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Hopf Galois Structures on Degree p^2 Cyclic Extensions of Local Fields

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To Alex Rosenberg on his 70th birthday

Abstract. Let L be a Galois extension of K, finite field extensions of $\mathbb{Q}_p$, p odd, with Galois group cyclic of order p^2. There are p distinct K-Hopf algebras A_d, $d = 0, \ldots, p-1$, which act on L and make L into a Hopf Galois extension of K. We describe these actions. Let R be the valuation ring of K. We describe a collection of R-Hopf orders E_v in A_d, and find criteria on E_v for E_v to be the associated order in A_d of the valuation ring S of some L. We find criteria on an extension L/K for S to be E_v-Hopf Galois over R for some E_v, and show that if S is E_v-Hopf Galois over R for some E_v, then the associated order $\mathcal{A}_d$ of S in A_d is Hopf, and hence S is $\mathcal{A}_d$-free, for all d. Finally we parametrize the extensions L/K whose ramification numbers are $\equiv -1 \pmod{p^2}$ and determine the density of the parameters of those L/K for which the associated order of S in KG is Hopf.

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Let p be an odd prime, and let K be a finite extension of $\mathbb{Q}_p$ which contains a primitive p th root of unity ζ, and with valuation ring R. Let L be a Galois extension of K with Galois group G and valuation ring S. Relative Galois module theory seeks to understand S as a module over the group ring RG, or more generally over the associated order $\mathcal{A}$ of S in KG, $\mathcal{A} = \{\alpha \in KG | \alpha S \subset S\}$. Then $\mathcal{A} = RG$ and S is RG-free of rank one if and only if L/K is tamely ramified. For wildly ramified extensions, the only general criterion available is that if the associated order $\mathcal{A}$ is a Hopf order over R in KG, then S is $\mathcal{A}$-free of rank one [Ch87]. (The converse is far from true.)

Since the work of Greither and Pareigis [GP87], one knows that L/K may be a Hopf Galois extension with respect to different Hopf Galois actions on L. In

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fact, Byott has recently shown that for a Galois extension L/K with group G, the classical Hopf Galois structure is unique if and only if the order g of G is coprime to $\phi(g)$ (Euler's function) [By96]. In case L is a cyclic Galois extension of K of order p^n, then L/K has exactly p^{n-1} distinct Hopf Galois structures [Ko96]. Thus when $n = 2$ there are p distinct Hopf algebras A_d, $d = 0, \dots, p-1$, which give a Hopf Galois structure on L/K.

The existence of different Hopf Galois structures on L/K raises the possibility that S may have different Galois module properties with respect to one structure than another. For example, in [CM94] we found that the associated order of the valuation ring of $\mathbb{Q}(2^{\frac{1}{4}})$ in one Hopf Galois structure was Hopf and the associated order in the other structure was not. N. Byott [By96b] found a cyclotomic Lubin-Tate extension of local fields which has two Hopf Galois structures: one associated order is Hopf, while the second associated order $\mathcal{B}$ is not Hopf and the valuation ring is not free over $\mathcal{B}$.

In this paper we describe as algebras the Hopf algebras A_d which make L/K Hopf Galois, and their actions on L. Following [Gr92], we construct a collection of Hopf orders E_v over R inside each A_d. We find criteria on L/K in order that S be a Hopf Galois extension of R for some E_v. This implies, by [Ch87], that E_v is the associated order of S in A_d. In contrast to the examples just described, however, it turns out that if S is Hopf Galois over R for E_v, a Hopf order in A_d for some d, then the associated order of S in A_d for every d is Hopf, in particular for $A_0 = KG$. Thus in the case of cyclic Galois extensions of degree p^2, the non-classical Hopf Galois structures on L do not "tame" the wild extension L/K better than the classical structure given by the Galois group.

We apply Greither [Gr92] to find necessary and sufficient conditions on an order E_v to be realizable: that is, to be the associated order of the valuation ring of some extension L/K: the congruence condition on v is the same as for Hopf orders in KG as found by Greither. Finally, we quantify the remark in [Gr92, Remark (c), page 63] that congruence conditions on the ramification numbers of a cyclic totally ramified extension L/K of degree p^2 are "badly insufficient" for deciding whether the valuation ring S of L is Hopf Galois over R.

The concept of Hopf Galois extension of commutative rings arose in [CS69] as a merger of M. Sweedler's work on Hopf algebras and the development of Galois theory of commutative rings by S. U. Chase, D. K. Harrison and Alex Rosenberg [CHR65].

1. Hopf Galois Structures on Galois Field Extensions

We begin by recalling the main result of Greither and Pareigis [GP87].

Greither-Pareigis. *If L is a Galois extension of K with group G, then there is a bijection between Hopf Galois structures on L/K and regular subgroups of $Perm(G)$ normalized by $\lambda(G)$.*

Here $Perm(G)$ is the group of permutations of the set G, $\lambda(G)$ is the image of G in $Perm(G)$ given by left translation, and a subgroup N of $Perm(G)$ is regular if N acts transitively, has order equal to the order of G, and the stabilizer in N of any element of G is trivial. (Any two of these last conditions implies the third.)

If N is a regular subgroup of $Perm(G)$, then the group ring LN acts on $GL := Map(G, L)$ by $a\eta(f)(\sigma) = af(\eta^{-1}(\sigma))$ for a in L, σ in G, f in GL, η in N. Thus if

e_σ is the function which sends σ to 1 and τ to 0 if $\tau \neq \sigma$ in G, and η is in N, then $\eta(e_\sigma) = e_{\eta(\sigma)}$. This yields a map

$$LN \times GL \to GL.$$

The Hopf Galois structure on L is obtained by taking the fixed rings of LN and GL under the action of G, where G acts on GL by $\sigma(ae_\tau) = \sigma(a)e_{\sigma\tau}$, and acts on LN by $\sigma(a\eta) = \sigma(a)\sigma(\eta)$: the action of σ in G on η in N is by conjugation by $\lambda(\sigma)$ in $Perm(G)$.

