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Deformed Enveloping Algebras

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Abstract. We construct deformed enveloping algebras without using generators and relations via a generalized semidirect product construction. We give two Hopf algebraic constructions, the first one for general Hopf algebras with triangular decomposition and the second one for the special case that the outer tensorands are dual. The first construction generalizes Radford's biproduct and Majid's double crossproduct, the second one Drinfel'd's Double construction. The second construction is applied in the last section to construct deformed enveloping algebras in the setting created by G. Lusztig.

Contents

1. Introduction

Deformed enveloping algebras were defined by V. G. Drinfel'd at the International Congress of Mathematicians 1986 in Berkeley [2]. His definition uses a system of generators and relations which is in a sense a deformation of the system of generators and relations that defines the enveloping algebras of semisimple Lie algebras considered by J. P. Serre [15] in 1966 and known since then as Serre's relations. Serre's relations consist of two parts, the first part interrelating the three types of generators and thereby leading to the triangular decomposition, the second, more important one being relations between generators of one type. In 1993, G. Lusztig gave a construction of the deformed enveloping algebras that did not use the second part of Serre's relations [4]. Lusztig's approach was interpreted by P. Schauenburg as a kind of symmetrization process in which the braid group replaces the symmetric group [13]. In this paper, we give a construction of deformed enveloping algebras without referring to generators and relations at all.

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The paper is organized as follows: In Section 2, we recall the notion of a Yetter-Drinfel'd bialgebra and review some of their elementary properties that will be needed in the sequel. In Section 3, we carry out the first construction which leads to a Hopf algebra which has a two-sided cosmash product as coalgebra structure. We show that many Hopf algebras with triangular decomposition are of this form. As special cases, we obtain Radford's biproduct and Majid's double crossproduct. In Section 4, we carry out the second construction which applies to a pair of Yetter-Drinfel'd Hopf algebras which are in a sense dual to each other. In Section 5, we explain how Lusztig's algebra $'\mathbf{f}$ which corresponds to the nilpotent part of a semisimple Lie algebra is a Yetter-Drinfel'd Hopf algebra and how the second construction can be used to construct deformed enveloping algebras.

2. Yetter-Drinfel'd modules

2.1. In this preliminary section we recall some very well known facts on Yetter-Drinfel'd modules. Suppose that H is a bialgebra over a field K with comultiplication Δ_H and counit ϵ_H. We use the following Sweedler notation: $\Delta_H(h) = h_1 \otimes h_2$. Recall the notion of a left Yetter-Drinfel'd module (cf. [17], [7, Definition 10.6.10]): This is a left H-comodule V which is also a left H-module such that the following compatibility condition is satisfied:

$$h_1 v^1 \otimes (h_2 \to v^2) = (h_1 \to v)^1 h_2 \otimes (h_1 \to v)^2$$

for all $h \in H$ and $v \in V$. Here we have used the following Sweedler notation for the coaction: $\delta(v) = v^1 \otimes v^2 \in H \otimes V$. The arrow $\to$ denotes the module action.

2.2. We also define right Yetter-Drinfel'd modules, which are the left Yetter-Drinfel'd modules over the opposite and coopposite bialgebra. They are right comodules and right modules that satisfy:

$$(v^1 \leftarrow h_1) \otimes v^2 h_2 = (v \leftarrow h_2)^1 \otimes h_1 (v \leftarrow h_2)^2$$

Of course one can also define left-right and right-left Yetter-Drinfel'd modules, but they are not used in this article.

2.3. The tensor product of two Yetter-Drinfel'd modules becomes again a Yetter-Drinfel'd module if it is endowed with the diagonal module and the codiagonal comodule structure (cf. [7, Example 10.6.14], [12, Theorem 4.2]). The left Yetter-Drinfel'd modules, and also the right ones, therefore constitute a monoidal category (cf. [3]). But these categories also possess pre-braidings, which are in the left case given by

$$\sigma_{V,W} : V \otimes W \longrightarrow W \otimes V$$
$$v \otimes w \mapsto (v^1 \to w) \otimes v^2.$$

The corresponding formula in the right case reads: $\sigma_{V,W}(v \otimes w) = w^1 \otimes (v \leftarrow w^2)$. These mappings are bijective if H is a Hopf algebra with bijective antipode, but we do not assume this.

2.4. Suppose that V is a left Yetter-Drinfel'd module and that W is a right Yetter-Drinfel'd module. We define a Yetter-Drinfel'd form to be a bilinear form

$$V \times W \to K, (v, w) \mapsto \langle v, w \rangle$$

such that the following conditions are satisfied for all $v \in V$, $w \in W$ and $h \in H$:

(1) $\langle h \rightarrow v, w \rangle = \langle v, w \leftarrow h \rangle$
(2) $\langle v, w^1 \rangle w^2 = v^1 \langle v^2, w \rangle$

If V is a finite dimensional left Yetter-Drinfel'd module, then the dual vector space $W := V^*$ is in a unique way a right Yetter-Drinfel'd module such that the natural pairing

$$V \times V^* \to K, (v, f) \mapsto \langle v, f \rangle := f(v)$$

is a Yetter-Drinfel'd form. The comodule structure is in this case given by the formula:

$$\delta_{V^*}(f) = \sum_{i=1}^{n} v^{(i)*} \otimes f({v_{(i)}}^2) {v_{(i)}}^1$$

where $v_{(1)}, \ldots, v_{(n)}$ is a basis of V with dual basis $v^{(1)*}, \ldots, v^{(n)*}$. However, in our main application we consider the infinite dimensional case.

2.5. The transpose of an H-linear and colinear map between finite-dimensional left Yetter-Drinfel'd modules is linear and colinear. If $\langle \cdot, \cdot \rangle_1 : V_1 \times W_1 \to K$ and $\langle \cdot, \cdot \rangle_2 : V_2 \times W_2 \to K$ are Yetter-Drinfel'd forms, then

$$(V_1 \otimes V_2) \times (W_1 \otimes W_2) \to K, (v_1 \otimes v_2, w_1 \otimes w_2) \mapsto \langle v_1, w_1 \rangle_1 \langle v_2, w_2 \rangle_2$$

is also a Yetter-Drinfel'd form. The pre-braidings are mutually adjoint with respect to this bilinear form.

2.6. Since we have the notion of a bialgebra inside a pre-braided monoidal category (cf. [11], [7, p. 203]), it is meaningful to speak of left Yetter-Drinfel'd bialgebras (or Hopf algebras). Suppose that A is a left Yetter-Drinfel'd bialgebra and that B is a right Yetter-Drinfel'd bialgebra. We say that a Yetter-Drinfel'd form is a bialgebra form if the following conditions are satisfied:

(1) $\langle a \otimes a', \Delta_B(b) \rangle = \langle aa', b \rangle$
(2) $\langle a, bb' \rangle = \langle \Delta_A(a), b \otimes b' \rangle$
(3) $\langle 1, b \rangle = \epsilon_B(b)$, $\langle a, 1 \rangle = \epsilon_A(a)$

for all $a, a' \in A$ and all $b, b' \in B$. The bilinear form on the tensor products is defined as in Subsection 2.5. If B is the dual vector space of a finite-dimensional Yetter-Drinfel'd bialgebra A, then the natural pairing considered in Subsection 2.4 is a bialgebra form. If A and B possess antipodes, they are interrelated as follows:

Proposition 2.1. *If A and B are Yetter-Drinfel'd Hopf algebras with antipodes S_A resp. S_B and $\langle \cdot, \cdot \rangle : A \times B \to K$ is a bialgebra form, we have for all $a \in A$ and $b \in B$: $\langle S_A(a), b \rangle = \langle a, S_B(b) \rangle$.*

Proof. This follows from the fact that the mappings $a \otimes b \mapsto \langle S_A(a), b \rangle$ and $a \otimes b \mapsto \langle a, S_B(b) \rangle$ are left resp. right inverses of the mapping $a \otimes b \mapsto \langle a, b \rangle$ inside the convolution algebra $(A \otimes B)^*$, and these two inverses must coincide. □

2.7. We consider next the situation that the bilinear form is degenerate. We consider the left radical $R_A = \{a \in A \mid \forall b \in B : \langle a, b\rangle = 0\}$ and the right radical $R_B = \{b \in B \mid \forall a \in A : \langle a, b\rangle = 0\}$ of the form:

Proposition 2.2. *We have:*

(1) *R_A is an H-submodule and an H-subcomodule.*
(2) *R_A is a two-sided ideal and a two-sided coideal.*

Proof. We only prove the subcomodule-property. Suppose that $a \in R_A$ is nonzero. We write $\delta_A(a) = \sum_{i=1}^{k} h^{(i)} \otimes a^{(i)}$ where δ_A denotes the comodule operation. By choosing k minimal we can assume that the $h^{(i)}$'s and the $a^{(i)}$'s are linearly independent. We have for $b \in B$:

$$\sum_{i=1}^{k} \langle a^{(i)}, b\rangle h^{(i)} = \langle a^2, b\rangle a^1 = \langle a, b^1\rangle b^2 = 0$$

and therefore $\langle a^{(i)}, b\rangle = 0$ for all i. Therefore we have $a^{(i)} \in R_A$. □

Since $\epsilon_A(a) = \langle a, 1\rangle$, the counit vanishes on the radical. It is now clear that $\bar{A} = A/R_A$ is a Yetter-Drinfel'd bialgebra.

