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Multiple Rokhlin Tower Theorem: A Simple Proof

S. J. Eigen and V. S. Prasad

Abstract. S. Alpern has proved that an invertible antiperiodic measurable measure preserving transformation of a Lebesgue probability space can be represented by k towers of heights $n_1, \dots, n_k$, with prescribed measures, provided that the heights have greatest common divisor 1. In this paper we give a simple proof of Alpern's theorem. It is elementary in the sense that it involves no limits and uses Kakutani's easy proof of Rokhlin's Lemma.

Contents

1. Alpern's Multiple Rokhlin Tower Theorem

In this paper we show that Kakutani's proof of Rokhlin's Lemma [Hal56] can be used to give a short, elementary proof of the following Multiple Rokhlin Tower Theorem of Alpern's [Alp79, Cor 2].

Theorem 1.1 (Alpern). *For any $k \geq 2$, let $n_1, n_2, \dots, n_k$ be relatively prime positive integers, and let $q_1, \dots, q_k$ be positive numbers such that $n_1 q_1 + \cdots + n_k q_k = 1$. Then for any antiperiodic invertible measure preserving transformation T of a Lebesgue probability space (X, Σ, μ), there exist sets $Q_i \in \Sigma, i = 1, \dots, k$ with $\mu(Q_i) = q_i$ and such that $\{T^j(Q_i) : i = 1, \dots, k,\ j = 0, \dots, n_i - 1\}$ is a partition of X (into k columns of heights $n_1, \dots, n_k$ and μ-widths $q_1, \dots, q_k$).*

Alpern applied this result to prove that in the space of measure preserving homeomorphisms of a compact connected manifold with the topology of uniform convergence, any measure theoretic property which is generic (dense G_δ) in the weak topology on the space of invertible measure preserving transformations of the underlying probabilty measure space is also generic (dense G_δ) in the uniform convergence topology in the group of measure preserving homeomorphisms. The latter result is a far-reaching generalization of the classical (1940) Oxtoby-Ulam Theorem [OU40], where it is proved that ergodicity is generic in the space of measure

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preserving homeomorphisms. Furthermore this contains Katok and Stepin's 1970 result [KS70] that weak mixing is generic for measure preserving homeomorphisms.

Very recently, N. Ormes [Orm] has used Alpern's Multiple Rokhlin Tower Theorem as one step in obtaining the following result about realizing an ergodic measure preserving system as a minimal homeomorphism of the Cantor set within a given topological orbit equivalence class: Let Y be a Cantor set, S a minimal homeomorphism of Y and ν a uniquely ergodic S-invariant Borel probability measure. Let T be an ergodic invertible measure preserving transformation of the Lebesgue probability space (X, Σ, μ). Then there is a topological realization (S', ν) of the ergodic system (T, μ), where S' is a minimal homeomorphism of the Cantor set Y strongly orbit equivalent to S if and only if the finite rotations which are topological factors of S are measurable factors of T.

Two homeomorphisms, S and S', of the Cantor set Y, are said to be strongly orbit equivalent if there is a homeomorphism $h : Y \to Y$ and integer valued maps $m, n : Y \to Z$ such that $hS^{m(x)}(x) = S'h(x), hS(x) = (S')^{n(x)}h(x)$ and m and n have no more than one point of discontinuity. Strong orbit equivalence of two minimal Cantor homeomorphisms has been identified in the work of Giordano, Putnam and Skau [GPS96, Theorem 2.2] as a necessary and sufficient condition for the isomorphism of the crossed product C^*-algebras $(C(Y) \times_S Z$, and $C(Y) \times_{S'} Z)$ associated with the minimal homeomorphisms .

Multiple Rokhlin Towers arise naturally in the study of minimal homeomorphisms S of the Cantor set in the following manner. Given any clopen set A in the Cantor set Y, consider $r_A : A \to N$, the first return time function to A for the homeomorphism S, where $r_A(x)$ is the smallest positive integer such that $S^{r_A(x)}(x) \in A$. Then the continuity of this function implies there is a finite set of positive integers $n_1, \ldots, n_k$ which is the range of r_A. This gives a Multiple Rokhlin Tower partition of Y into clopen sets (into k towers of heights $n_1, \ldots, n_k$ over the base A). It is this tower for S which Ormes "copies" for the ergodic T using Alpern's theorem. While it need not be true that the $\gcd\{n_1, \ldots, n_k\} = 1$, Ormes notes [Orm, Cor 4.2] that if $\gcd\{n_1, \ldots, n_k\} = p$, then there is periodic clopen set for S of period p. Thus to have a similar tower picture for T, there must also be a T-periodic set in X of order p.

