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An Example of Fourier Transforms of Orbital Integrals and their Endoscopic Transfer

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Abstract. For the Lie algebra $\mathfrak{sl}_2$ over a $\mathfrak{p}$-adic field, the Fourier transform of a regular orbital integral is expressed as an integral over all regular orbital integrals, with explicit coefficients. This expression, unlike the Shalika germ expansion, is not restricted to orbits of small elements. The result gives quite an elementary access to a simple example of Waldspurger's recent theorem on endoscopic transfer of the Fourier transforms.

Contents

1. Introduction

Integrals over conjugation classes in Lie groups, called *orbital integrals*, are fundamental objects of harmonic analysis. They occur in the "geometric side" of Selberg's *trace formula*; the "spectral side" contains traces of representations. For applications to automorphic forms, it is useful to consider not just Lie groups but both real and p-adic (and also adelic) algebraic groups. Howe [11] conjectured and Clozel [7] proved, for reductive p-adic Lie groups or Lie algebras, that the invariant distributions of given compactly generated support, restricted to a space of uniformly locally constant functions, make a finite dimensional space. Harish-Chandra [10] uses this principle in his study of Fourier transforms and characters of admissible representations. *Shalika germs* [24] describe the orbital integrals of small regular elements as combinations of unipotent orbital integrals, with coefficients difficult to get hold of.

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Endoscopic transfer tries to relate instable orbital integrals of a group and stable orbital integrals of so-called endoscopic groups attached to it via L-groups. It was developed by Shelstad, Langlands, Labesse, Kottwitz, and stated in final though conjectural form by [20]. To explain it roughly: Given an algebraic group G over a field F, with algebraic closure $\overline{F}$; elements in $\mathrm{G}(F)$ which are conjugate in $\mathrm{G}(\overline{F})$ make a "stable orbit", which may contain several orbits of $\mathrm{G}(F)$-conjugacy (more precisely, see [14]). A stable orbital integral is the sum of integrals of the orbits in one stable orbit; certain other linear combinations of these integrals, such as their difference when the stable orbit contains two orbits, are called instable orbital integrals. Stable orbits have the advantage of being compatible when two groups are inner forms of each other. Transfer means that an instable integral of a function on $\mathrm{G}(F)$ may be expressed as stable integral of another function on another group, called an endoscopic group; the construction of that group is based on L-groups, defined by a duality of root systems using the structure theory of reductive algebraic groups. For $\mathrm{G} = \mathrm{SL}_2$ the non-trivial endoscopic groups are one-dimensional algebraic tori defined by quadratic extensions $E|F$ and a stable orbital integral in such a torus is just evaluation at an element.

Arthur developed trace formulae introducing truncation operators and weighted orbital integrals. Waldspurger [29] established a p-adic trace formula and suggested a transfer formula for the *Fourier transforms* of regular orbital integrals in the Lie algebra, (9) below, which would imply the transfer for the orbital integrals themselves; recently he reduced this conjecture to the fundamental lemma and proved it for many groups [30, Théorème 1.5], among them SL_n; in general, transfer remains a conjecture.

Let F be a p-adic field ($p \neq 2$). Fourier transformation was thoroughly studied for orbital integrals on the group $\mathrm{SL}_2(F)$ by Sally and Shalika [22], and for nilpotent orbital integrals on the Lie algebra $\mathfrak{sl}_\ell(F)$ by Assem [4]. Now for $X \in \mathfrak{sl}_2(F)$ semi-simple regular, we consider the orbital integral over $\mathrm{Ad}(\mathrm{SL}_2(F)) \cdot X$. Its Fourier transform is invariant with respect to $\mathrm{Ad}(\mathrm{SL}_2(F))$ and is known to be an integral (over the space of regular orbits) of orbital integrals; we explicitly specify the coefficients (Propositions 4 and 10). This expression, unlike Shalika's germ expansion, is not limited to orbits of small elements. In the last two sections we translate our Corollary 11 into the context of formula (9) recently proved by [30] for certain groups, of which SL_2 is but one example; our elementary computation yields this example directly.

Our discussion, up to Corollary 11, does not presume knowledge of the works mentioned above; in the last two sections we quote from [20] and [29] respectively.

2. Conjugacy Classes

Matrices in the Lie algebra $\mathfrak{sl}_2$ will always be denoted

$$X = \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \quad \text{or} \quad U = \begin{pmatrix} u & v \\ w & -u \end{pmatrix},$$

and the letters U, u, v, w, X, x, y, z, bare or decorated like X' or X_m, will always mean this. The action of GL_2 on $\mathfrak{sl}_2$ by $\mathrm{Ad}(g)(X) := gXg^{-1}$ leaves the bilinear form

$$\langle U, X\rangle := 2ux + vz + wy$$

invariant; it will be used for Fourier transforms.