Let G be cyclic of order p^n. Then Kohl [Ko96] has shown that the only regular subgroups N of $Perm(G)$ normalized by $\lambda(G)$ are isomorphic to G, and hence (cf. also [By96, Lemma 1, (i)]) there are exactly p^{n-1} such N.

We restrict to the case $n = 2$. Then we have

Proposition 1.1. *The subgroups of $Perm(G)$ normalized by $\lambda(G)$ are N_d for $d = 0, 1, \ldots, p-1$, where $N_d = \langle\eta\rangle$ with $\eta(\sigma^i) = \sigma^{(i-1)(1+pd)}$.*

These groups were found by using [By96, Proposition 1], a refinement of [Ch89, Proposition 1].

Proof. Clearly η is in $Perm(G)$. One verifies by induction that for any r,

$$\eta^r(\sigma^i) = \sigma^{(i-r)+(ir-\frac{r(r+1)}{2})pd}.$$

Hence η has order p^2 and the stabilizer in N_d of any σ^i is trivial. So N_d is regular. Also, for any d, $N_d \subset Perm(G)$ is normalized by $\lambda(G)$. In fact,

$$\lambda(\sigma)\eta\lambda(\sigma^{-1}) = \eta^{1+pd}.$$

For

$$\begin{aligned}\lambda(\sigma)\eta\lambda(\sigma^{-1})(\sigma^i) &= \lambda(\sigma)\eta(\sigma^{i-1}) \\ &= \lambda(\sigma)(\sigma^{(i-2)(1+pd)}) \\ &= \sigma^{(i-1)+(i-2)pd},\end{aligned}$$

while

$$\begin{aligned}\eta^{1+pd}(\sigma^i) &= \sigma^{i-(1+pd)+(i-1)pd} \\ &= \sigma^{(i-1)+(i-2)pd}.\end{aligned}$$

□

Example 1.2. For $p = 3$, set $d = 1$, then η is the permutation which sends σ^i to $\sigma^{4(i-1)}$; its cycle representation is

$$(0, 5, 7, 6, 2, 4, 3, 8, 1).$$

We have an action $LN \times GL \to GL$, which we will describe below. Looking at the fixed elements under the action of G, we have, first, that

$$(GL)^G = \left\{ \sum_\tau a_\tau e_\tau : \sum a_\tau e_\tau = \sum \sigma(a_\tau) e_\tau \right\}$$
$$= \left\{ \sum_\tau a_\tau e_\tau : a_{\sigma\tau} = \sigma(a_\tau) \right\}$$
$$= \left\{ \sum_\sigma \sigma(a) e_\sigma \right\}$$

This is isomorphic to L under the map sending a in L to $\sum \sigma(a) e_\sigma$.

Now identify σ in G with $\lambda(\sigma)$ in $Perm(G)$. Then,

$$LN^G = \left\{ \sum a_i \eta^i : \sum a_i \eta^i = \sum \sigma(a_i) \sigma(\eta^i) \right\}$$

where $\sigma(\eta^i)$ means the element η_0 of N so that $\eta_0 = \lambda(\sigma)\eta^i\lambda(\sigma)^{-1}$ in $Perm(G)$. Now

$$\sigma(\eta) = \sigma\eta\sigma^{-1} = \eta^{1+pd}$$

as we observed above, and hence $\sigma(\eta^i) = \eta^{i(1+dp)}$, and so $\sigma^k(\eta^i) = \eta^{i(1+kdp)}$. In particular, η^p is fixed under the action of G.

Let $N^p = \langle \eta^p \rangle$ and let

$$e_s = (1/p) \sum_{i=0}^{p-1} \zeta^{-si} \eta^{pi}$$

in KN^p. The e_s for $s = 0, \ldots, p-1$ are the pairwise orthogonal idempotents of KN^p corresponding to the distinct irreducible representations of KN^p: $\eta^p e_s = \zeta^s e_s$ for all s.

For v in L, set $a_v = \sum_{s=0}^{p-1} v^s e_s$. These elements, defined by Greither [Gr92], are the elements of LN^p corresponding to the tuple $(1, v, v^2, \ldots, v^{p-1})$ under the isomorphism between LN^p and $L \times L \times \cdots \times L$ induced by $\eta^p \to (1, \zeta, \zeta^2, \ldots, \zeta^{p-1})$. Thus $a_{vw} = a_v a_w$ for all v, w in L.

Proposition 1.3. *Let $L^{\langle \sigma^p \rangle} = M = K[z]$ where z^p is in K and $\sigma(z) = \zeta z$. Let LN^G correspond to the embedding β of G into $Hol(N)$ so that $\beta(\sigma) = \eta\gamma$ where $\gamma\eta\gamma^{-1} = \eta^{1+pd}$. Then $LN^G = K[\eta^p, a_v\eta]$ where $v = z^{-d}$.*

Proof. We have that $\sigma^k(\eta) = \eta^{1+kpd}$, so $\sigma^p(\eta) = \eta^{1+p^2d} = \eta$. So σ^p fixes the elements of N, and $LN^G = MN^G$. Since G fixes η^p and

$$e_s = (1/p) \sum_{i=0}^{p-1} \zeta^{-si} \eta^{pi},$$

G fixes the idempotents e_s for all s. Hence

$$\begin{aligned}\sigma(a_{z^{-d}}\eta) &= \eta^{1+pd}\sum_{s=0}^{p-1}\sigma(z^{-ds})e_s \\ &= \eta\sum_{s=0}^{p-1}\zeta^{-ds}z^{-ds}\eta^{pd}e_s \\ &= \eta\sum_{s=0}^{p-1}\zeta^{-ds}z^{-ds}\zeta^{ds}e_s \\ &= \eta\sum_{s=0}^{p-1}z^{-ds}e_s \\ &= a_{z^{-d}}\eta.\end{aligned}$$

Thus $K[\eta^p, a_v\eta] \subset LN^G$. But by Galois descent, LN^G has rank p^2 over K, and since a_{v^p} is in $K[\eta^p]$, one easily sees that $(a_v\eta)^p$ is in $K[\eta^p]$, hence $K[\eta^p, a_v\eta]$ has rank p^2 over K, hence equality. □

We observe for later use that $K[\eta^p, a_v\eta] = K[\eta^p, a_{vc}\eta]$ for any c in K. For $a_{vc} = a_v a_c$, so $a_{vc}\eta = a_c \cdot a_v\eta$, and a_c is in $K[\eta^p]$.