Of course, one can show similarly that $\bar{B} = B/R_B$ is a right Yetter-Drinfel'd bialgebra. The induced pairing $\bar{A} \times \bar{B} \to K$, $(\bar{a}, \bar{b}) \mapsto \langle a, b\rangle$ is also a bialgebra form.

2.8. The following lemma is often useful in verifying that a certain bilinear form is in fact a bialgebra form (cf. [4, Proposition 1.2.3]).

Lemma 2.3. *Suppose that A (resp. B) is a left (resp. right) Yetter-Drinfel'd bialgebra. Suppose that $B' \subset B$ generates B as an algebra. We further assume that a bilinear form $\langle \cdot, \cdot \rangle : A \times B \to K$ is given which satisfies axiom (2) in Subsection 2.6 for all $a \in A$ and all $b, b' \in B$. Now suppose that the other axioms (1), (3) of Subsection 2.6 and (1), (2) of Subsection 2.4 are satisfied for all $a, a' \in A$ and all $h \in H$, but only for all $b \in B'$. Then the bilinear form is a bialgebra form.*

Proof. Since these verifications are rather similar, we only show 2.6 (1). (However, 2.4 (1) and 2.4 (2) must be shown first.) Since among the assumptions we have in 2.6 (3) that $\langle a, 1\rangle = \epsilon_A(a)$, this holds if $b = 1$. If 2.6 (1) holds for $b, b' \in B$, it also holds for bb':

$$\begin{aligned}
\langle a \otimes a', \Delta_B(bb')\rangle &= \langle a \otimes a', b_1 {b'_1}^1 \otimes (b_2 \leftarrow {b'_1}^2) b'_2\rangle \\
&= \langle a, b_1 {b'_1}^1\rangle \langle a', (b_2 \leftarrow {b'_1}^2) b'_2\rangle \\
&= \langle a_1, b_1\rangle \langle a_2, {b'_1}^1\rangle \langle a'_1, b_2 \leftarrow {b'_1}^2\rangle \langle a'_2, b'_2\rangle \\
&= \langle a_1, b_1\rangle \langle {a_2}^2, b'_1\rangle \langle {a_2}^1 \to a'_1, b_2\rangle \langle a'_2, b'_2\rangle \\
&= \langle a_1({a_2}^1 \to a'_1), b\rangle \langle {a_2}^2 a'_2, b'\rangle \\
&= \langle \Delta_A(aa'), b \otimes b'\rangle = \langle aa', bb'\rangle.
\end{aligned}$$

Here, the equality of the third and fourth lines follows from 2.4 (1) and 2.4 (2). □

2.9. We have already noted in Subsection 2.2 the correspondence between left and right Yetter-Drinfel'd modules. This implies the following correspondence for Yetter-Drinfel'd bialgebras:

Lemma 2.4. *We have:*

(1) *If A is a left Yetter-Drinfel'd bialgebra over H, then the opposite and coopposite bialgebra $A^{op\ cop}$ is a right Yetter-Drinfel'd bialgebra over $H^{op\ cop}$.*
(2) *If B is a right Yetter-Drinfel'd bialgebra over H, then $B^{op\ cop}$ is a left Yetter-Drinfel'd bialgebra over $H^{op\ cop}$.*

The proof is omitted.

3. The first construction

3.1. In this section, A (resp. B) is a fixed left (resp. right) Yetter-Drinfel'd bialgebra over a bialgebra H. Δ_A (resp. Δ_B) and ϵ_A (resp. ϵ_B) denote the comultiplication and the counit. The aim is to investigate under which circumstances the two-sided cosmash product is a bialgebra.

3.2. We first define the two-sided cosmash product.

Proposition 3.1. *$A \otimes H \otimes B$ is a coalgebra by the following comultiplication and counit:*

$$\Delta : A \otimes H \otimes B \to (A \otimes H \otimes B) \otimes (A \otimes H \otimes B)$$
$$a \otimes h \otimes b \mapsto (a_1 \otimes {a_2}^1 h_1 \otimes {b_1}^1) \otimes ({a_2}^2 \otimes h_2 {b_1}^2 \otimes b_2)$$
$$\epsilon : A \otimes H \otimes B \to K$$
$$a \otimes h \otimes b \mapsto \epsilon_A(a)\epsilon_H(h)\epsilon_B(b)$$

This coalgebra structure is called the two-sided cosmash product.

Proof. This follows by direct computation. □

3.3. We now introduce certain structure elements which will be used to turn the two-sided cosmash product into a bialgebra.

Definition 3.2. A pair (A, B) consisting of a left and a right Yetter-Drinfel'd bialgebra together with linear mappings $\rightharpoonup : B \otimes A \to A$, $\leftharpoonup : B \otimes A \to B$ and $\sharp : B \otimes A \to H$ is called a Yetter-Drinfel'd bialgebra pair if:

(a) A is a left B-module via $\rightharpoonup$.
(b) B is a right A-module via $\leftharpoonup$.

and the following compatibility conditions are satisfied:

(1) $\Delta_A(b \rightharpoonup a) = ({b_1}^1 \rightharpoonup a_1) \otimes ({b_1}^2 \to (b_2 \rightharpoonup a_2))$
$\quad\ \Delta_B(b \leftharpoonup a) = ((b_1 \leftharpoonup a_1) \leftarrow {a_2}^1) \otimes (b_2 \leftharpoonup {a_2}^2)$
(2) $\Delta_H(b \sharp a) = ({b_1}^1 \sharp a_1){a_2}^1 \otimes {b_1}^2 (b_2 \sharp {a_2}^2)$
(3) $b \rightharpoonup (aa') = ({b_1}^1 \rightharpoonup a_1)({b_1}^2 (b_2 \sharp a_2){a_3}^1 \to [(b_3 \leftharpoonup {a_3}^2) \rightharpoonup a'])$
$\quad\ (bb') \leftharpoonup a = ([b \leftharpoonup ({b'_1}^1 \rightharpoonup a_1)] \leftarrow {b'_1}^2 (b'_2 \sharp a_2){a_3}^1)(b'_3 \leftharpoonup {a_3}^2)$
(4) $b \sharp (aa') = (b_1 \sharp a_1){a_2}^1 ((b_2 \leftharpoonup {a_2}^2) \sharp a')$
$\quad\ (bb') \sharp a = (b \sharp ({b'_1}^1 \rightharpoonup a_1)){b'_1}^2 (b'_2 \sharp a_2)$
(5) $\epsilon_H(b \sharp a) = \epsilon_A(a)\epsilon_B(b)$
(6) $b \rightharpoonup 1 = \epsilon_B(b)1, \quad 1 \leftharpoonup a = \epsilon_A(a)1$
(7) $b \sharp 1 = \epsilon_B(b)1, \quad 1 \sharp a = \epsilon_A(a)1$

(8) $({b_1}^1 \rightharpoonup a_1)^1 {b_1}^2 (b_2 \sharp a_2) \otimes ({b_1}^1 \rightharpoonup a_1)^2 = ({b_1}^1 \sharp a_1) {a_2}^1 \otimes ({b_1}^2 \rightarrow (b_2 \rightharpoonup {a_2}^2))$
$(b_2 \leftharpoonup {a_2}^2)^1 \otimes (b_1 \sharp a_1) {a_2}^1 (b_2 \leftharpoonup {a_2}^2)^2 = (({b_1}^1 \leftharpoonup a_1) \leftarrow {a_2}^1) \otimes {b_1}^2 (b_2 \sharp {a_2}^2)$

(9) $b \rightharpoonup (h \rightarrow a) = h_1 \rightarrow ((b \leftarrow h_2) \rightharpoonup a)$
$(b \leftarrow h) \leftharpoonup a = (b \leftharpoonup (h_1 \rightarrow a)) \leftarrow h_2$

(10) $(b \sharp (h_1 \rightarrow a)) h_2 = h_1 ((b \leftarrow h_2) \sharp a)$

(11) $(b_1 \rightharpoonup a_1) \otimes (b_2 \leftharpoonup a_2) = ({b_1}^2 \rightarrow (b_2 \rightharpoonup {a_2}^2)) \otimes (({b_1}^1 \leftharpoonup a_1) \leftarrow {a_2}^1)$

These conditions are of course required for all a, $a' \in A$, b, $b' \in B$ and $h \in H$.