Let us classify the *orbits* of $G := \mathrm{SL}_2(F)$ acting on $\mathfrak{g} := \mathfrak{sl}_2(F)$, where F is a field in which $2 \neq 0$; integrals over these orbits will then be the basic objects to study. The group G acts in the set of regular semi-simple elements

$$\mathfrak{g}^{\mathrm{reg}} = \{X \in \mathfrak{g} | \det X \neq 0\}$$

and in each of the *stable orbits*

$$\mathfrak{g}_t := \{X \in \mathfrak{g} \smallsetminus \{0\} | \det X = -t\}, \quad t \in F.$$

Each $X \in \mathfrak{g}$ is G-conjugate to some $\begin{pmatrix} 0 & t/\beta \\ \beta & 0 \end{pmatrix}$ where $t = -\det X$ and $\beta \in F^\times$, the latter unique up to a factor in

$$\mathcal{N}_t := \mathcal{N}_{tF^{\times 2}} := \left\{\lambda^2 - \mu^2 t \middle| \lambda, \mu \in F\right\} \smallsetminus \{0\} \subseteq F^\times,$$

which is the norm group of the algebra

$$E_t := F[\vartheta]/(\vartheta^2 - t);$$

in fact the orbits are parametrized by the bijection

$$\begin{aligned} \mathrm{Orb}(\mathfrak{g} \smallsetminus \{0\}) := \{(t,b) | t \in F, b \in F^\times/\mathcal{N}_t\} &\xrightarrow{\sim} \mathrm{Ad}(G)\backslash(\mathfrak{g} \smallsetminus \{0\}) \\ (t,b) &\mapsto \mathfrak{g}_{t,b} := \{X \in \mathfrak{g}_t | z \in b \cup \{0\} \ni -y\}, \end{aligned}$$

and $\mathrm{Orb}(\mathfrak{g}^{\mathrm{reg}}) := \{(t,b) | t \neq 0\}$ corresponds to $\mathrm{Ad}(G)\backslash\mathfrak{g}^{\mathrm{reg}}$. Each stable orbit $\mathfrak{g}_t$ is the union of all the $\mathfrak{g}_{t,b}$ where $b \in F^\times/\mathcal{N}_t$, and

$$\text{(1)} \qquad \forall g \in \mathrm{GL}_2(F): \qquad \mathrm{Ad}(g)\mathfrak{g}_{t,b} = \mathfrak{g}_{t,\det(g)b}.$$

(Each group $F^\times/\mathcal{N}_t$ can also be considered as Galois cohomology group of the stabilizer in SL_2 of the matrix $\left(\begin{smallmatrix} 0 & t \\ 1 & 0 \end{smallmatrix}\right) \in \mathfrak{g}_t$; in this guise they appear in [18, p. 728] when orbits in G are parametrized.)

From now on let F be a non-archimedean local field [6]; it comes with the valuation ring $\mathcal{O} \subset F$, the maximal ideal $\mathfrak{p} \subset \mathcal{O}$, the residue field $\kappa = \mathcal{O}/\mathfrak{p}$ with q elements, the valuation exponent $\nu : F \twoheadrightarrow \mathbb{Z} \cup \{+\infty\}$ and the absolute value $|\cdot|$ so that $|\pi| = 1/q$ for any uniformizing element $\pi \in \mathfrak{p} \smallsetminus \mathfrak{p}^2$. We shall assume $2 \notin \mathfrak{p}$.

(Example: $F = \mathbb{Q}_p$, $\mathcal{O} = \mathbb{Z}_p$, $\mathfrak{p} = p\mathbb{Z}_p$, $\kappa = \mathbb{F}_p$, $\pi = p$.)

The group $F^\times/F^{\times 2}$ has four elements and fits in the exact sequence

$$\begin{array}{ccccc} 1 \to & \mathcal{O}^\times/\mathcal{O}^{\times 2} & \longrightarrow F^\times/F^{\times 2} \xrightarrow{\nu} \mathbb{Z}/2\mathbb{Z} \to 0 \\ & \downarrow\cong & \\ & \kappa^\times/\kappa^{\times 2} & \end{array}$$

and for each $\tau = tF^{\times 2} \in F^\times/F^{\times 2}$ there is a canonical isomorphism

$$F^\times/\mathcal{N}_\tau \cong \begin{cases} \{1\} & \text{if } 1 \in \tau, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } \tau = \nu^{-1}(2\mathbb{Z}) \smallsetminus F^{\times 2}, \\ \kappa^\times/\kappa^{\times 2} & \text{if } \nu(\tau) = \mathbb{Z} \smallsetminus 2\mathbb{Z}. \end{cases}$$

Thus for $t \in F^\times \smallsetminus F^{\times 2}$, the stable orbit $\mathfrak{g}_t$ contains two orbits $\mathfrak{g}_{t,b}$.

3. Orbital Integrals

On each G-orbit in $\mathfrak{g}$, we need a G-invariant measure to define integrals. Instead of referring to the general fact that a quotient space of a unimodular locally compact group G by a unimodular closed subgroup has a G-invariant positive measure unique up to a positive constant, we shall construct such measures on each $\mathfrak{g}_{t,b}$ so as to have an explicit formula to work with.

The additive Haar measure on F such that $\mathrm{vol}(\mathcal{O}) = 1$, as well as its restriction to subsets of F, will be denoted dt, or dx, dy, dz respectively; for example, $dt/|t|$ is a Haar measure on the multiplicative group $F^\times$. The F-rational points of a smooth algebraic variety over F form a totally disconnected topological space, on which measures can be defined by means of differential forms, as explained in [31, p. 13–14]. For any complex-valued locally constant compactly supported function (in short, C_c^∞ function), integrals are finite sums. The characteristic function of an open set A will be called f_A.

