. When this happens, we combine the corresponding terms in the above sum. One can show easily that if $\mathcal{R}$ is r-valued and $\mathcal{S}$ is s-valued then $\mathcal{S} \circ \mathcal{R}$ is rs-valued.

The intuition behind the notion of a system of non-singular algebraic relations is that the term $n_j R_j$ represents n_j copies of R_j which are "infinitely close". We also note that an algebraic variety $R \subseteq X \times Y$ which is not necessarily irreducible but $R = \bigcup_j R_j$ where R_j is a non-singular algebraic relation, is naturally represented by a system of non-singular algebraic relations $\sum_j R_j$.

Compositions of non-singular algebraic relations may naturally be systems of algebraic relations, as formula 14 demonstrates. We interpret the calculations leading to formula 14 by writing

$$\phi \circ \phi^{-1} = S + 2\Delta, \tag{16}$$

where S is the graph of the 2-valued map $(x, y) \mapsto (x \pm 2\sqrt{1+y^2}, y)$ and Δ is the diagonal (the graph of id). This formalism can be developed further and made completely rigorous and analogous to the notion of a divisor, but we will not pursue this level of generality here.

We claim that regardless of whether $\phi^{-1}|V_1$ is multi-valued or not, there is a branch of $\phi^{-1} \circ \phi$ (single- or multi-valued) different from id for which V_1 is invariant. Let us consider both cases separately:

1. There exist two single-valued branches of $\phi^{-1}|V_1$. In this case, one of them maps V_1 to V_0. The composition of this branch with ϕ_1 (the outer function is ϕ_1) is a 2-valued map $V_1 \to V_1$ and is given by (14) for some choice of the signs. Moreover, since the resulting multi-valued map has two distinct values for some values of the argument, there must be a branch of it given by the formula:

$$\phi \circ \phi^{-1}(x, y) = (x \pm 2\sqrt{1+y^2}, y). \tag{17}$$

   In view of the fact that $\sqrt{1+y^2}$ has two single-valued branches on V_1, the above formula represents two single-valued maps, one of which preserves V_1. The two maps are inverses of each other and thus both preserve V_1.
2. The map $\phi^{-1}|V_1$ is 2-valued. In this case, for every point $(x, y) \in V_1$ and for all choices of signs in (14) the right-hand side belongs to V_1. Thus, all branches of $\phi \circ \phi^{-1}$ preserve V_1.

In both cases there is a branch of the map $(x, y) \mapsto (x \pm 2\sqrt{1+y^2}, y)$ (single-valued in the first case and 2-valued in the second case) which preserves V_1. In particular, by iterating this branch we show that for every integer n the multi-valued map $(x, y) \mapsto (x \pm 2n\sqrt{1+y^2}, y)$ has a branch that preserves V_1. Thus V_1 has an infinite intersection with a line $\mathbb{C} \times \{y_0\}$ for some $y_0 \neq \pm i$ and therefore it must coincide with this line. This is a contradiction with our assumptions. □

We note that if there exists an algebraic integral $F(x, y)$ then only a finite number of its level curves can contain a vertical or horizontal line or F is constant. Perhaps with a little more work we could show that ϕ^n does not preserve any vertical or horizontal line. This result is not needed in the proof of our main theorem, but it is of interest while pursuing the sharpest result possible.

6. The Functional Equation

Let us assume that $V = \bigcup_{j=0}^{r-1} V_j$ is an invariant curve for the map ϕ and that $\phi_j : V_j \to V_{j+1}$ is a bi-rational map between the variety V_j and V_{j+1} and a branch of ϕ. The arithmetic of the index j is modulo r.

For a given algebraic curve W, let $\hat{W}$ denote the Riemann surface associated with the curve W. Let $\pi_W : \hat{W} \to W$ be the natural projection. There exist many constructions of $(\hat{W}, \pi_W)$ but this pair is determined uniquely up to an isomorphism by the following property: There exists a finite subset $B \subset W$ such that if $\hat{B} = \pi_W^{-1}(B)$ then the restriction $\pi_W|\hat{W}\backslash\hat{B} : \hat{W}\backslash\hat{B} \to W\backslash B$ is a biholormorphic map. Moreover, if W is a projective algebraic curve then $\hat{W}$ is compact. We will implicitly