Let A_d denote the K-Hopf algebra $K[\eta^p, a_v\eta]$ with $v = z^{-d}$. We examine the action of $A_d = LN^G$ on L.

Since L/K is a Galois extension with Galois group $G = C_{p^2} = \langle\sigma\rangle$ and K contains ζ, a primitive pth root of unity, we can assume that $M = L^{\langle\sigma^p\rangle} = K[z]$ with z^p in K and $\sigma(z) = \zeta z$, and $L = M[x]$ with x^p in M and $\sigma^p(x) = \zeta x$. Let $v = cz^{-d}$,with c in K and $0 \leq d \leq p-1$.

Proposition 1.4. *$A_d = K[\eta^p, a_v\eta]$ acts on $L = K[z][x]$ by*

$$\eta^p = \sigma^p$$

and for a in $K[z]$

$$(a_v\eta)(ax^m) = v^m\sigma(ax^m).$$

In particular, $A_0 = K[\eta]$ with $\eta(s) = \sigma(s)$ for s in L, the classical action by the group ring of the Galois group G.

Proof. We identify L as a subset of $GL = Map(G, L)$ via the isomorphism

$$a \rightarrow \sum_{i=0}^{p-1}\sigma^i(a)e_i$$

where $e_i = e_{\sigma^i}$. Then as we observed in the proof of Proposition 1.1,

$$\eta^r(e_i) = e_{i-r-pd(ir-\frac{r(r+1)}{2})}.$$

In particular, $\eta^{pk}(e_i) = e_{i-pk}$, so

$$\begin{aligned}\eta^p\left(\sum \sigma^i(a)e_i\right) &= \sum \sigma^i(a)e_{i-p}\\ &= \sum \sigma^{i+p}(a)e_i\\ &= \sum \sigma^i(\sigma^p(a))e_i\end{aligned}$$

which corresponds to $\sigma^p(a)$ in L.

Now for a in $K[z]$,

$$\begin{aligned}(a_v\eta)(ax^m) &= \left(\sum_{s,k}\frac{1}{p}v^s\zeta^{-ks}\eta^{kp+1}\right)(ax^m)\\ &= \sum_{s,k}\frac{1}{p}v^s\zeta^{-ks}\eta^{kp+1}\left(\sum_i \sigma^i(ax^m)e_i\right)\\ &= \sum_{i,s,k}\frac{1}{p}v^s\zeta^{-ks}\sigma^i(ax^m)e_{(i-kp-1)+pd(i-1)}.\end{aligned}$$

The subscript on e is mod p^2, so if we set

$$j = i(1+pd) - (1+kp+dp),$$

then

$$\begin{aligned}i &\equiv j(1-pd) + (1+kp) \pmod{p^2}\\ &= (j+1) + p(k-jd)\end{aligned}$$

and the sum becomes

$$= \sum_{j,s,k}\frac{1}{p}v^s\zeta^{-ks}\sigma^{(j+1)+p(k-jd)}(ax^m)e_j.$$

Since σ^p fixes a in $M = K[z]$, this is

$$\begin{aligned}&= \sum_{j,s,k}\frac{1}{p}v^s\zeta^{-ks}\sigma^{j+1}(ax^m)\zeta^{(k-jd)m}e_j\\ &= \sum_j\sum_s v^s\left(\frac{1}{p}\sum_k \zeta^{-ks+km}\right)\sigma^{j+1}(ax^m)\zeta^{-jdm}e_j.\end{aligned}$$

The sum over k is p if $s = m$ and 0 otherwise. So the sum over j and s becomes

$$= \sum_j v^m\zeta^{-jdm}\sigma^{j+1}(ax^m)e_j.$$

Now $v = cz^{-d}$, so

$$\begin{aligned}\sigma^j(v^m) &= c^m\zeta^{-jdm}(z^{-dm})\\ &= \zeta^{-jdm}v^m.\end{aligned}$$

Thus the sum

$$\begin{aligned}&= \sum_j \sigma^j(v^m)\sigma^{j+1}(ax^m)e_j\\ &= \sum_j \sigma^j(v^m\sigma(ax^m))e_j\end{aligned}$$

which corresponds to $v^m\sigma(ax^m)$ in L. That is,

$$(a_v\eta)(ax^m) = v^m\sigma(ax^m).$$

□

2. Hopf Orders

Now suppose K is a finite extension of $\mathbb{Q}_p$, with valuation ring R and parameter π. Let e be the absolute ramification index of K. Assume K contains a primitive pth root of unity ζ. Then $(\zeta - 1)R = \pi^{e'}R$ and $(p-1)e' = e$.

Let $M = K[z]$ with $z^p = b$ in R, and let T be the valuation ring of M. Then we may consider the K-Hopf algebras $A_d = K[\eta^p, a_v\eta]$, where $v = z^{-d}$, as described in Section 1. (Recall that for any c in K, $K[\eta^p, a_v\eta] = K[\eta^p, a_{vc}\eta]$). In this section we extend work of Greither [Gr92][GC96] to construct a collection of Hopf orders over R in A_d for each d with $0 \le d \le p-1$. These Hopf orders are parametrized by integers i, j with $0 \le i, j \le e'$ and a unit c in R.