For each $t \in F$, the variety $\{X \in \mathfrak{sl}_2 \smallsetminus \{0\} \,|\, x^2 + yz = t\}$ carries a GL_2-invariant algebraic 2-form

$$\frac{dy \wedge dz}{2x} = \frac{dx \wedge dy}{y} = \frac{dz \wedge dx}{z}$$

(the relation $0 = d(x^2 + yz) = 2x\,dx + z\,dy + y\,dz$ implies that all three terms are equal and invariant). This 2-form defines a measure dX on $\mathfrak{g}_t$. The orbital integrals

$$I_{t,b}(f) := \int_{\mathfrak{g}_{t,b}} f(X)dX \in \mathbb{C} \quad (f \in \mathrm{C}_c^\infty(\mathfrak{g}),\ (t,b) \in \mathrm{Orb}(\mathfrak{g} \smallsetminus \{0\}))$$

are finite sums if $t \neq 0$, or still convergent for $t = 0$, see the proof of Proposition 2 below. Let $\Lambda_0 := \mathfrak{sl}_2(\mathcal{O}) = \{X \,|\, x, y, z \in \mathcal{O}\}$ and $\Lambda_n := \mathfrak{p}^n \Lambda_0$ for each $n \in \mathbb{Z}$.

Proposition 1. *Let $m < \ell \in \mathbb{Z}$, $X_m \in \Lambda_m \smallsetminus \Lambda_{m+1}$, and $(t,b) \in \mathrm{Orb}(\mathfrak{g} \smallsetminus \{0\})$. If*

(i) $t + \det(X_m) \in \mathfrak{p}^{\ell+m}$ *and*
(ii) $\{z_m, -y_m\} \subset b \cup \mathfrak{p}^{m+1}$,

then $I_{t,b}(\mathrm{f}_{X_m+\Lambda_\ell}) = q^{m-2\ell}$. Otherwise the integral is zero.

Proposition 2. *Let $\ell \in \mathbb{Z}$, $(t,b) \in \mathrm{Orb}(\mathfrak{g} \smallsetminus \{0\})$. Then*

$$I_{t,b}(\mathrm{f}_{\Lambda_\ell}) = \begin{cases} \frac{1}{2}q^{-\ell} & \text{if } t = 0,\ \ell \in \nu(b) \\ \frac{1}{2}q^{-\ell-1} & \text{if } t = 0,\ \ell \notin \nu(b) \\ 0 & \text{if } 2\ell > \nu(t) \\ q^{-\ell} + q^{-1-\ell} & \text{if } 2\ell \leq \nu(t),\ t \in F^{\times 2} \\ q^{-\ell} - q^{-1-h} & \text{if } 2\ell \leq \nu(t) = 2h,\ 0 \neq t \notin F^{\times 2},\ \ell \in \nu(b) \\ q^{-1-\ell} - q^{-1-h} & \text{if } 2\ell \leq \nu(t) = 2h,\ 0 \neq t \notin F^{\times 2},\ \ell \notin \nu(b) \\ (q^{-\ell} - q^{-h})\frac{q+1}{2q} & \text{if } 2\ell < \nu(t) = 2h - 1. \end{cases}$$

Corollary 3. *For every $f \in \mathrm{C}_c^\infty(\mathfrak{g}^{\mathrm{reg}})$ the function*

$$\mathrm{Orb}(\mathfrak{g}^{\mathrm{reg}}) \ni (t,b) \mapsto I_{t,b}(f)$$

belongs to $\mathrm{C}_c^\infty(\mathrm{Orb}(\mathfrak{g}^{\mathrm{reg}}))$. Explicitly, let $n \in \mathbb{Z}$, $X_n \in \Lambda_n \smallsetminus \Lambda_{n+1}$, $t = -\det X_n \neq 0$, $j > \nu(t) - n$. Then $f = \mathrm{f}_{X_n+\Lambda_j}$ corresponds to the function $q^{n-2j}\mathrm{f}_{(t+\mathfrak{p}^{n+j})\times\{b\}}$.

Proofs. For Proposition 1, fix a prime element $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$, no matter which, so that $\pi\mathcal{O} = \mathfrak{p}$, let $X_m = \pi^m X_0$ and let X' run through Λ_0 so that $X = X_m + \pi^\ell X'$ runs through $X_m + \Lambda_\ell$. Then

$$\pi^{-\ell-m}(t + \det X) = \pi^{-\ell-m}(t + \det X_m) - \langle X_0, X' \rangle + \pi^{\ell-m} \det X'.$$

If this wants to be zero, the condition (i) is necessary.

If $y_0, z_0 \in \mathfrak{p}$, then $x_0 \in \mathcal{O}^\times$ and one can solve the equation $t = -\det X$ for x'. Because of $t \in \pi^{2m}(x_0^2 + \mathfrak{p}) \subset F^{\times 2}$ we have $\mathfrak{g}_{t,b} = \mathfrak{g}_t$. The integral is

$$\int\limits_{\mathfrak{g}_t \cap (X_m + \Lambda_\ell)} \frac{dy\, dz}{|2x|} = \int\limits_{\mathcal{O}\times\mathcal{O}} \frac{|\pi^\ell| dy' \cdot |\pi^\ell| dz'}{|x_m|} = q^{m-2\ell}.$$

If $y_0 \in \mathcal{O}^\times$ one can solve for z' and then $-y\mathcal{N}_t = -y_m\mathcal{N}_t$. If this class is the same as b, the integral is again

$$\int\limits_{\mathfrak{g}_{t,b} \cap (X_m + \Lambda_\ell)} \frac{dx\, dy}{|y|} = q^{m-2\ell},$$

otherwise it is zero. Likewise in the case $z_0 \in \mathcal{O}^\times$.

The second proposition follows from the first one applied to each term of the sum

$$\mathrm{f}_{\Lambda_m} - \mathrm{f}_{\Lambda_{m+1}} = \sum_{0 \notin X \in \Lambda_m/\Lambda_{m+1}} \mathrm{f}_X,$$

by descending induction on ℓ if $t \neq 0$, or with a geometric series if $t = 0$.