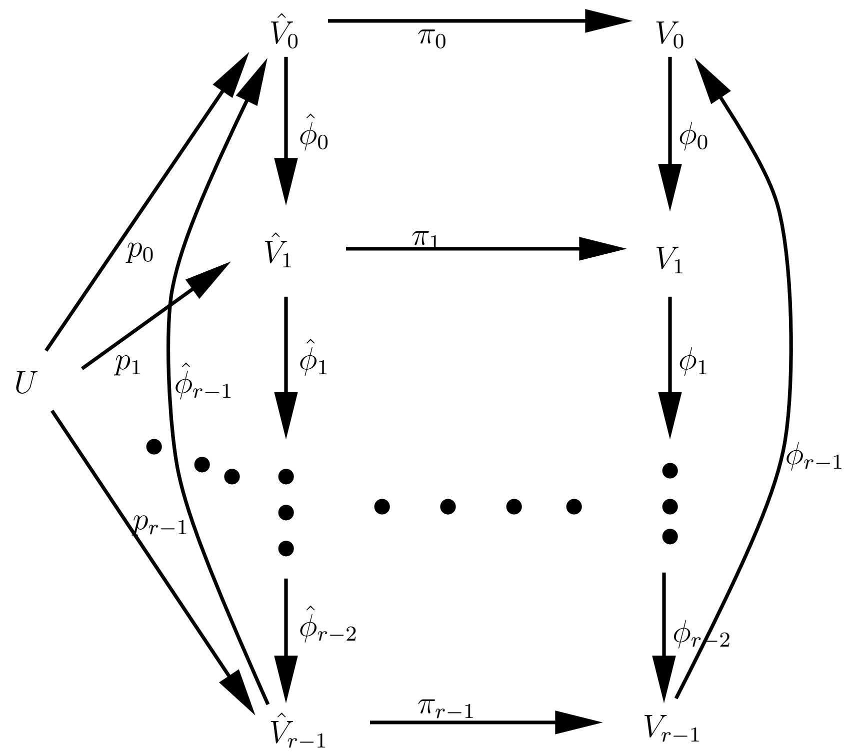


FIGURE 4. The diagram of maps and curves

assume that each algebraic curve which we are dealing with in the current paper is a projective curve, i.e., that the points at ∞ have been added to it.

Using Uniformization Theory, we may translate the condition that V is invariant into a sequence of functional equations for meromorphic functions on simply connected Riemann surfaces: $\mathbb{C}$, $\mathbb{D}$ or $\mathbb{P}^1(\mathbb{C})$ (Riemann sphere). Indeed, we have a commuting diagram pictured in Figure 4. In this diagram U is one of the three models ($\mathbb{C}$, $\mathbb{D}$ or $\mathbb{P}^1(\mathbb{C})$) of simply connected Riemann surfaces which is a universal covering space of $\hat{V}_0$. As $\hat{V}_0$ is isomorphic to each of $\hat{V}_j$, we also have universal covering maps $p_j : U \to \hat{V}_j$. For every covering map p_j, let Γ_j be the corresponding group of deck transformations. Thus $\Gamma_j \subset \mathrm{Aut}(U)$, where $\mathrm{Aut}(U)$ stands for the group of automorphisms of U.

Each of the bi-rational maps ϕ_j lifts to a bi-holomorphic map $\hat{\phi}_j : \hat{V}_j \to \hat{V}_{j+1}$ and to an automorphism $\mu_j : U \to U$. Moreover, for $j = 0, 1, \ldots, r-1$ we have $\Gamma_{j+1} = \mu_j \Gamma_j \mu_j^{-1}$.

The map $\mu = \mu_{r-1} \circ \mu_{r-2} \circ \cdots \circ \mu_0$ is an automorphism of U. We may assume that

1. If $U = \mathbb{P}^1(\mathbb{C})$ then $\mu(z) = az$ is a multiplication by a complex number $a \neq 0$ (elliptic, hyperbolic and loxodromic case) or a translation $\mu(z) = z + 1$ (the parabolic case).

2. If $U = \mathbb{C}$ then V_0 is an elliptic curve. There is a 2-dimensional lattice $\Gamma \subset \mathbb{C}$ and the automorphism $\mu(z) = az + b$ is a composition of a translation and a periodic rotation, where $a^n = 1$ and $n = 1, 2$ or 3.
3. If $U = \mathbb{D}$ then μ is an element of the Poincaré group and moreover, since μ must have a fixed point as a lift of an automorphism of a compact Riemann surface of genus ≥ 2, we may assume that this fixed point is 0. Hence, $\mu(z) = az$ is a multiplication by a number again. Moreover, there is n such that $a^n = 1$, i.e., a is a root of unity. We note that every automorphism of a compact Riemann surface is periodic because the group of automorphisms of such a surface is finite [GH78, p. 275].