Extensions of Alpern's Multiple Rokhlin Tower Theorem to denumerably many columns and to nonsingular aperiodic transformations may be found in [Alp81] and [AP90]. In addition applications of these extensions to coding Markov chains and approximate conjugacy theorems can also be found in those papers.

2. Proof of Alpern's Theorem

Step 1: Prescribing return times: We find a set $A \subset X$ such that the set of first return times to A under T are exactly $n_1, \ldots, n_k$.

Proof Step 1: Since $\gcd\{n_1, \ldots, n_k\} = 1$, let R be a positive integer such that every integer $r \geq R$, can be expressed as nonnegative integer multiples of $n_1, \ldots, n_k$.

Following Kakutani's easy proof of Rokhlin's Lemma, let E be a sweep out set (i.e., $X = \cup_{i=0}^{\infty} T^i(E)$), such that the set of first return times back to E are all greater than R (in Step 2 we will require that E has small measure). This is elementary in the case that T is ergodic. See [Hal56] for a proof when T is antiperiodic. Write

$E = \cup_{m \geq R} E_m$, where $E_m = \{x \in E : \text{the first return time to } E \text{ is } m\}$. The column of height m over E is the set $C(E_m) = \cup_{i=1}^{m} T^{i-1} E_m$.

For each $m \geq R$ write $m = qN + r$ where $R \leq r < R + N$ and $N = n_1 n_2 \cdots n_k$. Then we can break up the column of height m over E, $(C(E_m))$, into q new subcolumns of height N and one new subcolumn of height r as follows: The first N floors of $C(E_m)$ are labeled $(N, 1), \ldots, (N, N)$, as are the next N floors. After labeling the first qN floors of $C(E_m)$ into q many N-columns in this manner, the last r floors are labelled to form an r-column (i.e, as $(r, 1), \ldots, (r, r)$). After doing this for each $m \geq R$ and combining all of the new columns of the same height (i.e, grouping all sets with the same label), we end up with one (large) column of height N and at most N "remainder" columns of heights $R, R+1, \ldots R+N-1$.

For each r, $R \leq r < R+N$, since we can write $r = r_1 n_1 + \cdots + r_k n_k$ where the r_i are nonnegative integers, we can break up each remainder r-column as follows: the first $r_1 n_1$ floors are broken up into r_1 columns of height n_1; the next block of $r_2 n_2$ floors are grouped into r_2 columns of height n_2; continuing, the last $r_k n_k$ floors of the r-column is broken into r_k columns of height n_k.

This partitions the space into a group of columns, one of height N and the others of heights $n_1, \ldots, n_k$. The set A which is the base of these columns, has return times N and $n_1, \ldots, n_k$. Note that the column of height N can be decomposed (labeled) into as many columns of height $n_1, \ldots, n_k$ as we wish, since N is a multiple of n_i for each i.

Step 2: Controlling the distribution: We note this has been a measure free construction thus far. If we wish to prescribe the distribution of the return times we note that the total measure of the "remainder" columns of height $r, R \leq r < R+N$ is less than $(R+N)\mu(E)$ where E was the base of the original skyscraper. By choosing E small enough so that $(R+N)\mu(E) < \min \{n_i q_i : i = 1, \ldots, k\}$ we can guarantee that after breaking up the remainder columns of height r into columns of heights $n_1, \ldots, n_k$ no column of height n_i has used up more than its "total allowance" of measure $n_i q_i$.

Now partition the N-column into k disjoint vertical columns, one for each $i = 1, \ldots, k$, in the following manner: Suppose that the measure of the n_i-column used in paving all the "remainder" r-columns $(R \leq r < R+N)$ totals up to p_i. From the column of height N we take a vertical strip of measure $n_i q_i - p_i$. Then this part of the N-column is partitioned into columns of height n_i. Choosing disjoint vertical strips from the N-column for the different i's, the partition of X into columns of height $n_i, i = 1, \ldots, k$, has the required distribution.

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Northeastern University, Boston MA 02115
eigen@neu.edu http://www.math.neu.edu/~eigen/

University of Massachusetts, Lowell MA 01854
vidhu_prasad@uml.edu