The corollary follows immediately. □

In view of the two propositions, one recognizes the Shalika germs: Let $\ell \in \mathbb{Z}$, $(t, b) \in \mathrm{Orb}(\mathfrak{g}^{\mathrm{reg}})$, $\nu(t) \geq 2\ell$, and $f \in \mathrm{C}_c^\infty(\mathfrak{g})$ a function that factors through $\mathfrak{g}/\Lambda_\ell$. Then

$$I_{t,b}(f) = (\mathrm{const})\sqrt{|t|} \cdot f(0) \; + \sum_{\beta \in b/F^{\times 2}} I_{0,\beta}(f)$$

where (const) depends only on $tF^{\times 2}$; the right hand side consists of contributions from the nilpotent orbits, which are the four $\mathfrak{g}_{0,b}$ and $\{0\}$.

4. Fourier Transforms

Let $\Psi : F \to \mathbb{C}^\times$ be an unramified additive character; then

$$\Psi(\mathcal{O}) = 1, \qquad \sum_{a \in \mathfrak{p}^{-1}/\mathcal{O}} \Psi(a) = 0. \tag{2}$$

The Fourier transform of functions on $\mathfrak{g}$ is defined as

$$\mathrm{C}_c^\infty(\mathfrak{g}) \xrightarrow{\;\sim\;} \mathrm{C}_c^\infty(\mathfrak{g}), \quad \widetilde{f}(U) := \int\limits_{\mathfrak{g}} f(X)\Psi\langle U, X\rangle\, dX$$

where dX is the Haar measure such that $\operatorname{vol}(\Lambda_0) = 1$, which is self-dual in the sense that $\widetilde{\widetilde{f}}(X) = f(-X)$.

For each orbit, labeled $(s, a) \in \operatorname{Orb}(\mathfrak{g}^{\text{reg}})$, the distribution

$$\mathrm{C}_c^\infty(\mathfrak{g}^{\text{reg}}) \to \mathbb{C}, \quad f \mapsto I_{s,a}(\widetilde{f}),$$

is G-invariant in the sense that $I_{s,a}(\widetilde{f \circ \operatorname{Ad}(g)}) = I_{s,a}(\widetilde{f})$ for each $g \in G$. This invariant distribution can be expanded in terms of all the $I_{t,b}$:

Proposition 4. *There exists a unique locally constant function*

$$c_G : \operatorname{Orb}(\mathfrak{g}^{\text{reg}}) \times \operatorname{Orb}(\mathfrak{g}^{\text{reg}}) \to \mathbb{C}$$

such that

$$\forall f \in \mathrm{C}_c^\infty(\mathfrak{g}^{\text{reg}}) : \quad I_{s,a}(\widetilde{f}) = \int_{F^\times} \sum_{b \in F^\times/\mathcal{N}_t} c_G(s, a; t, b) I_{t,b}(f) \frac{dt}{\sqrt{|t|}}. \tag{3}$$

Explicitly, for $n \in \mathbb{Z}$ and $X_n \in \mathfrak{g}_{t,b} \cap \Lambda_n \smallsetminus \Lambda_{n+1}$:

$$\frac{c_G(s, a; t, b)}{\sqrt{|t|}} = I_{s,a}(\mathrm{f}_{\Lambda_{-n}}) + \sum_{i > n} \sum_{\Lambda_{1-i} \not\ni U_{-i} \in \Lambda_{-i}/\Lambda_{-n}} \Psi \langle U_{-i}, X_n \rangle \, I_{s,a}(\mathrm{f}_{U_{-i}}), \tag{4}$$

and the sum over i is actually finite.

The proof will be finished before Corollary 8. First we show that (3) implies (4). Suppose c_G locally constant. Given n and X_n, let $j \gg \nu(t) - n$ and $c \in \mathbb{C}$ so that

$$\forall X' \in X_n + \Lambda_j : \quad c_G(s, a; -\det X', b) = c\sqrt{|t|}.$$

Corollary 3 yields

$$\begin{aligned} c &= q^{3j} \cdot \operatorname{vol}(t + \mathfrak{p}^{n+j}) \cdot c \cdot q^{n-2j} \\ &= q^{3j} \cdot I_{s,a}(\widetilde{\mathrm{f}_{X_n + \Lambda_j}}) \\ &= I_{s,a}\big(\mathrm{f}_{\Lambda_{-n}} + \sum_{n < i \leq j} \sum_{\Lambda_{1-i} \not\ni U_{-i} \in \Lambda_{-i}/\Lambda_{-n}} \Psi \langle U_{-i}, X_n \rangle \, \mathrm{f}_{U_{-i}}\big); \end{aligned} \tag{5}$$

this does not depend on j and therefore the $i \gg \nu(t) - n$ do not contribute to (4).

It remains to show that there exists a function with the property (3). (We might refer to [10] but the lemmas below are needed anyway for Proposition 10.) For each $(t, b) \in \operatorname{Orb}(\mathfrak{g}^{\text{reg}})$ let us fix $n \in \mathbb{Z}$, $y \in \mathcal{O}$, $z \in \mathcal{O}^\times$ such that

$$X_n = \pi^n \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix} \in \mathfrak{g}_{t,b}\,, \ \nu(y) \leq 2, \ \text{ and } (t \in F^{\times 2} \implies y \in \mathcal{O}^{\times 2}). \tag{6}$$

Let c_G be defined by (4) with this choice of X_n. The relations (2) imply that the apparently infinite sum (4) is actually finite as follows.