Let $\pi_j \circ p_j = (f_j, g_j)$, where $f_j, g_j : U \to \mathbb{C}$ are meromorphic functions. The equation

$$\phi(f_j(z), g_j(z)) = (f_{j+1}(\mu_j z), g_{j+1}(\mu_j z)) \tag{18}$$

follows directly from our definitions. Thus

$$\begin{aligned} f_{j+1}(\mu_j z) &= \sqrt{1 + f_j(z)^2} - g_j(z), \\ g_{j+1}(\mu_j z) &= f_j(z). \end{aligned}$$

The function g_j can be eliminated from the first equation by using the second one, and we arrive at the following functional equation:

$$f_{j+1}(\mu_j z) + f_{j-1}(\mu_{j-1}^{-1} z) = \sqrt{1 + f_j(z)^2} \tag{19}$$

valid for $j = 0, 1, \ldots, r - 1$. Moreover, without loss of generality, we may assume that $\mu_0 = \mu_1 = \ldots = \mu_{r-2} = 1$ and $\mu_{r-1} = \mu$. This is obtained by a simple change of coordinates involving compositions of μ_j. Also, in view of the fact that there is always an automorphism λ satisfying $\lambda^r = \mu$, we may change coordinates again and obtain an equation involving λ only:

$$f_{j+1}(\lambda z) + f_{j-1}(\lambda^{-1} z) = \sqrt{1 + f_j(z)^2}. \tag{20}$$

This is the form of the functional equation that we are going to use in the future. The standarization of λ is compatible with the following standarization of Γ_j: we may assume that f_j is invariant under an action of a discrete subgroup $\Gamma_j \subset \mathrm{Aut}(U)$, acting discretely and co-compactly. For $j = 0, 1, \ldots, r-1$ we have $\Gamma_{j+1} = \lambda \Gamma_j \lambda^{-1}$, thus the groups Γ_j, $j = 0, 1, \ldots, r - 1$ are all conjugate.

In each case, U is an open subset of the Riemann sphere and $f_j : U \to \mathbb{C}$ are meromorphic functions. Furthermore, we may make the following assumptions:

1. If $U = \mathbb{P} = \mathbb{C} \cup \{\infty\}$ then f_j are rational functions, and λ is a multiplication by a complex number $a \neq 0$ or a translation by 1.
2. If $U = \mathbb{C}$ then $\lambda(z) = az + b$ is a composition of a translation and a periodic rotation ($a^{rn} = 1$, where $n = 1, 2$ or 3). The group Γ_i is a 2-dimensional lattice in $\mathbb{C}$.
3. If $U = \mathbb{D}$ then $\lambda(z) = az$, $|a| = 1$ and a is a root of unity. In this case, the group Γ_i is a discrete subgroup of the Poincaré group, acting discretely and co-compactly on $\mathbb{D}$, so that $\mathbb{D}/\Gamma$ is a compact Riemann surface of genus ≥ 2.

7. The Poles of the Solution

The system of equations (20) permits one to draw conclusions about the poles of the functions f_j. Since each of these functions is meromorphic on a compact Riemann surface, the poles and the principal parts of f_j *at a finite number of these poles* determine f_j completely. Thus, this information provides us with rather sharp criteria for the existence of solutions.

Lemma 7.1. *Let $(f_j)_{j=0}^{r-1}$ be a solution to the system of equations (20) and let $z_0 \in U$ be a pole of f_j for some j. Then z_0 is a periodic point of λ.*

Proof. Let ν be the maximal order of z_0 as a pole of f_j for $j = 0, 1, \dots, r-1$. Let us write $z_m = \lambda^m z_0$ for all integer m. Let us consider the leading term of the Laurent series of f_j at z_j:

$$f_j(z) = \frac{A_j}{(z - z_j)^\nu} + \cdots . \tag{21}$$

We compose the functional equation (20) with λ^j and obtain

$$f_{j+1}(\lambda^{j+1}z) + f_{j-1}(\lambda^{j-1}z) = \sqrt{1 + f_j(\lambda^j z)^2}. \tag{22}$$