For i an integer, $0 \le i \le e'$, let $i' = e' - i$.

Theorem 2.1. *Let i, j be integers with $0 < i, j \le e'$. Let $H_i = R\left[\frac{\eta^p - 1}{\pi^i}\right]$, a Hopf order in $K[\eta^p]$. For $v = z^{-d}c, c$ in R, let $y = \frac{a_v\eta - 1}{\pi^j}$. Then the R-algebra $E = H_i[y]$ is an R-Hopf order in $A_d = K[\eta^p, a_v\eta]$ and a Hopf algebra extension of H_j by H_i if and only if*

$$\zeta b^{-d}c^p \equiv 1 \pmod{\pi^{i'+pj}R}$$

and

$$b^{-d}c^p \equiv 1 \pmod{\pi^{pi'+j}R}.$$

Recall that the H_i for $0 \le i \le e'$ are all the Hopf orders in the group ring $K[\eta^p]$ by Tate-Oort [TO70]. This description of the H_i goes back to Larson [La76].

Proof. The canonical map from $K[N]$ to $K[N/N^p]$ sends η^p to 1, and sends a_v to 1 and H_i to R, so the image of E is $R[\frac{\bar{\eta}-1}{\pi^j}] = H_j$. To show that E is a Hopf algebra extension of H_j by H_i, we need to show that $E \cap K[\eta^p] = H_i$. This is equivalent to showing that the monic polynomial of degree p satisfied by y over $K[\eta^p]$ has coefficients in H_i. We follow [GC96, Section 2] and utilize [Gr92, I, section 3].

Now $a_v\eta = 1 + \pi^j y$, so

$$\begin{aligned}(a_v\eta)^p &= (1+\pi^j y)^p \\ &= 1 + \sum_{r=1}^{p-1} \binom{p}{r} \pi^{jr} y^r + \pi^{jp} y^p,\end{aligned}$$

hence

$$y^p + \pi^{-jp} \sum_{r=1}^{p-1} \binom{p}{r} \pi^{jr} y^r + \frac{1-(a_v\eta)^p}{\pi^{jp}} = 0.$$

Note that $(a_v\eta)^p = a_{v^p}\eta^p$, and $\eta^p = a_\zeta$, so $(a_v\eta)^p = a_{v^p\zeta}$. Thus y satisfies a monic polynomial with coefficients in H_i if and only if in H_i,

1) π^{jp} divides $p\pi^{jr}$ for $r = 1, \ldots, p-1$;
2) π^{jp} divides $1 - a_{v^p\zeta}$.

Condition 1) is equivalent to $jp \le e + j$, or $j \le e'$.
Condition 2) is the same as

$$a_{v^p\zeta} \equiv 1 \pmod{\pi^{jp} H_i},$$

which, by [Gr92, I 3.2b], is equivalent to

$$v^p\zeta \equiv 1 \pmod{\pi^{i'+pj} R},$$

or, since $v^p = b^{-d}c^p$,

$$b^{-d}c^p\zeta \equiv 1 \pmod{\pi^{i'+pj} R}.$$

Note that if $j \le e'$ then $\frac{1-(a_v\eta)^p}{\pi^{pj}} \in E \cap K[\eta^p]$, so if $\frac{1-(a_v\eta)^p}{\pi^{pj}} \notin H_i$ then $E \cap K[\eta^p] \neq H_i$.

Now we show that E is closed under comultiplication if and only if $v^p \equiv 1 \pmod{\pi^{pi+j} R}$.

Recall that $A_d = K[\eta^p, a_v\eta]$ and T is the valuation ring of M. Let $E = R[t][y] = H_i[y]$ with $t = \frac{\eta^p - 1}{\pi^i}$, $y = \frac{a_v\eta - 1}{\pi^j}$. Since Δ is an algebra homomorphism, to show E is a coalgebra, it suffices to show that $\Delta(y) \in E \otimes E$.

Now $\Delta(y) \in A_d \otimes A_d = K \otimes_R (E \otimes_R E)$ and R is integrally closed. If we show that $\Delta(y) \in T \otimes_R (E \otimes_R E) = TE \otimes_T TE$, then, since E and therefore $E \otimes_R E$ are free R-modules,

$$(T \otimes_R (E \otimes_R E)) \cap (K \otimes_R (E \otimes_R E)) = E \otimes_R E,$$

and so $\Delta(y) \in E \otimes E$.

We will show, in fact, that

$$\Delta(y) \in C \otimes C$$

where $C = H_i \cdot 1 + H_i \cdot y$. Again, it is enough to show that $\Delta(y) \in TC \otimes_T TC$.

Now

$$\Delta(y) = \Delta\left(\frac{a_v\eta - 1}{\pi^j}\right) = \frac{\Delta(\alpha_v\eta) - a_v\eta \otimes a_v\eta}{\pi^j} + y \otimes (1 + \pi^j y) + 1 \otimes y$$

and the last two terms are in $C \otimes C$. So it suffices to show that

$$\frac{\Delta(a_v\eta) - a_v\eta \otimes a_v\eta}{\pi^j} \in TC \otimes_T TC.$$

Now a_v is a unit of TH_i. For since $v^p \in U_{pi'+j}(R)$, then $v \in U_{pi'+j}(T)$, hence by [Gr92, I 3.2(b)], $a_v \in 1 + \pi^{j/p}H_i$. Since $j > 0, a_v$ is a unit of TH_i. Since $a_v\eta = 1 + \pi^j t \in TH_i \cdot 1 + TH_i \cdot t = TC$, therefore $\eta \in TC$. So

$$\left(\frac{\Delta(a_v) - a_v \otimes a_v}{\pi^j}\right)(\eta \otimes \eta) \in TC \otimes_T TC$$

if and only if

$$\frac{\Delta(a_v) - a_v \otimes a_v}{\pi^j} \in TH_i \otimes_T TH_i.$$

To decide if

$$\frac{\Delta(a_v) - a_v \otimes a_v}{\pi^j} \in TH_i \otimes_T TH_i$$

we identify elements of $M[\eta^p] \otimes_M M[\eta^p]$ as $p \times p$ matrices as in [Gr92, I, Section 3].