3.4. In this situation, we can carry out the first construction:

Theorem 3.3. *Given a Yetter-Drinfel'd bialgebra pair, the two-sided cosmash product $A \otimes H \otimes B$ is a bialgebra with multiplication*

$$\mu : (A \otimes H \otimes B) \otimes (A \otimes H \otimes B) \rightarrow A \otimes H \otimes B$$

$$(a \otimes h \otimes b) \otimes (a' \otimes h' \otimes b') \mapsto$$

$$a(h_1 \rightarrow ({b_1}^1 \rightharpoonup a'_1)) \otimes h_2 {b_1}^2 (b_2 \sharp a'_2) {a'_3}^1 h'_1 \otimes ((b_3 \leftharpoonup {a'_3}^2) \leftarrow h'_2) b'$$

and unit element $1 \otimes 1 \otimes 1$.

This will be proved in Subsections 3.5 and 3.6.

3.5. We first prove that the multiplication is associative:

$$((a \otimes h \otimes b)(a' \otimes h' \otimes b'))(a'' \otimes h'' \otimes b'') =$$
$$a(h_1 \rightarrow ({b_1}^1 \rightharpoonup a'_1))(h_2 {b_1}^2 (b_2 \sharp a'_2)_1 {a'_3}^1 h'_1 \rightarrow ([((b_3 \leftharpoonup {a'_3}^3) \leftarrow h'_3) b']_1{}^1 \rightharpoonup a''_1)) \otimes$$
$$h_3 {b_1}^3 (b_2 \sharp a'_2)_2 {a'_3}^2 h'_2 [((b_3 \leftharpoonup {a'_3}^3) \leftarrow h'_3) b']_1{}^2 ([((b_3 \leftharpoonup {a'_3}^3) \leftarrow h'_3) b']_2 \sharp a''_2) {a''_3}^1 h''_1 \otimes$$
$$(([((b_3 \leftharpoonup {a'_3}^3) \leftarrow h'_3) b']_3 \leftharpoonup {a''_3}^2) \leftarrow h''_2) b''$$

By condition (2) of Definition 3.2, this is equal to

$$a(h_1 \rightarrow ({b_1}^1 \rightharpoonup a'_1))(h_2 {b_1}^2 ({b_2}^1 \sharp a'_2) {a'_3}^1 {a'_4}^1 h'_1 \rightarrow ([((b_4 \leftharpoonup {a'_4}^3) \leftarrow h'_3)_1 {b'_1}^1]^1 \rightharpoonup a''_1)) \otimes$$
$$h_3 {b_1}^3 {b_2}^2 (b_3 \sharp {a'_3}^2) {a'_4}^2 h'_2 [((b_4 \leftharpoonup {a'_4}^3) \leftarrow h'_3)_1 {b'_1}^1]^2$$
$$([(((b_4 \leftharpoonup {a'_4}^3) \leftarrow h'_3)_2 \leftarrow {b'_1}^2) {b'_2}^1] \sharp a''_2) {a''_3}^1 h''_1 \otimes$$
$$(([(((b_4 \leftharpoonup {a'_4}^3) \leftarrow h'_3)_3 \leftarrow {b'_1}^3 {b'_2}^2) b'_3] \leftharpoonup {a''_3}^2) \leftarrow h''_2) b''$$

This is in turn equal to

$$a(h_1 \rightarrow ({b_1}^1 \rightharpoonup a'_1))(h_2 {b_1}^2 ({b_2}^1 \sharp a'_2) {a'_3}^1 {a'_4}^1 h'_1 \rightarrow ([((b_4 \leftharpoonup {a'_4}^3)_1 \leftarrow h'_3)^1 {b'_1}^1] \rightharpoonup a''_1)) \otimes$$
$$h_3 {b_1}^3 {b_2}^2 (b_3 \sharp {a'_3}^2) {a'_4}^2 h'_2 [((b_4 \leftharpoonup {a'_4}^3)_1 \leftarrow h'_3)^2 {b'_1}^2]$$
$$([((b_4 \leftharpoonup {a'_4}^3)_2 \leftarrow h'_4 {b'_1}^3) {b'_2}^1] \sharp a''_2) {a''_3}^1 h''_1 \otimes$$
$$(([((b_4 \leftharpoonup {a'_4}^3)_3 \leftarrow h'_5 {b'_1}^4 {b'_2}^2) b'_3] \leftharpoonup {a''_3}^2) \leftarrow h''_2) b''$$

By condition (1) of Definition 3.2, this is equal to

$$a(h_1 \rightarrow ({b_1}^1 \rightharpoonup a'_1))$$
$$(h_2 {b_1}^2 ({b_2}^1 \sharp a'_2) {a'_3}^1 {a'_4}^1 h'_1 \rightarrow ([((b_4 \leftharpoonup {{a'_4}^3}_1) \leftarrow {{{a'_4}^3}_2}^1 {{{a'_4}^3}_3}^1 h'_3)^1 {b'_1}^1] \rightharpoonup a''_1)) \otimes$$
$$h_3 {b_1}^3 {b_2}^2 (b_3 \sharp {a'_3}^2) {a'_4}^2 h'_2 [((b_4 \leftharpoonup {{a'_4}^3}_1) \leftarrow {{{a'_4}^3}_2}^1 {{{a'_4}^3}_3}^1 h'_3)^2 {b'_1}^2]$$
$$([((b_5 \leftharpoonup {{{a'_4}^3}_2}^2) \leftarrow {{{a'_4}^3}_3}^2 h'_4 {b'_1}^3) {b'_2}^1] \sharp a''_2) {a''_3}^1 h''_1 \otimes$$

$$(([((b_6 \leftharpoondown a_4'{}^3{}_3{}^3) \leftarrow h_5' b_1'{}^4 b_2'{}^2) b_3'] \leftharpoondown a_3''{}^2) \leftarrow h_2'') b''$$

This equals

$$\begin{gathered} a(h_1 \to ({b_1}^1 \rightharpoonup a_1')) \\ (h_2 {b_1}^2 ({b_2}^1 \sharp a_2') a_3'{}^1 a_4'{}^1 a_5'{}^1 a_6'{}^1 h_1' \to ([((b_4 \leftharpoondown a_4'{}^3) \leftarrow a_5'{}^3 a_6'{}^3 h_3')^1 b_1'{}^1] \rightharpoonup a_1'')) \otimes \\ h_3 {b_1}^3 {b_2}^2 (b_3 \sharp a_3'{}^2) a_4'{}^2 a_5'{}^2 a_6'{}^2 h_2' [((b_4 \leftharpoondown a_4'{}^3) \leftarrow a_5'{}^3 a_6'{}^3 h_3')^2 b_1'{}^2] \\ ([((b_5 \leftharpoondown a_5'{}^4) \leftarrow a_6'{}^4 h_4' b_1'{}^3) b_2'{}^1] \sharp a_2'') a_3''{}^1 h_1'' \otimes \\ (([((b_6 \leftharpoondown a_6'{}^5) \leftarrow h_5' b_1'{}^4 b_2'{}^2) b_3'] \leftharpoondown a_3''{}^2) \leftarrow h_2'') b'' \end{gathered}$$