Calculating the most singular term of the Laurent expansion at z_0 leads to the following equation:

$$A_{j+1}\left((\lambda^{j+1})'(z_0)\right)^{-\nu} + A_{j-1}\left((\lambda^{j-1})'(z_0)\right)^{-\nu} = \pm A_j\left((\lambda^j)'(z_0)\right)^{-\nu}. \tag{23}$$

Let $B_j = A_j\left((\lambda^j)'(z_0)\right)^{-\nu}$. We obtain:

$$B_{j+1} + B_{j-1} = \pm B_j. \tag{24}$$

Thus,

$$\begin{bmatrix} B_{j+1} \\ B_j \end{bmatrix} = \begin{bmatrix} \pm 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} B_j \\ B_{j-1} \end{bmatrix}. \tag{25}$$

Thus, the vector (B_{j+1}, B_j) is obtained from (B_j, B_{j-1}) by multiplying by a non-singular matrix. Hence, if $(B_{j+1}, B_j) \neq (0,0)$ for some j then the same is true for all j. Hence, if all z_j are distinct then for some k the function f_k has infinitely many poles of the form z_{jr+k}. But this is impossible since the poles are isolated and there is only a finite number of them on the compact Riemann surface U/Γ_k. Thus, for some k and $l \neq k$ we have $z_k = z_l$. In particular $\lambda^{k-l}(z_0) = z_0$. □

8. Periodicity Considerations

In this section we develop criteria for establishing periodicity of a non-singular algebraic relation. A single-valued map $\phi : X \to X$ is periodic if for some n we have $\phi^n = id$. The following definition will be useful when ϕ is multi-valued:

Definition 8.1. A non-singular algebraic relation $R \subset X \times X$ is said to have a *periodic component* of period n iff $\Delta \subseteq R^n$ where $\Delta = \mathrm{graph}(id)$ is the diagonal of $X \times X$. If there exists n such that $R^n = \Delta$ then R is called a *periodic non-singular algebraic relation* of period n.

The following weaker definition of periodicity can also be considered: R is periodic if for every finite sequence of local branches $(\phi_j)_{j=0}^{p-1}$ such that the composition $\phi_{p-1} \circ \phi_{p-2} \circ \cdots \circ \phi_0$ is defined on a non-empty open set there exists $q \geq p$ and local branches $(\phi_j)_{j=p}^{q-1}$ such that $\phi_{q-1} \circ \phi_{q-2} \circ \cdots \circ \phi_0 \equiv id$ on an open set. It can be shown that this definition is equivalent to the previous one.

The main goal of this section is the proof of the following:

Proposition 8.2. *The mapping ϕ is periodic on every invariant variety V, i.e., for every invariant variety V the non-singular algebraic relation* $\text{graph}(\phi) \cap (V \times V)$ *is periodic.*

Before proceeding with the proof of Proposition 8.2 we will prove a corollary which motivates the importance of this proposition.

Corollary 8.3 (of Proposition 8.2)**.** *If there exists an algebraic integral for the the map ϕ then ϕ is periodic, i.e., the non-singular algebraic relation* $\text{graph}(\phi)$ *is periodic.*

Proof of Corollary 8.3. By definition of algebraic integrability, if ϕ has an algebraic integral then there exists a non-constant polynomial $G(x, y, z)$ with complex coefficients such that for every $c \in \mathbb{C}$ the curve

$$V_c = \{(x, y) \in \mathbb{C}^2 : G(x, y, c) = 0\}$$

is invariant in the sense of Definition 4.1. We apply Proposition 8.2 to each curve V_c. Let $R = \text{graph}(\phi)$. The restricted relation $R \cap (V_c \times V_c)$ is periodic. For every non-negative integer n let $\mathcal{C}_n$ denote the set of these $c \in \mathbb{C}$ for which $\Delta_c \subseteq R^n$. There exists n such that $\mathcal{C}_n$ is (Zariski) dense in $\mathbb{C}$, where Δ_c is the diagonal of $V_c \times V_c$. R^n has only a finite number of components. Hence, there exists a single irreducible component $S \subseteq R^n$ such that $\Delta_c \subseteq S \cap (V_c \times V_c)$ for a (Zariski) dense set of $c \in \mathcal{C}_n$. It is easy to see that we must have $S = \Delta$, where Δ is the diagonal of $\mathbb{C}^2 \times \mathbb{C}^2$.