We have

$$\frac{\Delta(a_v) - a_v \otimes a_v}{\pi^j} = \frac{1}{\pi^j}\sum_{s=0}^{p-1}\left[\Delta(v^s e_s) - \sum_{0\le r,t<p, r+t\equiv s \pmod p} v^r e_r \otimes v^t e_t\right] = \sum_{s=1}^{p-1} v^s \sum_{r+t\ge p, r+t\equiv s \pmod p}\left[\frac{1-v^p}{\pi^j} e_r \otimes e_t\right].$$

Let $\frac{1-v^p}{\pi^j} = w$. Then

$$\frac{\Delta(a_v) - a_v \otimes a_v}{\pi^j}$$

corresponds to the matrix $M = \{M_{a,b}\}$ where $M_{a,b}$ is the coefficient of $e_a \otimes e_b$. Here, $M_{a,b} = 0$ if $a + b < p$, and $M_{a,b} = wv^s$ where $a + b = p + s$ for $a + b \ge p$.

Now $\frac{\Delta(a_v)-a_v\otimes a_v}{\pi^j} \in TH_i \otimes TH_i$ is equivalent, by [Gr92, I, Lemma 3.3] to: for all k, k^* with $0 \le k, k^* < p$, $\pi^{i'(k+k^*)}$ divides

$$d^{k,k^*}(M) = \sum_{a=0}^{k}\sum_{b=0}^{k^*}\binom{k}{a}\binom{k^*}{b}(-1)^{a+b}M_{a,b} = \sum_{s=0}^{l}\sum_{a+b=p+s}\binom{k}{a}\binom{k^*}{b}(-1)^{a+b}M_{a,b}$$

where $k+k^*=p+l$. Since $M_{a,b}=wv^s$ for $a+b=p+s$, this is

$$
\begin{aligned}
&= w\sum_{s=0}^{l}\sum_{a+b=p+s}\binom{k}{a}\binom{k^*}{b}(-1)^{p+s}v^s \\
&= w\sum_{s=0}^{l}\binom{k+k^*}{p+s}(-1)^{p+s}v^s.
\end{aligned}
$$

Now since $s<p$,

$$\binom{k+k^*}{p+s}=\binom{p+l}{p+s}\equiv\binom{l}{s} \pmod p,$$

so

$$
\begin{aligned}
&\equiv w\sum_{s=0}^{l}\binom{l}{s}(-1)^{p+s}v^s \pmod p \\
&\equiv -w(1-v)^l \pmod p.
\end{aligned}
$$

Thus $M\in TH_i\otimes TH_i$ if and only if $\pi^{i'(k+k^*)}=\pi^{i'(p+l)}$ divides $w(1-v)^l$ for all $l\geq 0$.

For $l=0$ the condition is: $\pi^{i'p}$ divides $w=\frac{1-v^p}{\pi^j}$, or $v^p\equiv 1 \pmod{\pi^{pi'+j}}$. Assuming $v^p\equiv 1 \pmod{\pi^{pi'+j}}$, then, since $v\in U_{pi'+j}(T)$,

$$v-1\in\pi^{i'+\frac{j}{p}}T$$

(recall: π is the parameter for R), so

$$(v-1)^l\in\pi^{i'l+\frac{jl}{p}}T.$$

Also $w\in\pi^{pi'}R$, so

$$w(1-v)^l\in\pi^{pi'+i'l+\frac{jl}{p}}T.$$

Since $i'(k+k^*)=pi'+i'l$, therefore $\pi^{i'(k+k^*)}$ divides $d^{k+k^*}(M)$ for all k,k^*.

Thus

$$\frac{\Delta(a_v)-a_v\otimes a_v}{\pi^j}\in TH_i\otimes TH_i$$

if and only if $v^p\equiv 1 \pmod{\pi^{pi'+j}}$. That completes the proof. □

Suppose i,j satisfy $0<i,j\leq e'$ and consider the two conditions

$$
\begin{aligned}
v^p &\equiv 1 \pmod{\pi^{pi'+j}}; \\
\zeta v^p &\equiv 1 \pmod{\pi^{i'+pj}}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\zeta v^p-1 &= \zeta v^p-v^p+v^p-1 \\
&= (\zeta-1)v^p+(v^p-1)
\end{aligned}
$$

we must have two of $ord_R(\zeta v^p - 1)$, $ord_R(v^p - 1)$ and e' equal, and both $\leq$ the third (isosceles triangle inequality). For E to be a Hopf algebra and a free H_i-module requires

$$ord_R(\zeta v^p - 1) \geq i' + pj$$

and

$$ord_R(v^p - 1) \geq pi' + j.$$

Thus $i' + pj \leq e'$ or $pi' + j \leq e'$. The first is equivalent to $i \geq pj$; the second to $j' \geq pi'$. Hence:

Corollary 2.2. *In order that E be a Hopf algebra, i and j must satisfy: $0 < i, j \leq e'$ and $i \geq pj$ or $j' \geq pi'$.* □

Note: $i \geq pj$ is the condition of [Gr92, I 3.6] and [Gr92, II], cf. [Un94].
If $i + j \leq e'$, then $i' + pj \leq pi' + j$, so if $ord_R(v^p - 1) \geq pi' + j$, then

$$\begin{aligned} ord_R(\zeta v^p - 1) &\geq min\{e', ord_R(v^p - 1)\} \\ &\geq min\{e', pi' + j\} \geq i' + pj. \end{aligned}$$

So we have

Corollary 2.3. *If $i, j > 0, i + j \leq e'$ and $i \geq pj$, then E is a Hopf order with $E \cap K[\eta^p] = H_i$ if and only if $ord_R(v^p - 1) \geq pi' + j$.* □

The Hopf algebras E presumably fit within the classification of [By93], but the description of the E here is rather different that that of Byott.