By the Yetter-Drinfel'd condition in Subsection 2.2, this is

$$\begin{gathered} a(h_1 \to ({b_1}^1 \rightharpoonup a_1')) \\ (h_2 {b_1}^2 ({b_2}^1 \sharp a_2') a_3'{}^1 a_4'{}^1 a_5'{}^1 a_6'{}^1 h_1' \to ([((b_4 \leftharpoondown a_4'{}^3)^1 \leftarrow a_5'{}^2 a_6'{}^2 h_2') b_1'{}^1] \rightharpoonup a_1'')) \otimes \\ h_3 {b_1}^3 {b_2}^2 (b_3 \sharp a_3'{}^2) a_4'{}^2 (b_4 \leftharpoondown a_4'{}^3)^2 a_5'{}^3 a_6'{}^3 h_3' b_1'{}^2 \\ ([((b_5 \leftharpoondown a_5'{}^4) \leftarrow a_6'{}^4 h_4' b_1'{}^3) b_2'{}^1] \sharp a_2'') a_3''{}^1 h_1'' \otimes \\ (([((b_6 \leftharpoondown a_6'{}^5) \leftarrow h_5' b_1'{}^4 b_2'{}^2) b_3'] \leftharpoondown a_3''{}^2) \leftarrow h_2'') b'' \end{gathered}$$

And this equals

$$\begin{gathered} a(h_1 \to ({b_1}^1 \rightharpoonup a_1')) \\ (h_2 {b_1}^2 ({b_2}^1 \sharp a_2') a_3'{}^1 a_4'{}^1 a_5'{}^1 h_1' \to ([((b_4 \leftharpoondown a_3'{}^2{}_2{}^2)^1 \leftarrow a_4'{}^2 a_5'{}^2 h_2') b_1'{}^1] \rightharpoonup a_1'')) \otimes \\ h_3 {b_1}^3 {b_2}^2 (b_3 \sharp a_3'{}^2{}_1) a_3'{}^2{}_2{}^1 (b_4 \leftharpoondown a_3'{}^2{}_2{}^2)^2 a_4'{}^3 a_5'{}^3 h_3' b_1'{}^2 \\ ([((b_5 \leftharpoondown a_4'{}^4) \leftarrow a_5'{}^4 h_4' b_1'{}^3) b_2'{}^1] \sharp a_2'') a_3''{}^1 h_1'' \otimes \\ (([((b_6 \leftharpoondown a_5'{}^5) \leftarrow h_5' b_1'{}^4 b_2'{}^2) b_3'] \leftharpoondown a_3''{}^2) \leftarrow h_2'') b'' \end{gathered}$$

By condition (8) of Definition 3.2, this is

$$\begin{gathered} a(h_1 \to ({b_1}^1 \rightharpoonup a_1')) \\ (h_2 {b_1}^2 ({b_2}^1 \sharp a_2') a_3'{}^1 a_4'{}^1 a_5'{}^1 a_6'{}^1 h_1' \to ([(({b_3}^1 \leftharpoondown a_3'{}^2) \leftarrow a_4'{}^2 a_5'{}^2 a_6'{}^2 h_2') b_1'{}^1] \rightharpoonup a_1'')) \otimes \\ h_3 {b_1}^3 {b_2}^2 {b_3}^2 (b_4 \sharp a_4'{}^3) a_5'{}^3 a_6'{}^3 h_3' b_1'{}^2 ([((b_5 \leftharpoondown a_5'{}^4) \leftarrow a_6'{}^4 h_4' b_1'{}^3) b_2'{}^1] \sharp a_2'') a_3''{}^1 h_1'' \otimes \\ (([((b_6 \leftharpoondown a_6'{}^5) \leftarrow h_5' b_1'{}^4 b_2'{}^2) b_3'] \leftharpoondown a_3''{}^2) \leftarrow h_2'') b'' \end{gathered}$$

By condition (9) of Definition 3.2, this gives

$$\begin{gathered} a(h_1 \to ({b_1}^1 \rightharpoonup a_1')) \\ (h_2 {b_1}^2 ({b_2}^1 \sharp a_2') a_3'{}^1 \to [({b_3}^1 \leftharpoondown a_3'{}^2) \rightharpoonup ((a_4'{}^1 a_5'{}^1 a_6'{}^1 h_1') \to (b_1'{}^1 \rightharpoonup a_1''))]) \otimes \\ h_3 {b_1}^3 {b_2}^2 {b_3}^2 (b_4 \sharp a_4'{}^2) a_5'{}^2 a_6'{}^2 h_2' b_1'{}^2 ([((b_5 \leftharpoondown a_5'{}^3) \leftarrow a_6'{}^3 h_3' b_1'{}^3) b_2'{}^1] \sharp a_2'') a_3''{}^1 h_1'' \otimes \\ (([((b_6 \leftharpoondown a_6'{}^4) \leftarrow h_4' b_1'{}^4 b_2'{}^2) b_3'] \leftharpoondown a_3''{}^2) \leftarrow h_2'') b'' \end{gathered}$$

By condition (4) of Definition 3.2, this is

$$\begin{gathered} a(h_1 \to ({b_1}^1 \rightharpoonup a_1')) \\ (h_2 {b_1}^2 ({b_2}^1 \sharp a_2') a_3'{}^1 \to [({b_3}^1 \leftharpoondown a_3'{}^2) \rightharpoonup ((a_4'{}^1 a_5'{}^1 a_6'{}^1 h_1') \to (b_1'{}^1 \rightharpoonup a_1''))]) \otimes \end{gathered}$$

$$h_3 {b_1}^{3} {b_2}^{2} {b_3}^{2} (b_4 \sharp {a'_4}^{2}) {a'_5}^{2} {a'_6}^{2} h'_2 {b'_1}^{2}$$
$$([(b_5 \leftharpoonup {a'_5}^{3}) \leftarrow {a'_6}^{3} h'_3 {b'_1}^{3}] \sharp ({{b'_2}^{1}}_{1}{}^{1} \rightharpoonup a''_2)) {{b'_2}^{1}}_{1}{}^{2} ({b'_2}^{1}{}_{2} \sharp a''_3) {a''_4}^{1} h''_1 \otimes$$
$$((([((b_6 \leftharpoonup {a'_6}^{4}) \leftarrow h'_4 {b'_1}^{4} {b'_2}^{2}) b'_3] \leftharpoonup {a''_4}^{2}) \leftarrow h''_2) b''$$

By condition (3) of Definition 3.2, this is

$$a(h_1 \rightarrow ({b_1}^{1} \rightharpoonup a'_1))$$
$$(h_2 {b_1}^{2} ({b_2}^{1} \sharp a'_2) {a'_3}^{1} \rightarrow [({b_3}^{1} \leftharpoonup {a'_3}^{2}) \rightharpoonup (({a'_4}^{1} {a'_5}^{1} {a'_6}^{1} h'_1) \rightarrow ({b'_1}^{1} \rightharpoonup a''_1))]) \otimes$$
$$h_3 {b_1}^{3} {b_2}^{2} {b_3}^{2} (b_4 \sharp {a'_4}^{2}) {a'_5}^{2} {a'_6}^{2} h'_2 {b'_1}^{2}$$
$$([(b_5 \leftharpoonup {a'_5}^{3}) \leftarrow {a'_6}^{3} h'_3 {b'_1}^{3}] \sharp ({{b'_2}^{1}}_{1}{}^{1} \rightharpoonup a''_2)) {{b'_2}^{1}}_{1}{}^{2} ({b'_2}^{1}{}_{2} \sharp a''_3) {a''_4}^{1} h''_1 \otimes$$
$$(((([((b_6 \leftharpoonup {a'_6}^{4}) \leftarrow h'_4 {b'_1}^{4} {b'_2}^{2}) \leftharpoonup ({b'_3}^{1} \rightharpoonup {a''_4}^{2}{}_{1})]$$
$$\leftarrow {b'_3}^{2} (b'_4 \sharp {a''_4}^{2}{}_{2}) {{a''_4}^{2}}_{3}{}^{1}) (b'_5 \leftharpoonup {{a''_4}^{2}}_{3}{}^{2})) \leftarrow h''_2) b''$$