By Lemma 3.3, R^n is an irreducible variety, and thus if $\Delta \subseteq R^n$ then $R^n = \Delta$, i.e., R is periodic of period n. □

Corollary 8.3 results in a contradiction because R^n is not periodic. A detailed argument will be presented in Lemma 8.8 and Section 9.

The next lemma deals with invariant curves of high genus.

Lemma 8.4. *If V is an invariant variety of ϕ and V has genus ≥ 2 then* $\text{graph}(\phi) \cap (V \times V)$ *is a periodic algebraic relation.*

Proof. If V has genus ≥ 2 then V does not contain a line $\mathbb{C} \times \{y_0\}$ or $\{x_0\} \times \mathbb{C}$. Hence, we can apply Lemma 5.1 and without loss of generality we may assume that ϕ restricted to V is single-valued. Let $V = \bigcup_{j=0}^{r-1} V_j$ and let $\phi_j : V_j \to V_{j+1}$ be the single-valued and thus bi-rational branch of ϕ restricted to V (see the beginning of Section 6). Let $\hat{V}_j$ be the corresponding Riemann surfaces and $\hat{\phi}_j : \hat{V}_j \to \hat{V}_{j+1}$ the corresponding bi-holomorphic maps. Thus, the composition $\hat{\psi} = \hat{\phi}_{r-1} \circ \hat{\phi}_{r-2} \circ \cdots \circ \hat{\phi}_0$ is an element of $\text{Aut}(\hat{V}_0)$. It is well-known that the group of automorphisms of a compact Riemann surface of genus ≥ 2 is finite. Thus, $\hat{\psi}$ has a finite rank, i.e., for some n we have $\hat{\psi}^n = id_{\hat{V}_0}$. It is easy to see that this is equivalent to ϕ restricted to V being periodic of period nr. □

The case of genus 1 can also be settled easily.

Lemma 8.5. *If V is an invariant variety of ϕ and V has genus 1 then* $\operatorname{graph}(\phi)\cap(V\times V)$ *is a periodic algebraic relation.*

Proof. If the genus of V is 1 then the automorphism λ obtained in the previous section is of the form $\lambda(z)=az+b$ where a is a root of unity. We know that there is at least one pole of any non-constant function f_j. Thus, there exists a point which is fixed by λ^n for some n. But this means that λ^n is equivalent to a rotation by a^n and thus for some $m\geq n$ we have $\lambda^m=id$. □

The least trivial case is that of genus 0. But in this case we have the following:

Lemma 8.6. *Let $U=\mathbb{P}^1(\mathbb{C})=\mathbb{C}\cup\{\infty\}$. Let us suppose that λ has infinite order, i.e., $\lambda^m\neq id$ for all integers m. Then there are no rational solutions to the system of equations 20, except for the constant ones.*

Proof. We note that if for some $j\in\{0,1,\dots,r-1\}$ the functions f_j have a pole different from 0 and ∞ then $\lambda^m=id$ (Lemma 7.1). Thus, the only poles of f_j can be 0 and ∞. Hence, we may write

$$f_j(z)=\sum_{l=\nu_1}^{\nu_2} f_{j,l}z^l \tag{26}$$

where $\nu_1\leq 0$ and $\nu_2\geq 0$.

Equation (20) can be transformed into

$$(f_{j+1}(\lambda z)+f_{j-1}(\lambda^{-1}z))^2-f_j(z)^2=1. \tag{27}$$

After factoring we obtain:

$$\big(f_{j+1}(\lambda z)+f_{j-1}(\lambda^{-1}z)+f_j(z)\big)\big(f_{j+1}(\lambda z)+f_{j-1}(\lambda^{-1}z)-f_j(z)\big)=1. \tag{28}$$

We claim that there exists a sequence of $\epsilon_j\in\{-1,1\}$, $j=0,1,\dots,r-1$ such that the equation

$$f_{j+1}(\lambda z)+f_{j-1}(\lambda^{-1}z)=\epsilon_j f_j(z)+u_j(z) \tag{29}$$

is satisfied, where $u_j(z)$ is a function analytic at 0. We may say that the principal part of the equation

$$f_{j+1}(\lambda z)+f_{j-1}(\lambda^{-1}z)=\epsilon_j f_j(z) \tag{30}$$

is satisfied.