3. Hopf Galois Structures

Now we consider a cyclic extension L/K with Galois group $G = \langle\sigma\rangle$ of order p^2, and see when S/R is E_v-Galois for some v.

We assume throughout this section that $i, j > 0$, $0 \leq i + j \leq e'$ and $i \geq pj$. Under these hypotheses, $p(i' + j) \leq pj' + 1$. For since $pj \leq i$, we have

$$pi \geq p^2 j > 2pj - 1$$

so

$$\begin{gathered} 1 - pj > -pi + pj, \\ 1 + pe' - pj > pe' - pi + pj, \end{gathered}$$

which is

$$pj' + 1 > p(i' + j).$$

Suppose S/R is E_v-Galois. Then T/R is H_j-Galois and S/T is $T \otimes H_i$-Galois, by [Gr92]. Since $i, j > 0$, M/K and L/M are totally, hence wildly ramified.

If T/R is H_j-Galois, then (cf. [Ch87]) $M = K[z]$ with $z^p = 1 + u\pi^{pj'+1}$ and $t = \frac{z-1}{\pi^{j'}}$ is a parameter for T, so $T = R[t]$. Since $\sigma(t) = \frac{\zeta-1}{\pi^{j'}}z + t = t + ut^{pj}$ for u some unit of T, the ramification number $t_1^{G/H} = pj - 1$. The converse also holds: c.f [Ch87] or [Gr92]. By [Se62, Ch. V, Sec. 1, Cor. to Prop. 3], $t_1^{G/H} = t_1^G$, so $t_1^G = pj - 1$.

Similarly, if S/T is $T \otimes H_i$-Galois, M/K is totally ramified, and t is a parameter for T, we may find x in L so that $L = M[x]$ with $\sigma^p(x) = \zeta x$ and $x^p = \gamma = 1 + ut^{p^2i'+1}$ for some unit u of T. Then $w = \frac{x-1}{\pi^{i'}}$ is a parameter for S, and

$$\sigma^p(w) = \frac{\zeta - 1}{\pi^{i'}}x + w = w + w^{p^2i}u'$$

for some unit u' of S. So the ramification number for L/M is $t_1^H = p^2i - 1$, and conversely. Since $t_1^H = t_2^G$, we have $t_2^G = p^2i - 1$.

Now L is a Galois extension of K with group $G = \langle\sigma\rangle$, cyclic of order p^2, so $\sigma(x) = \beta x$ for some β in T with $N_{M/K}(\beta) = \zeta$. If $ord_T(x^p - 1) = p^2i' + 1$, then $\sigma(w) = \frac{\beta-1}{\pi^{i'}}x + w$, so since $t_1^G = pj - 1$, $ord_L(\frac{\beta-1}{\pi^{i'}}) = pj$. Thus

$$ord_L(\beta - 1) = p^2i' + pj$$

and so

$$ord_M(\beta^p - 1) = p^2i' + pj.$$

Lemma 3.1. *β is unique modulo $t^{pi'+pj}T$.*

Proof. Let $\gamma = x^p = 1 + ut^{p^2i'+1}$ for some unit u of T.

Suppose we replace x by $x\alpha$ for some $\alpha \in T$. Then

$$(x\alpha)^p = \gamma\alpha^p = (1 + ut^{p^2i'+1})\alpha^p.$$

If $ord_T((x\alpha)^p - 1) = p^2i' + 1$, then $ord_T(\alpha^p - 1) \geq p^2i' + 1$. If $ord_T(\alpha - 1) = s$, then $ord_T(\alpha^p - 1) = ps$ unless $pe' \leq s$. Assuming $s \leq pe'$, then we require

$$ps \geq p^2i' + 1,$$

so

$$s \geq pi' + 1.$$

Now if we replace x by $x\alpha$, then $\sigma(x\alpha) = \beta\frac{\sigma(\alpha)}{\alpha}(x\alpha)$, so β is replaced by $\beta\frac{\sigma(\alpha)}{\alpha}$. If $ord_T(\alpha - 1) = s$ then by [Wy69, Theorem 22],

$$\begin{aligned} ord_T\left(\frac{\sigma(\alpha)}{\alpha} - 1\right) &\geq s + pj - 1 \\ &\geq pi' + 1 + pj - 1 = p(i' + j). \end{aligned}$$

So $\beta\frac{\sigma(\alpha)}{\alpha} \equiv \beta \pmod{t^{p(i'+j)}T}$.

Thus β is unique modulo $t^{p(i'+j)}T$. □

Given L/K with ramification numbers $t_1^G = pj - 1$ and $t_2^G = p^2 i - 1$, when is there some E_v so that S/R is E_v-Galois? Since the discriminant over R of S equals the discriminant of the dual of E_v, S will be E_v-Galois if and only if E_v acts on S (see [Gr92, II, Section 1]), that is, $\xi \cdot s$ is in S (not just in L) for all $\xi \in E_v$ and $s \in S$. Equivalently, $E_v \subset \mathcal{A}$, the associated order of S in A_d.