And this equals

$$a(h_1 \rightarrow ({b_1}^{1} \rightharpoonup a'_1))$$
$$(h_2 {b_1}^{2} ({b_2}^{1} \sharp a'_2) {a'_3}^{1} \rightarrow [({b_3}^{1} \leftharpoonup {a'_3}^{2}) \rightharpoonup (({a'_4}^{1} {a'_5}^{1} {a'_6}^{1} h'_1) \rightarrow ({b'_1}^{1} \rightharpoonup a''_1))]) \otimes$$
$$h_3 {b_1}^{3} {b_2}^{2} {b_3}^{2} (b_4 \sharp {a'_4}^{2}) {a'_5}^{2} {a'_6}^{2} h'_2 {b'_1}^{2}$$
$$([(b_5 \leftharpoonup {a'_5}^{3}) \leftarrow {a'_6}^{3} h'_3 {b'_1}^{3}] \sharp ({b'_2}^{1} \rightharpoonup a''_2)) {b'_2}^{2} ({b'_3}^{1} \sharp a''_3) {a''_4}^{1} {a''_5}^{1} {a''_6}^{1} h''_1 \otimes$$
$$([((b_6 \leftharpoonup {a'_6}^{4}) \leftarrow h'_4 {b'_1}^{4} {b'_2}^{3} {b'_3}^{2}) \leftharpoonup ({b'_4}^{1} \rightharpoonup {a''_4}^{2})]$$
$$\leftarrow {b'_4}^{2} (b'_5 \sharp {a''_5}^{2}) {a''_6}^{2} h''_2) ((b'_6 \leftharpoonup {a''_6}^{3}) \leftarrow h''_3) b''$$

Reading the formulas in this calculation backwards, interchanging a's and b's, interchanging unprimed and doubleprimed symbols and turning around the numeration of the indices — the type of duality discussed in Subsection 2.9 — one can show that:

$$(a \otimes h \otimes b)((a' \otimes h' \otimes b')(a'' \otimes h'' \otimes b'')) =$$
$$a(h_1 \rightarrow ({b_1}^{1} \rightharpoonup a'_1))$$
$$(h_2 {b_1}^{2} ({b_2}^{1} \sharp a'_2) {a'_3}^{1} \rightarrow [({b_3}^{1} \leftharpoonup {a'_3}^{2}) \rightharpoonup (({a'_4}^{1} {a'_5}^{1} {a'_6}^{1} h'_1) \rightarrow ({b'_1}^{1} \rightharpoonup a''_1))]) \otimes$$
$$h_3 {b_1}^{3} {b_2}^{2} {b_3}^{2} (b_4 \sharp {a'_4}^{2}) {a'_5}^{2} ((b_5 \leftharpoonup {a'_5}^{3}) \sharp [{a'_6}^{2} h'_2 {b'_1}^{2} \rightarrow ({b'_2}^{1} \rightharpoonup a''_2)])$$
$${a'_6}^{3} h'_3 {b'_1}^{3} {b'_2}^{2} ({b'_3}^{1} \sharp a''_3) {a''_4}^{1} {a''_5}^{1} {a''_6}^{1} h''_1 \otimes$$
$$([((b_6 \leftharpoonup {a'_6}^{4}) \leftarrow h'_4 {b'_1}^{4} {b'_2}^{3} {b'_3}^{2}) \leftharpoonup ({b'_4}^{1} \rightharpoonup {a''_4}^{2})]$$
$$\leftarrow {b'_4}^{2} (b'_5 \sharp {a''_5}^{2}) {a''_6}^{2} h''_2) ((b'_6 \leftharpoonup {a''_6}^{3}) \leftarrow h''_3) b''$$

By condition (10) of Definition 3.2, this expression equals the last term in the above calculation.

3.6. We show next that the comultiplication is multiplicative. We have:

$$\Delta((a \otimes h \otimes b)(a' \otimes h' \otimes b')) =$$
$$[(a(h_1 \rightarrow ({b_1}^{1} \rightharpoonup a'_1)))_1 \otimes (a(h_1 \rightarrow ({b_1}^{1} \rightharpoonup a'_1)))_2{}^{1} h_2 {b_1}^{2} (b_2 \sharp a'_2)_1 {a'_3}^{1} h'_1 \otimes$$
$$(((b_3 \leftharpoonup {a'_3}^{3}) \leftarrow h'_3) b')_1{}^{1}] \otimes$$
$$[(a(h_1 \rightarrow ({b_1}^{1} \rightharpoonup a'_1)))_2{}^{2} \otimes h_3 {b_1}^{3} (b_2 \sharp a'_2)_2 {a'_3}^{2} h'_2 (((b_3 \leftharpoonup {a'_3}^{3}) \leftarrow h'_3) b')_1{}^{2} \otimes$$

$$(((b_3 \leftharpoondown {a'_3}^3) \leftarrow h'_3)b')_2]$$

This equals

$$[a_1({a_2}^1h_1 \rightarrow ({b_1}^1 \rightharpoonup a'_1)_1) \otimes ({a_2}^2(h_2 \rightarrow ({b_1}^1 \rightharpoonup a'_1)_2))^1 h_3 {b_1}^2({b_2}^1 \sharp a'_2){a'_3}^1{a'_4}^1h'_1 \otimes$$
$$(((b_4 \leftharpoondown {a'_4}^3)_1 \leftarrow h'_3){b'_1}^1)^1] \otimes$$
$$[({a_2}^2(h_2 \rightarrow ({b_1}^1 \rightharpoonup a'_1)_2))^2 \otimes h_4{b_1}^3{b_2}^2(b_3\sharp{a'_3}^2){a'_4}^2h'_2(((b_4 \leftharpoondown {a'_4}^3)_1 \leftarrow h'_3){b'_1}^1)^2 \otimes$$
$$((b_4 \leftharpoondown {a'_4}^3)_2 \leftarrow h'_4{b'_1}^2)b'_2]$$

By the conditions (1) and (2) in Definition 3.2, this is

$$[a_1({a_2}^1h_1 \rightarrow ({b_1}^1 \rightharpoonup a'_1)) \otimes {a_2}^2(h_2{b_1}^2 \rightarrow ({b_2}^1 \rightharpoonup a'_2))^1 h_3{b_1}^3{b_2}^2({b_3}^1\sharp a'_3){a'_4}^1{a'_5}^1{a'_6}^1h'_1 \otimes$$
$$((b_5 \leftharpoondown {a'_5}^3) \leftarrow {a'_6}^3h'_3)^1{b'_1}^1] \otimes$$
$$[{a_2}^3(h_2{b_1}^2 \rightarrow ({b_2}^1 \rightharpoonup a'_2))^2 \otimes h_4{b_1}^4{b_2}^3{b_3}^2(b_4\sharp{a'_4}^2){a'_5}^2{a'_6}^2h'_2((b_5 \leftharpoondown {a'_5}^3) \leftarrow {a'_6}^3h'_3)^2{b'_1}^2 \otimes$$
$$((b_6 \leftharpoondown {a'_6}^4) \leftarrow h'_4{b'_1}^3)b'_2]$$

By the Yetter-Drinfel'd conditions in Subsections 2.1 and 2.2, this is

$$[a_1({a_2}^1h_1 \rightarrow ({b_1}^1 \rightharpoonup a'_1)) \otimes {a_2}^2h_2{b_1}^2({b_2}^1 \rightharpoonup a'_2)^1{b_2}^2({b_3}^1\sharp a'_3){a'_4}^1{a'_5}^1{a'_6}^1h'_1 \otimes$$
$$((b_5 \leftharpoondown {a'_5}^3)^1 \leftarrow {a'_6}^2h'_2){b'_1}^1] \otimes$$
$$[{a_2}^3(h_3{b_1}^3 \rightarrow ({b_2}^1 \rightharpoonup a'_2)^2) \otimes h_4{b_1}^4{b_2}^3{b_3}^2(b_4\sharp{a'_4}^2){a'_5}^2(b_5 \leftharpoondown {a'_5}^3)^2{a'_6}^3h'_3{b'_1}^2 \otimes$$
$$((b_6 \leftharpoondown {a'_6}^4) \leftarrow h'_4{b'_1}^3)b'_2]$$