Indeed, equation (28) implies that if one of the factors in the left hand side has a pole at 0 then the other one is analytic. Hence, for every j we may fix ϵ_j so that (29) is satisfied. A similar argument with 0 and ∞ switched shows that

$$f_{j+1}(\lambda z)+f_{j-1}(\lambda^{-1}z)=\eta_j f_j(z)+v_j(z) \tag{31}$$

holds as well, but now $v_j(z)$ is analytic at ∞. Moreover, by analytic continuation along a path connecting 0 to ∞ we can see that $\eta_j=\epsilon_j$. Indeed, we may assume that γ is a path connecting 0 to ∞ and not passing through any roots of the equations $f_j(z)=\pm i$. The fact that we can take $\sqrt{1+f_j(z)^2}=f_j(z)\sqrt{1+f_j(z)^{-2}}$ and obtain a rational function means that the winding number of $f_j\circ\gamma$ with respect to

$\pm i$ is even. Thus, the analytic continuation of $\sqrt{\cdot}$ along the closed path $1+(f_j \circ \gamma)^{-2}$ results in no change of branch at 1. Hence $\epsilon_j = \eta_j$. Thus the function

$$f_{j+1}(\lambda z) + f_{j-1}(\lambda^{-1} z) - \epsilon_j f_j(z) \tag{32}$$

is a bounded meromorphic (rational) function, and therefore it is constant. By comparing this equation with the original equation (20) we conclude that for every j the function $\sqrt{1+f_j(z)^2} - f_j(z)$ must be constant. It is easy to see that this is only possible when $f_j(z)$ is constant. Indeed, we may rewrite the equation $\sqrt{1+f_j(z)^2} - f_j(z) = c$ as

$$1 + f_j(z)^2 = (c + f_j(z))^2 = c^2 + 2cf_j(z) + f_j(z)^2$$

which implies $c \neq 0$ and $f_j(z) = (1-c^2)/(2c)$ for all z. □

Corollary 8.7. *For every genus* 0 *invariant curve* V *of* ϕ*, the map* $\phi|V$ *has a periodic branch.*

We note that Proposition 8.2 follows from Lemmas 8.4, 8.5 and Corollary 8.7.

It is not difficult to verify that a non-singular algebraic relation is not periodic. The following argument can be used in the case of the map ϕ:

Lemma 8.8. *The mapping* ϕ *has uncountably many real-analytic invariant curves on which* ϕ *is non-periodic.*

Proof. This is a consequence of the KAM theory . For instance, the fixed point $(1/\sqrt{3}, 1/\sqrt{3})$ has an irrational rotation number satisfying the usual diophantine and non-degeneracy conditions. Hence, there are uncountably many invariant curves nearby. On each of these curves the dynamics is non-periodic. □

As we have mentioned in the introduction, in [AE90] we find a version of KAM theory suitable for proving that also there is a large set of closed, real-analytic invariant curves for ϕ near ∞, close to the integral curves of the vector field $X(x, y)$ given by (6).

9. The Conclusion of the Proof of the Main Theorem

Proposition 8.2 implies that every branch of ϕ is periodic on each of its algebraic invariant curves, except for a finite number of horizontal lines of the form $\mathbb{C} \times \{y_0\}$, where $y_0 \in \mathbb{C}$ (see Lemma 5.1). Corollary 8.3 states that if there exists an algebraic integral for ϕ then ϕ is periodic.

On the other hand, Lemma 8.8 implies that ϕ is non-periodic on a large set of 1-dimensional invariant curves. This contradiction concludes the proof of the main Theorem 1.1.

10. Open Problems

The map ϕ without a doubt has an integral which is analytic in the real domain. It is not clear how to prove this, except for explicitly writing down the integral.

Problem 10.1. Can the integral of ϕ be written in “closed form”?

Integrals written in closed form typically have only mild singularities. It is possible that the singularities of the integral in question are not mild. For instance, there could be a natural boundary for the natural continuation of the integral. If this is the case, our approach could be sharpened to eliminate solutions with increasingly complex singularities by considering more general analytic invariant curves. Such curves give rise to Riemann surfaces which are not compact. However, some arguments can be made even in this case, for instance, the functional equation (20) is still valid.

Problem 10.2. Generalize the proof of our theorem to exclude stronger than algebraic singularities. For instance, exclude integrals of the form $F(x, y) + \log G(x, y)$ where F and G are algebraic functions.

Finally, let us note that there are formal procedures for producing the Taylor expansion of the integral of ϕ near the finite fixed point or near ∞.

Problem 10.3. Prove convergence of these formal expansions.

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