Lemma 5. *Let $i > n$, $s' = \pi^{2i}s + \mathfrak{p}^{i-n}$, $h_1 = i - n - 1$, $h_2 = h_1 - \nu(y)$. The $U_{-i} = \pi^{-i}U$ such that*

(i) $\nu(v) = 0 < h_1 \wedge (s' = 0 \vee h_1 + \nu(w) > \min\{\nu(s' - vw), h_1\})$ *or*
(ii) $\nu(w) = 0 < h_2 \wedge (s' = 0 \vee h_2 + \nu(v) > \min\{\nu(s' - vw), h_1\})$

contribute all in all nothing to $c_G(s, a; t, b)$ in (4).

Corollary 6. *The only contributions to the sum (4) come from values of i such that*

$$\max\{n+1, -\frac{\nu(s)}{2}\} \leq i \leq i_{\max} := \max\{n+1+\nu(y), -1-n-\nu(s)\}. \tag{7}$$

Proofs. Let us assemble the U_{-i} with property (i) as follows: v runs through one class $\bmod \mathfrak{p}^{h_1}$, w remains fixed, and u varies so that the equation $s' = u^2 + vw$ remains valid. The $I_{s,a}(\mathrm{f}_{U_{-i}})$ are all equal and the $\Psi\langle U_{-i}, X_n\rangle = \Psi(\pi^{n-i}(vz+wy))$ make a total of zero.

Likewise, we can assemble the U_{-i} such that (ii) $\wedge \neg$(i); this time w varies by $\mathfrak{p}^{h_2}$, and v remains fixed.

The upper bound in the corollary follows from the lemma because

$$i \geq -n - \nu(s) \implies s' = 0 \implies v\mathcal{O} + w\mathcal{O} = \mathcal{O}.$$

The two lower bounds come from (4) and from condition (i) of Proposition 1. □

By replacing X and U by $\lambda^{-1}X$ and λU we see that the definition of c_G does not depend on the choice in (6) and that

$$\forall \lambda \in F^\times : \qquad c_G(\lambda^{-2}s, a; \lambda^2 t, b) = c_G(s, a; t, b). \tag{8}$$

Lemma 7. *The function $c_G : \mathrm{Orb}(\mathfrak{g}^{\mathrm{reg}}) \times \mathrm{Orb}(\mathfrak{g}^{\mathrm{reg}}) \to \mathbb{C}$ is locally constant.*

Proof. First let us fix (t, b) and a, and vary s. In view of Propositions 1 and 2, each term of (4) allows a neighbourhood where s can move. The set of terms which are not excluded by Corollary 6 is finite and depends only on $\nu(s)$ and (t, b). Therefore c_G is locally constant with respect to s. Using (8) we can barter a small change of t for a small change of s. Hence the lemma. □

Going back via (5), we now find that (3) is true for $f = \mathrm{f}_{X_n+\Lambda_j}$ when the X_n are chosen as in (6) and when $j \geq i_{\max}$ as defined in (7). These functions f, and their transforms $f \circ \mathrm{Ad}(g)$ by all $g \in \mathrm{SL}_2(F)$, span the whole linear space $\mathrm{C}_c^\infty(\mathfrak{g}^{\mathrm{reg}})$. The condition (3) being linear and $\mathrm{SL}_2(F)$-invariant, Proposition 4 follows.

Corollary 8. $\forall (s, a), (t, b) \in \mathrm{Orb}(\mathfrak{g}^{\mathrm{reg}})\ \forall \lambda \in F^\times : c_G(s, \lambda a; t, \lambda b) = c_G(s, a; t, b)$.

Proof. All the ingredients of Proposition 4 being $\mathrm{GL}_2(F)$-invariant, so is c_G; and we remember (1). □

Corollary 9. *If $sF^{\times 2} = tF^{\times 2}$ then $c_G(s, a; t, b)$ depends on a and b only via $a \cdot b$. If $sF^{\times 2} \neq tF^{\times 2}$ then $c_G(s, a; t, b)$ does not at all depend on a or b.*

Proof. This follows from Corollary 8; if $sF^{\times 2} \neq tF^{\times 2}$ one can choose λ so as to change a without changing b, or vice versa. □

We shall need auxiliary complex numbers depending on Ψ, namely

$$\gamma(c) := 1 \quad \text{if} \quad c \in F,\ \nu(c) \in 2\mathbb{Z},$$

$$\gamma(c) := \Big(1 + 2 \sum_{cF^{\times 2} \supset a \in \mathfrak{p}^{-1}/\mathcal{O}} \Psi(a)\Big)/\sqrt{q} \qquad \text{if} \quad c \in F,\ \nu(c) \notin 2\mathbb{Z}.$$

If $c \in F^\times$ and $\nu(c) = -1$ we have

$$\sqrt{q} \cdot \gamma(c) = \sum_{\lambda \in \kappa} \Psi(c\lambda^2),$$

$$\gamma(c \cdot (\mathcal{O}^\times \smallsetminus \mathcal{O}^{\times 2})) = -\gamma(c), \quad \gamma(-c) = \overline{\gamma(c)} = (-1)^{\frac{q-1}{2}} \cdot \gamma(c), \quad |\gamma(c)| = 1.$$

In view of lemma 5 and Corollary 6 there remain only quite few U_{-i} contributing to (4) for the X_n chosen in (6). Counting thoroughly and using the relations (2), we obtain:

Proposition 10. *Given* $(s,a),(t,b) \in \mathrm{Orb}(\mathfrak{g}^{\mathrm{reg}})$, *let*

$$S = \sum_{r^2 = st} \Psi(2r),\ Q_1 = \frac{1}{q}\sqrt{|st|},\ Q_2 = \frac{q+1}{2q}\sqrt{|st|/q}.$$

Then $c_G(s,a;t,b)$ *is as follows:*

<table>
<tr><td></td><td>$t \in F^{\times 2}$</td><td>$t \notin F^{\times 2},\ 2 \mid \nu(t)$</td><td>$2 \nmid \nu(t)$</td></tr>
<tr><td>$s \in F^{\times 2}$</td><td>S</td><td colspan="2">0</td></tr>
<tr><td>$st \in \mathfrak{p}^{-1}$</td><td>$1 - Q_1$</td><td>$(-1)^{\nu(ab)} - Q_1$</td><td>$-Q_1$</td></tr>
<tr><td>$st \notin \mathfrak{p}^{-1}$
$2 \mid \nu(s)$
$s \notin F^{\times 2}$</td><td>0</td><td>$(-1)^{\nu(ab)} S$
if $\frac{\nu(st)}{2} \in \nu(ab)$
0 otherwise</td><td>0</td></tr>
<tr><td>$st \in \mathfrak{p}^{-1}$</td><td>$1 - Q_2$</td><td>$-Q_2$</td><td>$\gamma(-ab) - Q_2$ if $st \in F^{\times 2}$
$-Q_2$ otherwise</td></tr>
<tr><td>$st \notin \mathfrak{p}^{-1}$
$2 \nmid \nu(s)$</td><td colspan="2">0</td><td>$\gamma(-ab) \sum_{r^2 = st, r \in -ab} \Psi(2r)$
if $st \in F^{\times 2}$
0 otherwise</td></tr>
</table>

For each $s \in F^\times$ let $\varepsilon_s : F^\times/\mathcal{N}_s \hookrightarrow \{\pm 1\}$ be the character. From Proposition 10 we extract the following corollary (actually some of the boxes of the table are not needed here; in particular those where $s \notin F^{\times 2} \not\ni st$ cancel out by Corollary 9.)

Corollary 11. *Let $s \in F^\times$ and $(t,b) \in \mathrm{Orb}(\mathfrak{g}^{\mathrm{reg}})$. Then*

$$c_G^\varepsilon(s;t,b) := \sum_{a \in F^\times/\mathcal{N}_s} \varepsilon_s(a) \cdot c_G(s,a;t,b) = \varepsilon_t(b) \cdot \gamma(t) \cdot \sum_{r^2 = st} \Psi(2r);$$

this is zero if $st \notin F^{\times 2}$.

The corollary describes the ε_s-instable integral

$$I_s^\varepsilon(\widetilde{f}) := \sum_{a \in F^\times/\mathcal{N}_s} \varepsilon_s(a) I_{s,a}(\widetilde{f}) = \int_{F^\times} \sum_{b \in F^\times/\mathcal{N}_t} c_G^\varepsilon(s;t,b) I_{t,b}(f) \frac{dt}{\sqrt{|t|}}$$

for $f \in \mathrm{C}_c^\infty(\mathrm{Orb}(\mathfrak{g}^{\mathrm{reg}}))$.

5. Transfer Factors

In order to derive (9) below from Corollary 11, we first spell out the definition of transfer factors which occur in $I^{G,H}$. As mentioned in the introduction, an instable integral in our case is the difference of two orbital integrals. The transfer factor will simply assign the signs $+1$ and -1 to the two orbital integrals; its definition in [20] is quite sophisticated because they make a canonical choice for all reductive groups and in general there occur coefficients other than ± 1.

Let $E|F$ be a quadratic extension, $\vartheta \in E^\times$ an element of trace zero, and $\varepsilon = \chi_{E|F}$ the quadratic character on $F^\times$ so that $\ker \chi_{E|F} = \mathcal{N}_{E|F} E^\times$. Let $\mathrm{H} = \ker(\mathrm{R}_{E|F}\mathbb{G}_m \xrightarrow{\mathcal{N}} \mathbb{G}_m)$ the corresponding one-dimensional algebraic torus (so that $\mathrm{H}(F) \subset E^\times$ is the norm-one group); its Lie algebra $\mathfrak{h}$ is naturally identified with the imaginary line $\ker(\mathrm{Tr}_{E|F}) \subset E$.

Now we borrow notations from [20, p. 222–223]. Let $h = \left(\begin{smallmatrix} 1/2 & -\vartheta \\ 1/(2\vartheta) & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(E)$,

$$\iota : \mathrm{H} \xrightarrow{\sim} \mathrm{T} := h\mathbb{T}h^{-1} \subset \mathrm{B} := h\mathbb{B}h^{-1} \subset \mathrm{G} := \mathrm{SL}_2,$$

$$\iota(a + b\vartheta) := h \begin{pmatrix} a + b\vartheta & 0 \\ 0 & a - b\vartheta \end{pmatrix} h^{-1} = \begin{pmatrix} a & b\vartheta^2 \\ b & a \end{pmatrix}.$$

Let

$$s := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathcal{T} := \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset \mathcal{B} := \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset \widehat{\mathrm{G}} = \mathrm{PGL}_2(\mathbb{C})$$

and

$$\xi : \mathcal{H} \to (\mathcal{T} \cdot \{1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}) \rtimes W \subset {}^L\mathrm{G}$$

the embedding such that $\xi^{-1} : \mathcal{T} \to \widehat{\mathrm{H}}$ is dual to

$$\mathrm{H} \to \mathbb{T}, \quad a + b\vartheta \mapsto h^{-1}\iota(a + b\vartheta)h.$$

Then $(\mathrm{H}, \mathcal{H}, s, \xi)$ are endoscopic data and ι is an admissible embedding. The algebraic character on T over E,

$$\alpha : \begin{pmatrix} a & b\vartheta^2 \\ b & a \end{pmatrix} \mapsto (a + b\vartheta)^2,$$

is the root of T in B; let the a-data consist of $a_\alpha := 2\vartheta$, $\quad a_{-\alpha} := -2\vartheta$, and choose whichever $\chi_\alpha : E^\times \to \mathbb{C}^\times$ extending $\chi_{E|F}$.