By condition (8) in Definition 3.2, this gives

$$[a_1({a_2}^1h_1 \rightarrow ({b_1}^1 \rightharpoonup a'_1)) \otimes {a_2}^2h_2{b_1}^2({b_2}^1\sharp a'_2){a'_3}^1{a'_4}^1{a'_5}^1{a'_6}^1h'_1 \otimes$$
$$(({b_4}^1 \leftharpoondown {a'_4}^2) \leftarrow {a'_5}^2{a'_6}^2h'_2){b'_1}^1] \otimes$$
$$[{a_2}^3(h_3{b_1}^3{b_2}^2 \rightarrow ({b_3}^1 \rightharpoonup {a'_3}^2)) \otimes h_4{b_1}^4{b_2}^3{b_3}^2{b_4}^2(b_5\sharp{a'_5}^3){a'_6}^3h'_3{b'_1}^2 \otimes$$
$$((b_6 \leftharpoondown {a'_6}^4) \leftarrow h'_4{b'_1}^3)b'_2]$$

We now calculate the other side of the equation:

$$\Delta(a \otimes h \otimes b)\Delta(a' \otimes h' \otimes b') =$$
$$[a_1({a_2}^1h_1 \rightarrow ({b_1}^1 \rightharpoonup a'_1)) \otimes {a_2}^2h_2{b_1}^2({b_2}^1\sharp a'_2){a'_3}^1{a'_4}^1{a'_5}^1{a'_6}^1h'_1 \otimes$$
$$(({b_3}^1 \leftharpoondown {a'_3}^2) \leftarrow {a'_4}^2{a'_5}^2{a'_6}^2h'_2){b'_1}^1] \otimes$$
$$[{a_2}^3(h_3{b_1}^3{b_2}^2{b_3}^2 \rightarrow ({b_4}^1 \rightharpoonup {a'_4}^3)) \otimes h_4{b_1}^4{b_2}^3{b_3}^3{b_4}^2(b_5\sharp{a'_5}^3){a'_6}^3h'_3{b'_1}^2 \otimes$$
$$((b_6 \leftharpoondown {a'_6}^4) \leftarrow h'_4{b'_1}^3)b'_2]$$

Both expressions are equal by condition (11) in Definition 3.2. The other bialgebra-axioms are easily verified. Observe that from the conditions (4) and (5) in Definition 3.2 we have:

$$\epsilon_A(b \rightharpoonup a) = \epsilon_B(b)\epsilon_A(a) = \epsilon_B(b \leftharpoondown a).$$

3.7. We omit the proof of the following proposition.

Proposition 3.4. *If A and B are Yetter-Drinfel'd Hopf algebras with antipodes S_A and S_B over the Hopf algebra H with antipode S_H, then $A \otimes H \otimes B$ is a Hopf algebra with antipode:*

$$S(a \otimes h \otimes b) = (1 \otimes 1 \otimes S_B(b^1))(1 \otimes S_H(a^1 h b^2) \otimes 1)(S_A(a^2) \otimes 1 \otimes 1)$$

3.8. This construction includes two constructions as special cases that have been considered earlier. The first one is Radford's biproduct (cf. [8], [7, Theorem 10.6.5]): Set $B = K$, the base field, regarded as a trivial Yetter-Drinfel'd module over H and as a trivial A-module via ϵ_A. By conditon (7) in Definition 3.2, $\sharp$ is forced to be: $1 \sharp a = \epsilon_A(a)1$. The compatibility conditions in Definition 3.2 are then satisfied. We identify $A \otimes H \otimes K$ with $A \otimes H$ and get a bialgebra structure on $A \otimes H$ with multiplication:

$$(a \otimes h)(a' \otimes h') = a(h_1 \rightarrow a') \otimes h_2 h'$$

and comultiplication

$$\Delta(a \otimes h) = (a_1 \otimes {a_2}^1 h_1) \otimes ({a_2}^2 \otimes h_2)$$

Of course, one can also set $A = K$ and obtain a bialgebra structure on $H \otimes B$ such that:

$$(h \otimes b)(h' \otimes b') = h h'_1 \otimes (b \leftarrow h'_2) b'$$
$$\Delta(h \otimes b) = (h_1 \otimes {b_1}^1) \otimes (h_2 {b_1}^2 \otimes b_2)$$

3.9. As a second special case, we set $H = K$. In this case Yetter-Drinfel'd bialgebras are ordinary bialgebras. As in Subsection 3.3, we assume that A is a left B-module and that B is a right A-module. We set: $b \sharp a = \epsilon_A(a)\epsilon_B(b)$. In this situation, the compatibility conditions (2), (5), (7), (8), (9) and (10) in Definition 3.2 are automatically satisfied. The remaining conditions (1), (3), (4), (6) and (11) take the following form:

(1) $\Delta_A(b \rightharpoonup a) = (b_1 \rightharpoonup a_1) \otimes (b_2 \rightharpoonup a_2)$
$\Delta_B(b \leftharpoonup a) = (b_1 \leftharpoonup a_1) \otimes (b_2 \leftharpoonup a_2)$
(2) $b \rightharpoonup (aa') = (b_1 \rightharpoonup a_1)((b_2 \leftharpoonup a_2) \rightharpoonup a')$
$(bb') \leftharpoonup a = (b \leftharpoonup (b'_1 \rightharpoonup a_1))(b'_2 \leftharpoonup a_2)$
(3) $\epsilon_A(b \rightharpoonup a) = \epsilon_B(b)\epsilon_A(a) = \epsilon_B(b \leftharpoonup a)$
(4) $b \rightharpoonup 1 = \epsilon_B(b)1, \quad 1 \leftharpoonup a = \epsilon_A(a)1$
(5) $(b_1 \rightharpoonup a_1) \otimes (b_2 \leftharpoonup a_2) = (b_2 \rightharpoonup a_2) \otimes (b_1 \leftharpoonup a_1)$

If these conditions are satisfied, we identify $A \otimes K \otimes B$ with $A \otimes B$ and get a bialgebra structure on $A \otimes B$ with multiplication:

$$(a \otimes b)(a' \otimes b') = a(b_1 \rightharpoonup a'_1) \otimes (b_2 \leftharpoonup a'_2) b'$$

and comultiplication:

$$\Delta(a \otimes b) = (a_1 \otimes b_1) \otimes (a_2 \otimes b_2)$$

This is Majid's double crossproduct ([5], cf. also [9]).

3.10. We show next that many bialgebras that admit a triangular decomposition are of the form given in the first construction:

Theorem 3.5. *Suppose that A and B are left (resp. right) Yetter-Drinfel'd bialgebras over the bialgebra H. Suppose that $A \otimes H \otimes B$ is a bialgebra in such a way that:*

(1) *The mappings*

$$A \otimes H \to A \otimes H \otimes B, a \otimes h \mapsto a \otimes h \otimes 1$$
$$H \otimes B \to A \otimes H \otimes B, h \otimes b \mapsto 1 \otimes h \otimes b$$

are bialgebra maps from the biproducts (cf. Subsection 3.8) to $A \otimes H \otimes B$.

(2) *For all $a \in A, h \in H$ and $b \in B$ we have:*

$$a \otimes h \otimes b = (a \otimes 1 \otimes 1)(1 \otimes h \otimes 1)(1 \otimes 1 \otimes b)$$

Then $A \otimes H \otimes B$ is a two-sided cosmash product as a coalgebra and there exist a left B-module structure on A, a right A-module structure on B and a mapping $\sharp : B \otimes A \to H$ such that A and B form a Yetter-Drinfel'd bialgebra pair and the multiplication is given as in Theorem 3.3.