We know $\mathcal{A}$ is an algebra. So to show $E_v \subset \mathcal{A}$ it suffices to show that

$$t = \frac{\eta^p - 1}{\pi^i} \in \mathcal{A}$$

and

$$y = \frac{a_v\eta - 1}{\pi^j} \in \mathcal{A}.$$

Now

$$\begin{aligned}\Delta(t) &= \frac{\eta^p \otimes \eta^p - 1 \otimes 1}{\pi^i} \\ &= \left(\frac{\eta^p - 1}{\pi^i}\right) \otimes \eta^p + 1 \otimes \left(\frac{\eta^p - 1}{\pi^i}\right) \\ &= t \otimes (1 + \pi^i t) + 1 \otimes t.\end{aligned}$$

Hence if

$$t\left(\frac{z-1}{\pi^{j'}}\right) \in S,$$

then since L is an A_d-module algebra,

$$t\left(R\left[\frac{z-1}{\pi^{j'}}\right]\right) \subset S,$$

so $tT \subset S$. Also, if

$$t\left(\frac{x-1}{\pi^{i'}}\right) \in S$$

then

$$t\left(T\left[\frac{x-1}{\pi^{i'}}\right]\right) \subset S,$$

so $tS \subset S$ and $t \in \mathcal{A}$. Hence $H_i \subset \mathcal{A}$.

Similarly, we showed in the proof of Theorem 2.1 that $C = H_i \cdot 1 + H_i \cdot y$ is a subcoalgebra of E_v. If

$$y\left(\frac{z-1}{\pi^{j'}}\right) \in S$$

then

$$C\left(\frac{z-1}{\pi^{j'}}\right) \subset S,$$

so $CT \subset S$. Also, if

$$y\left(\frac{x-1}{\pi^{i'}}\right) \in S$$

then

$$C\left(\frac{x-1}{\pi^{i'}}\right) \subset S,$$

so, since

$$S = R\left[\frac{z-1}{\pi^{j'}}\right]\left[\frac{x-1}{\pi^{i'}}\right],$$

$CS \subset S$. So $C \subset \mathcal{A}$. Since C generates E_v as an R-algebra, $E_v \subset \mathcal{A}$.

Thus E_v acts on S if and only if $t = \frac{\eta^p - 1}{\pi^i}$ and $y = \frac{a_v \eta - 1}{\pi^j}$ map $\frac{z-1}{\pi^{j'}}$ and $\frac{x-1}{\pi^{i'}}$ into S.

We see that

$$t\left(\frac{z-1}{\pi^{j'}}\right) = 0,$$

$$y\left(\frac{z-1}{\pi^{j'}}\right) = \frac{\sigma^{-1}(z) - z}{\pi^{e'}} = \frac{\zeta^{-1} - 1}{\pi^{e'}} z \in T,$$

and

$$t\left(\frac{x-1}{\pi^{i'}}\right) = \frac{\zeta^{-1} - 1}{\pi^{e'}} x \in S;$$

finally, by Proposition 1.4,

$$y\left(\frac{x-1}{\pi^{i'}}\right) = \frac{a_v \eta(x) - x}{\pi^{i'+j}} = \frac{v\sigma(x) - x}{\pi^{i'+j}} = \frac{v\beta - 1}{\pi^{i'+j}} x$$

is in S if and only if

$$\beta \equiv v^{-1} \pmod{\pi^{i'+j} T}.$$

From this we have

Proposition 3.2. *Let L/K be a Galois extension with group G cyclic of order p^2 and with ramification numbers $t_1 = pj - 1$ and $t_2 = p^2 i - 1$, where i, j satisfy the inequalities at the beginning of this section. Then the valuation ring S of L is E_v-Hopf Galois over R, and hence the associated order of S in A_d is Hopf, if and only if $\beta \equiv v^{-1} \pmod{\pi^{i'+j} T}$.* □

Now we observe

Lemma 3.3. *If $v \equiv z^{-d} c$ for some c in R, then $v \equiv c \pmod{\pi^{i'+j} T}$.*

Proof. We have

$$z = 1 + ut^{pj'+1},$$

u a unit of T. Since $pj' + 1 > p(i' + j)$,

$$z \equiv 1 \pmod{\pi^{i'+j} T = t^{p(i'+j)} T}.$$

□

Corollary 3.4. *With the hypotheses of Proposition 3.2, if S is E_v-Galois then p divides j.*

Proof. We have $ord_T(\beta-1) = pi'+j$, and so $ord_T(v^{-1}-1) = ord_T(v-1) = pi'+j$. Hence $ord_R(v^p-1) = pi'+j$.

Since $v = z^{-d}c$ and $pi'+j < pj'+1$, we have

$$ord_R(v^p-1) = pi'+j < pj'+1 = ord_R(z^p-1),$$

so $ord_R(v^p-1) = ord_R(c^p-1) = p\, ord_R(c-1)$. Hence $ord_R(c-1) = i'+j/p$, and p divides j. $\square$

Corollary 3.5. *With the hypotheses of Proposition 3.2, if S/R is Hopf Galois for some E_v, then S is free over the associated order in A_d for all d.*

Proof. We have that S/R is Hopf Galois for E_v, $v = z^{-d}c$, if and only if

$$\beta \equiv (z^{-d}c)^{-1} \pmod{\pi^{i'+j}T}.$$

But

$$z^{-d} \equiv 1 \pmod{\pi^{i'+j}T},$$

and hence

$$\beta \equiv (z^{-d}c)^{-1} \pmod{\pi^{i'+j}T}$$

for every d, and so E_v acts on S when $v = z^{-d}c$ for every d. Hence for any d, S/R is $E_{z^{-d}c}$-Hopf Galois, and so $E_{z^{-d}c}$ is the associated order of S in A_d for every d. $\square$

Corollary 3.6. *E_v is realizable if and only if $ord_T(v-1) = pi'+j$.*

Proof. If L/K realizes E_v, that is, E_v is the associated order of the valuation ring of the Galois extension L of K, then, as we showed, $\beta \equiv v^{-1} \pmod{\pi^{i'+j}T}$, so $ord_T(v-1) = pi'+j$. Conversely, if $ord_T(v-1) = pi'+j$, then since $v = cz^{-d}$ for some $c \in R$, $ord_T(c-1) = pi'+j$, so E_c is realizable by some L/K by [Gr92, Part II, Section 3]. But then, since $cz^{-d} \equiv c \pmod{\pi^{i'+j}T}$, we see that the extension L/K also realizes E_v by Proposition 3.2. $\square$

The problem raised at the beginning of this section can be precisely answered by the following corollary, in which the hypotheses on L are recapitulated.