Let $0 \neq Y = \eta\vartheta \in \mathfrak{h}$ and $X = \left(\begin{smallmatrix} x & y \\ z & -x \end{smallmatrix}\right) \in \mathfrak{g}_{\eta^2\vartheta^2}$. Following the "Remarques" of [29, p. 90], we evaluate the formulae (I), (II), (III_1) and (III_2) of [20] with $\gamma_{\mathrm{H}} = \exp(e^2 Y)$ and $\gamma = \exp(e^2 X)$ for any $e \in F^\times$ near 0. The cocycle $\lambda(T)$ of (I) is trivial. In (II) we have $\alpha(\gamma_T) - 1 \approx 2e^2\eta\vartheta$. For the cocycle in ($\mathrm{III}_1$) we take

$$g = \begin{pmatrix} \eta & x \\ 0 & z \end{pmatrix} \begin{pmatrix} a & b\vartheta^2 \\ b & a \end{pmatrix}$$

where $a, b \in E$, $a^2 - b^2\vartheta^2 = 1/\eta z$. In ($\mathrm{III}_2$) we need not know $\mathbf{a}$ because $\gamma_{\mathrm{T}} \approx 1$. Omitting (IV), we obtain the transfer factor

$$\Delta(Y, X) = (\Delta_I = 1) \cdot (\Delta_{II} = \varepsilon(\eta)) \cdot (\Delta_1 = \varepsilon(\eta z)) \cdot (\Delta_2 = 1) = \chi_{E|F}(z).$$

(For a split torus $\mathrm{H} \cong \mathbb{G}_m$ we simply have $\Delta(Y, X) = 1$ for all $Y \in \mathfrak{h}$, $X \in \mathfrak{g}_{Y^2}$.)

By identifying $\mathfrak{h}$ with the imaginary line of $E|F$ (or with F if H splits) we may write $Y \cdot Z \in F$ for $Y, Z \in \mathfrak{h}$. Then

$$\Delta(Y, X) = \begin{cases} \varepsilon_t(b) & \text{if } X \in \mathfrak{g}_{t,b} \text{ and } t = Y^2, \\ 0 & \text{otherwise.} \end{cases}$$

6. The Transfer Formula

For $f_\mathfrak{h} \in \mathrm{C}_c^\infty(\mathfrak{h})$ we define the Fourier transform $\widetilde{f_\mathfrak{h}}$ with respect to $\Psi(2Y \cdot Z)$.

Now we need formulae from [29]; quotations '...' will refer to that work.

In 'III.1' in the case $M = G$ the factor $v_M(x)$ is 1, so the J_G^G are the ordinary orbital integrals; so are the I_G^G by 'p.74, remarque (b)'. Now $I_H^{\mathrm{st}}(Y, f_h)$ and $I^{G,H}(Y, f)$ are defined by 'VIII.7, (6) and (7)'.

Next, there are $\lambda_1, \lambda_2 \in F^\times$ so that $\langle \cdot, \cdot \rangle_g = \lambda_1 \langle \cdot, \cdot \rangle$ and $\psi(c) = \Psi(\lambda_2 c)$. For the construction of $\langle \cdot, \cdot \rangle_h$ in 'VIII.6', we can simply take $\mathfrak{G}^* = \mathfrak{G} = \mathrm{SL}_2$ and the same endoscopic data as in the previous section; then the whole procedure simplifies to

$$\langle \vartheta, \vartheta \rangle_h = \left\langle \begin{pmatrix} 0 & \vartheta^2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \vartheta^2 \\ 1 & 0 \end{pmatrix} \right\rangle_g = 2\lambda_1 \cdot \vartheta \cdot \vartheta.$$

Consider the constant $\gamma_\psi(g)$ of 'VIII.5'. If $\lambda \in \mathcal{O}^\times$, we can take $r = \Lambda_0$ in 'VIII.1' and then $I(r) = 1$. If $\nu(\lambda) = -1$ we can still take $r = \Lambda_0$ and then

$$I(r) = q^{-3} \sum_{x,y,z \in \mathcal{O}/\mathfrak{p}} \Psi(\lambda \cdot (x^2 + yz)) = q^{-3} \sum_x \Psi(\lambda x^2) \cdot \sum_y \sum_z \Psi(\lambda y z);$$

by (2) the $\sum_z$ vanishes for each $y \neq 0$, so

$$I(r) = q^{-2} \sum_x \Psi(\lambda x^2)$$

whence $\gamma_\psi(g) = \gamma(\lambda)$. If $\nu(\lambda) \notin \{0, -1\}$ we can multiply λ by a square and adapt r accordingly.

The computation of $\gamma_\psi(h)$ is similar.