Proof. It is obvious that we have $(a \otimes h \otimes 1)(1 \otimes h' \otimes b) = (a \otimes hh' \otimes b)$ for all $a \in A,\ h, h' \in H$ and $b \in B$. We first derive the comultiplication:

$$\begin{aligned}\Delta(a \otimes h \otimes b) &= \Delta(a \otimes h \otimes 1)\Delta(1 \otimes 1 \otimes b)\\ &= (a_1 \otimes {a_2}^1 h_1 \otimes 1)(1 \otimes 1 \otimes {b_1}^1) \otimes ({a_2}^2 \otimes h_2 \otimes 1)(1 \otimes {b_1}^2 \otimes b_2)\\ &= (a_1 \otimes {a_2}^1 h_1 \otimes {b_1}^1) \otimes ({a_2}^2 \otimes h_2 {b_1}^2 \otimes b_2)\end{aligned}$$

We now define the following projections:

$$\begin{aligned}p_A &: A \otimes H \otimes B \to A, \quad a \otimes h \otimes b \mapsto a\epsilon_H(h)\epsilon_B(b)\\ p_H &: A \otimes H \otimes B \to H, \quad a \otimes h \otimes b \mapsto \epsilon_A(a)h\epsilon_B(b)\\ p_B &: A \otimes H \otimes B \to B, \quad a \otimes h \otimes b \mapsto \epsilon_A(a)\epsilon_H(h)b\end{aligned}$$

and use them to define:

$$\begin{aligned}b \rightharpoonup a &= p_A((1 \otimes 1 \otimes b)(a \otimes 1 \otimes 1))\\ b \sharp a &= p_H((1 \otimes 1 \otimes b)(a \otimes 1 \otimes 1))\\ b \leftharpoonup a &= p_A((1 \otimes 1 \otimes b)(a \otimes 1 \otimes 1))\end{aligned}$$

We now prove: $p_A((1 \otimes 1 \otimes b)(a \otimes 1 \otimes 1)(1 \otimes h \otimes 1)) = \epsilon_H(h)(b \rightharpoonup a)$. Write $(1 \otimes 1 \otimes b)(a \otimes 1 \otimes 1) = \sum_{i=1}^{n} a_{(i)} \otimes h_{(i)} \otimes b_{(i)}$. We have:

$$\begin{aligned}
&p_A((1 \otimes 1 \otimes b)(a \otimes 1 \otimes 1)(1 \otimes h \otimes 1)) \\
&\qquad = p_A(\sum_{i=1}^{n} (a_{(i)} \otimes 1 \otimes 1)(1 \otimes h_{(i)} \otimes b_{(i)})(1 \otimes h \otimes 1)) \\
&\qquad = p_A(\sum_{i=1}^{n} (a_{(i)} \otimes 1 \otimes 1)(1 \otimes h_{(i)} h_1 \otimes (b_{(i)} \leftharpoonup h_2))) \\
&\qquad = \sum_{i=1}^{n} a_{(i)} \epsilon_H(h_{(i)}) \epsilon_H(h) \epsilon_B(b_{(i)}) \\
&\qquad = \epsilon_H(h)(b \rightharpoonup a)
\end{aligned}$$

Similarly, one can show that:

$$\begin{aligned}
p_B((1 \otimes h \otimes 1)(1 \otimes 1 \otimes b)(a \otimes 1 \otimes 1)) &= \epsilon_H(h)(b \leftharpoonup a) \\
p_H((1 \otimes h \otimes b)(a \otimes h' \otimes 1)) &= h(b \sharp a) h'
\end{aligned}$$

Since $A \otimes H \otimes B$ is a coalgebra, $(A \otimes H \otimes B)^*$ is an algebra. It is easy to derive from the form of the comultiplication the formula:

$$(a^* \otimes h^* h'^* \otimes b^*) = (a^* \otimes h^* \otimes \epsilon_B)(\epsilon_A \otimes h'^* \otimes b^*)$$

for all $a^* \in A^*, h^*, h'^* \in H^*$ and $b^* \in B^*$. We use this to derive the form of the multiplication:

$$\begin{aligned}
&\langle a^* \otimes h^* \otimes b^*, (1 \otimes 1 \otimes b)(a \otimes 1 \otimes 1) \rangle \\
&\quad = \langle (a^* \otimes h^* \otimes \epsilon_B) \otimes (\epsilon_A \otimes \epsilon_H \otimes b^*), \Delta((1 \otimes 1 \otimes b)(a \otimes 1 \otimes 1)) \rangle \\
&\quad = \langle (a^* \otimes h^* \otimes \epsilon_B) \otimes (\epsilon_A \otimes \epsilon_H \otimes b^*), \Delta(1 \otimes 1 \otimes b) \Delta(a \otimes 1 \otimes 1) \rangle \\
&\quad = \langle a^* \otimes h^* \otimes \epsilon_B, (1 \otimes 1 \otimes {b_1}^1)(a_1 \otimes {a_2}^1 \otimes 1) \rangle \\
&\qquad\quad \langle \epsilon_A \otimes \epsilon_H \otimes b^*, (1 \otimes {b_1}^2 \otimes b_2)({a_2}^2 \otimes 1 \otimes 1) \rangle \\
&\quad = \langle a^* \otimes h^* \otimes \epsilon_B, (1 \otimes 1 \otimes {b_1}^1)(a_1 \otimes {a_2}^1 \otimes 1) \rangle \\
&\qquad\quad b^*(p_B((1 \otimes {b_1}^2 \otimes b_2)({a_2}^2 \otimes 1 \otimes 1))) \\
&\quad = \langle a^* \otimes h^* \otimes \epsilon_B, (1 \otimes 1 \otimes b_1)(a_1 \otimes {a_2}^1 \otimes 1) \rangle b^*(b_2 \leftharpoonup {a_2}^2)
\end{aligned}$$

By applying the same method to the tensorand $a^* \otimes h^* \otimes \epsilon_B = (a^* \otimes \epsilon_H \otimes \epsilon_B)(\epsilon_A \otimes h^* \otimes \epsilon_B)$, we arrive at the formula: $\langle a^* \otimes h^* \otimes b^*, (1 \otimes 1 \otimes b)(a \otimes 1 \otimes 1) \rangle = \langle a^* \otimes h^* \otimes b^*, ({b_1}^1 \rightharpoonup a_1) \otimes {b_1}^2 (b_2 \sharp a_2) {a_3}^1 \otimes (b_3 \leftharpoonup {a_3}^2) \rangle$, which implies that the multiplication is given by the formula in Theorem 3.3.

It remains to show the compatibility conditions in Definition 3.2. They follow by calculating both sides of the associative law resp. the multiplicativity of the comultiplication as in the Subsections 3.5 resp. 3.6 and projecting the resulting equations onto the tensor factors in all possible ways. As an example, we verify the second equation in condition (1). Observe first that p_B is obviously a coalgebra map. Projecting the multiplicativity of the comultiplication onto $B \otimes B$, we obtain:

$$\begin{aligned}\Delta_B(b \leftharpoonup a) &= \Delta_B(p_B((1 \otimes 1 \otimes b)(a \otimes 1 \otimes 1)))\\ &= (p_B \otimes p_B)(\Delta((1 \otimes 1 \otimes b)(a \otimes 1 \otimes 1)))\\ &= (p_B \otimes p_B)(\Delta(1 \otimes 1 \otimes b)\Delta(a \otimes 1 \otimes 1))\\ &= p_B((1 \otimes 1 \otimes {b_1}^1)(a_1 \otimes {a_2}^1 \otimes 1)) \otimes p_B((1 \otimes {b_1}^2 \otimes b_2)({a_2}^2 \otimes 1 \otimes 1))\\ &= ((b_1 \leftharpoonup a_1) \leftarrow {a_2}^1) \otimes (b_2 \leftharpoonup {a_2}^2)\end{aligned}$$

□

4. The second construction

4.1. In this section we apply the first construction to two dual Yetter-Drinfel'd Hopf algebras. In the whole section, we work in the following situation: H is a commutative and cocommutative Hopf algebra. Recall that in this case the antipode of H is an involution and therefore bijective. This implies that the prebraidings in the categories of left and right Yetter-Drinfel'd modules are actually braidings. We assume that A is a left Yetter-Drinfel'd Hopf algebra and that C is a right Yetter-Drinfel'd Hopf algebra. We assume that the antipodes of A and C are bijective. Furthermore, we suppose that a nondegenerate bialgebra form

$$\langle \cdot, \cdot \rangle_A : A \times C \to K$$

in the sense of Subsection 2.6 is given. And we impose the following main assumption on A:

$$\forall a, a' \in A : (a^1 \to a') \otimes a^2 = a'^2 \otimes (a'^1 \to a)$$

This condition says the following: Since H is commutative and cocommutative, left Yetter-Drinfel'd modules and right Yetter-Drinfel'd modules coincide, as noted in Subsection 2.2. However, the corresponding braidings do not coincide. Our main assumption now requires these braidings to coincide on $A \otimes A$, so that A is a left as well as a right Yetter-Drinfel'd Hopf algebra.