Corollary 3.7. *Let K be a finite extension of $\mathbb{Q}_p$ containing ζ_p, a primitive pth root of unity. Let L be a cyclic Galois extension of K with Galois group $G = \langle\sigma\rangle$ of degree p^2 with intermediate field M and with ramification numbers $t_1^G = pj-1$ and $t_2^G = p^2i-1$ where $0 < pj \leq i, p$ divides j, and $i+j \leq e' = e_{K/\mathbb{Q}_p}/(p-1)$. Let S, T and R be the valuation rings of L, M and K, respectively. Let $L = M[x]$ with $ord_M(x^p-1) = p^2i'+1$ and $\sigma(x) = \beta x$. Then S is an E_v-Hopf Galois extension of R if and only if β is congruent to an element of R modulo $t^{pi'+pj}T = \pi^{i'+j}T$.*

Proof. The ramification conditions on L/K are equivalent to T/R being H_j- Hopf Galois and S/T being $T \otimes H_i$-Hopf Galois. Then S is E_v-Hopf Galois for some v if and only if $\beta \equiv v^{-1} \pmod{t^{p(i'+j)}T}$ by Proposition 3.2, and

$$v \equiv c \pmod{\pi^{i'+j}T}$$

with $c \in R$ by Lemma 3.3. Thus S is E_v-Hopf Galois if and only if the element β which by Lemma 3.1 is uniquely associated to L is congruent to an element of R modulo $\pi^{i'+j}T$. □

Lemma 3.1 implies that there is a well-defined map from the set of cyclic extensions L of K containing M satisfying the hypotheses of Corollary 3.7 to

$$U_{pi'+j}(T)/U_{pi'+pj}(T),$$

and hence to

$$U_{pi'+j}(T)/U_{pi'+j+p-1}(T).$$

Call that map ϕ.

Corollary 3.8. *ϕ maps onto the classes $\bar{U}$ of $U_{pi'+j}(T)/U_{pi'+j+p-1}(T)$ represented by β in T with $ord_T(\beta-1) = pi'+j$.*

Proof. Let β be any element of T with $ord_T(\beta-1) = pi'+j$. We first show that β may be modified by an element of $U_{pi'+j+p-1}(T)$ to an element of norm ζ.

By [Wy69, Theorem 22], the map $\sigma - 1$ yields an isomorphism

$$U_{pi'+j+r-(pj-1)}(T)/U_{pi'+j+r+1-(pj-1)}(T) \to U_{pi'+j+r}(T)/U_{pi'+j+r+1}(T)$$

for all r such that $pi'+j+r-pj+1$ is not divisible by p. Since p divides j, we obtain such an isomorphism for $r = 0, 1, \ldots, p-2$. Thus any β_r in $U_{pi'+j+r}(T)$ is of the form $\beta_r = \frac{\sigma(\alpha_r)}{\alpha_r}\beta_{r+1}$ for some $\beta_{r+1} \in U_{pi'+j+r+1}(T)$. Making that observation for $r = 0, 1, \ldots, p-2$, we see that any β_0 with $ord_T(\beta_0 - 1) = pi'+j$ may be written as $\beta_0 = \frac{\sigma(\alpha)}{\alpha}\beta_{p-1}$ for some α in $U(T)$ and some β_{p-1} in $U_{pi'+j+p-1}(T)$. Thus every β in T with $ord_T(\beta-1) = pi'+j$ may be multiplied by an element of $U_{pi'+j+p-1}(T)$ to obtain an element β' of norm 1. That is, the class of any β_0 in $U_{pi'+j}(T)/U_{pi'+j+p-1}(T)$ contains an element of norm 1.

By [Gr92, Lemma 3.8], there exists an element $\delta \in U_{pi'+pj}(T)$ of norm ζ. Multiplying the representative in the class of β_0 with norm 1 by δ gives an element β in the class of β_0 of norm ζ.

Any β with $ord_T(\beta-1) = pi'+j$ and norm $= \zeta$ is in the image of ϕ. For by the proof of [Gr92, Lemma 3.9], we may find γ in $U(T)$ with $ord_T(\gamma-1) = p^2i'+1$ and $\frac{\sigma(\gamma)}{\gamma} = \beta^p$; such a γ yields a cyclic extension L/K of degree p^2 satisfying the hypotheses of Corollary 3.7 with $\sigma(x) = \beta x$.

Thus any class in $U_{pi'+j}(T)/U_{pi'+j+p-1}(T)$ represented by an element β with $ord_T(\beta) = pi'+j$ is represented by such a cyclic extension. □

Let $q = |R/\pi R|$. Then the number of elements of $U_{pi'+j}(T)/U_{pi'+j+p-1}(T)$ of order $pi'+j$ is easily seen to be $(q-1)q^{p-2}$ (expand elements of $U_{pi'+j}(T)$ t-adically).

Only $q-1$ of these have classes represented by units of R. Thus the field extensions L/K satisfying the hypotheses of Corollary 3.7 map by ϕ onto $\bar{U}$, but those whose valuation rings S are Hopf Galois over R map onto a subset of $\bar{U}$ of density $\frac{1}{q^{p-2}}$. This may illuminate Greither's remark [Gr92, Remark (c), p. 63] that congruence conditions on the ramification numbers are badly insufficient for insuring that S/R is Hopf Galois.

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