Summing up, we obtain the following dictionary:

in [29]	here
$I_H^{\mathrm{st}}(Y, f_h)$	$f_{\mathfrak{h}}(Y)$
$I^{G,H}(Y, f)$	$C_s \cdot I_s^\varepsilon(f), \quad s = Y^2$
$\langle \cdot, \cdot \rangle_g$	$\lambda_1 \cdot \langle \cdot, \cdot \rangle$
$\langle Y, Z \rangle_h$	$2\lambda_1 \cdot Y \cdot Z$
$\psi(c)$	$\Psi(\lambda_2 c)$
$\gamma_\psi(g)$	$\gamma(\lambda), \quad \lambda = \lambda_1 \lambda_2$
$\gamma_\psi(h)$	$\gamma(\lambda Y^2) \quad (\text{for any } Y \in \mathfrak{h} \smallsetminus \{0\})$
$\widehat{f}(X)$	$\lvert\lambda\rvert^{3/2}\, \widetilde{f}(\lambda X) \quad (f \in \mathrm{C}_c^\infty(\mathfrak{g}))$
$\widehat{f_h}(Y)$	$\lvert\lambda\rvert^{1/2}\, \widetilde{f}_{\mathfrak{h}}(\lambda Y) \quad (f_{\mathfrak{h}} \in \mathrm{C}_c^\infty(\mathfrak{h}))$

The constants $C_s > 0$ depend on the choice of Haar measures but C_s depends only on $sF^{\times 2}$; we do not need them explicitly.

For every function $f \in \mathrm{C}_c^\infty(\mathfrak{g})$, the function $f_{\mathfrak{h}}$ defined by

$$f_{\mathfrak{h}}(Y) := I^{G,H}(Y, f) \quad (0 \neq Y \in \mathfrak{h})$$

extends to a function $f_{\mathfrak{h}} \in \mathrm{C}_c^\infty(\mathfrak{h})$; this was known to [18] and it is also clear in view of Propositions 1 and 2.

Let $0 \neq Y \in \mathfrak{h}$ and $s = Y^2$, and $f \in \mathrm{C}_c^\infty(\mathfrak{g}^{\mathrm{reg}})$. The self-dual Haar measure dZ on $\mathfrak{h}$ for $f_{\mathfrak{h}} \mapsto \widetilde{f}_{\mathfrak{h}}$, restricted to $\mathfrak{h} \smallsetminus \{0\}$, corresponds, via $t = Z^2$, to $dt/\sqrt{|t|}$ on $sF^{\times 2} \subset F$. Then

$$\begin{aligned}
I^{G,H}(Y, \widehat{f}) &= C_s \cdot \sum_{a \in F^\times/\mathcal{N}_s} \varepsilon_s(a) \int_{X \sim \left(\begin{smallmatrix} 0 & s/a \\ a & 0 \end{smallmatrix}\right)} \widehat{f}(X) dX \\
&= C_s \cdot \sum_a \varepsilon_s(a) \int_{\lambda X \sim \left(\begin{smallmatrix} 0 & \lambda s/a \\ \lambda a & 0 \end{smallmatrix}\right)} |\lambda|^{3/2}\, \widetilde{f}(\lambda X) \frac{d(\lambda X)}{|\lambda|} \\
&= C_s \sqrt{|\lambda|} \cdot \sum_a \varepsilon_s(a) I_{\lambda^2 s, \lambda a}(\widetilde{f}) \\
&= C_s \sqrt{|\lambda|} \cdot \varepsilon_s(\lambda) I^\varepsilon_{\lambda^2 s}(\widetilde{f});
\end{aligned}$$

by Corollary 11 this is

$$\begin{aligned}
&= C_s \sqrt{|\lambda|} \cdot \varepsilon_s(\lambda) \int_{F^\times} \sum_{b \in F^\times/\mathcal{N}_t} \varepsilon_t(b) \gamma(t) \sum_{r^2 = \lambda^2 st} \Psi(2r) I_{t,b}(f) \frac{dt}{\sqrt{|t|}} \\
&= C_s \sqrt{|\lambda|} \cdot \varepsilon_s(\lambda) \gamma(s) \int_{\mathfrak{h} \smallsetminus \{0\}} \Psi(2\lambda Y \cdot Z) I^\varepsilon_{Z^2}(f) dZ \\
&= \varepsilon_s(\lambda) \gamma(s) \sqrt{|\lambda|} \widetilde{f}_{\mathfrak{h}}(\lambda Y) \\
&= \varepsilon_s(\lambda) \gamma(s) \widehat{f_h}(Y).
\end{aligned}$$

In the four cases according to $\nu(\lambda), \nu(s) \bmod 2$ one verifies that

$$\varepsilon_s(\lambda)\gamma(s)\gamma(\lambda) = \gamma(\lambda s),$$

and one recognizes the transfer formula of [29, VIII.7 Conj. 1] with his formulae '(6)' and '(7)' inserted:

$$\gamma_\psi(h) I_H^{\mathrm{st}}(Y, \widehat{f_h}) = \gamma_\psi(g) I^{G,H}(Y, \widehat{f}). \tag{9}$$

7. Remarks

- We have been supposing $f \in \mathrm{C}_c^\infty(\mathfrak{g}^{\mathrm{reg}})$ so that the integrals are finite sums. Due to [10, Th.8], it is known that the Fourier transforms of regular orbital integrals are locally integrable distributions. This allows all $f \in \mathrm{C}_c^\infty(\mathfrak{g})$.
- As long as $2 \notin \mathfrak{p}$, we allow F to have finite characteristic.

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