4.2. We now modify C in order to obtain a new right Yetter-Drinfel'd Hopf algebra called B in the following way: We set $B = C$ as an algebra and as an H-module. If δ_C and Δ_C denote the cooperation and comultiplication respectively, we define the cooperation and the comultiplication of B by:

$$\delta_B = (id_C \otimes S_H) \circ \delta_C, \Delta_B = \sigma_{C,C}^{-1} \circ \Delta_C$$

where σ is as in Subsection 2.3. We use the indicated Sweedler notation for δ_B whereas we write $\delta_C(c) = c^{(1)} \otimes c^{(2)}$. Similarly, we use the indicated Sweedler notation for Δ_B, not for Δ_C.

4.3. We shall also use the following notation:

$$\mu_A^{opp} = \mu_A \circ \sigma_{A,A}^{-1} \qquad \Delta_A^{copp} = \sigma_{A,A}^{-1} \circ \Delta_A$$
$$\mu_B^{opp} = \mu_B \circ \sigma_{B,B}^{-1} \qquad \Delta_B^{copp} = \sigma_{B,B}^{-1} \circ \Delta_B$$

where μ_A and μ_B denote the multiplication mappings of A and B.

4.4. We list the basic properties of B:

Proposition 4.1. *We have:*

(1) *B is a right Yetter-Drinfel'd bialgebra.*
(2) *B possesses the antipode $S_B = S_C^{-1}$.*
(3) $\langle S_A^{-1}(a), b\rangle_A = \langle a, S_B(b)\rangle_A$
(4) $\langle \sigma_{A,A}^{-1}(a \otimes a'), b \otimes b'\rangle_A = \langle a \otimes a', \sigma_{B,B}(b \otimes b')\rangle_A$*, where the form on the tensor products is defined as in Subsection 2.5.*
(5) $\langle \Delta_A(a), b \otimes b'\rangle_A = \langle a, bb'\rangle_A$
(6) $\langle aa', b\rangle_A = \langle a \otimes a', \Delta_B^{copp}(b)\rangle_A$
(7) $\langle \mu_A^{opp}(a \otimes a'), b\rangle_A = \langle a \otimes a', \Delta_B(b)\rangle_A$

Proof. The main assumption in Subsection 4.1 also implies that the inverses of the braidings agree on $A \otimes A$:

$$\forall a, a' \in A : a'^2 \otimes (S_H(a'^1) \to a) = (S_H(a^1) \to a') \otimes a^2$$

This implies (4) by direct computation. We now prove (1). Since H is commutative and cocommutative, the antipode is a Hopf algebra isomorphism. B is therefore a right Yetter-Drinfel'd module. From the bialgebra axioms, only the coassociativity and the fact that the comultiplication is an algebra homomorphism are not totally obvious. It is a standard fact on bialgebras in categories that if the comultiplication of a bialgebra C is changed to $\sigma_{C,C}^{-1} \circ \Delta_C$, then the resulting object is a bialgebra in the category with the modified braiding

$$\sigma_{W,V}^{-1} : V \otimes W \to W \otimes V$$

Since we have $\sigma_{B,B} = \sigma_{C,C}^{-1}$ by (4) and 2.5, this proves (1). The assertions (5), (6) and (7) are direct consequences of (4) and the definition in Subsection 2.6. Part (2) follows from the skew-antipode equation: $\mu_C(id_C \otimes S_C^{-1})\sigma_{C,C}^{-1}\Delta_C = \eta_C\epsilon_C$. From Proposition 2.1 in Subsection 2.6 and (2) we can directly prove (3). □

4.5. We define a second bilinear form:

$$\langle \cdot, \cdot \rangle_B : A \times B \to K, \quad (a, b) \mapsto \langle a, b\rangle_B := \langle S_A^{-1}(a), b\rangle_A$$

It follows directly from Proposition 4.1 and Subsection 2.6 that this form has properties which are in a sense dual to those of $\langle \cdot, \cdot \rangle_A$:

Proposition 4.2. *We have:*

(1) $\langle \Delta_A^{copp}(a), b \otimes b'\rangle_B = \langle a, bb'\rangle_B$
(2) $\langle \Delta_A(a), b \otimes b'\rangle_B = \langle a, \mu_B^{opp}(b \otimes b')\rangle_B$
(3) $\langle aa', b\rangle_B = \langle a \otimes a', \Delta_B(b)\rangle_B$
(4) $\langle 1, b\rangle_B = \epsilon_B(b), \quad \langle a, 1\rangle_B = \epsilon_A(a)$
(5) $\langle S_A^{-1}(a), b\rangle_B = \langle a, S_B(b)\rangle_B$
(6) $\langle h \to a, b\rangle_B = \langle a, b \leftarrow h\rangle_B$
(7) $\langle a, b\rangle_B = \langle a^2, b^1\rangle_B a^1 b^2$

4.6. We define now the left adjoint action of A on itself. This is the adjoint action in the category of Yetter-Drinfel'd modules using the inverse braiding. It is denoted by $\rightharpoonup$:

$$A \otimes A \to A, \quad a \otimes a' \mapsto (a \rightharpoonup a') := {a_2}^2 a'^2 S_A^{-1}(S_H({a_2}^1 a'^1) \to a_1)$$

This can also be written as:

$$a \rightharpoondown a' = \mu_A(\mu_A \otimes S_A^{-1})\sigma^{-1}_{A\otimes A,A}(\Delta_A \otimes id_A)(a \otimes a')$$

A is a left A-module via the left adjoint action. Similarly, B becomes a right B-module via the right adjoint action:

$$B \otimes B \to B, \quad b' \otimes b \mapsto (b' \leftharpoondown b) := S_B^{-1}(b_2 \leftarrow S_H(b'^2 {b_1}^2))b'^1 {b_1}^1$$

which can also be written as:

$$b' \leftharpoondown b = \mu_B(S_B^{-1} \otimes \mu_B)\sigma^{-1}_{B,B\otimes B}(id_B \otimes \Delta_B)(b' \otimes b)$$

4.7. We define the right coadjoint action of A on B as the action dual to the left adjoint action with respect to the form $\langle \cdot, \cdot \rangle_A$.

$$B \otimes A \to B, \quad b \otimes a \mapsto (b \leftharpoonup a)$$

with: $\langle a', b \leftharpoonup a\rangle_A = \langle a \rightharpoondown a', b\rangle_A$.

The dual action exists since the mappings involved possess adjoints by Proposition 4.1 in Subsection 4.4, and is unique since the bialgebra form is nondegenerate. Similarly, we define the left coadjoint action of B on A as the action dual to the right adjoint action with respect to the form $\langle \cdot, \cdot \rangle_B$.

$$B \otimes A \to A, \quad b \otimes a \mapsto (b \rightharpoonup a)$$

with: $\langle b \rightharpoonup a, b'\rangle_B = \langle a, b' \leftharpoondown b\rangle_B$. It is clear that these actions are module operations.

4.8. We are now ready to carry out the second construction.

Theorem 4.3. *$A \otimes H \otimes B$ is a Hopf algebra with comultiplication:*

$$\Delta : A \otimes H \otimes B \to (A \otimes H \otimes B) \otimes (A \otimes H \otimes B)$$
$$a \otimes h \otimes b \mapsto (a_1 \otimes {a_2}^1 h_1 \otimes {b_1}^1) \otimes ({a_2}^2 \otimes h_2 {b_1}^2 \otimes b_2)$$

and multiplication:

$$\mu : (A \otimes H \otimes B) \otimes (A \otimes H \otimes B) \to A \otimes H \otimes B$$

$$(a \otimes h \otimes b) \otimes (a' \otimes h' \otimes b') \mapsto$$
$$a(h_1 \to ({b_1}^1 \rightharpoonup a'_1)) \otimes h_2 {b_1}^2 (b_2 \sharp a'_2) {a'_3}^1 h'_1 \otimes ((b_3 \leftharpoonup {a'_3}^2) \leftarrow h'_2) b'$$

and counit:

$$\epsilon : A \otimes H \otimes B \to K, \quad a \otimes h \otimes b \mapsto \epsilon_A(a)\epsilon_H(h)\epsilon_B(b)$$

and unit $1 \otimes 1 \otimes 1$ *and antipode:*

$$S : A \otimes H \otimes B \to A \otimes H \otimes B$$
$$a \otimes h \otimes b \mapsto (1 \otimes 1 \otimes S_B(b^1))(1 \otimes S_H(a^1 h b^2) \otimes 1)(S_A(a^2) \otimes 1 \otimes 1)$$

where $\rightharpoonup$, $\leftharpoonup$ *are the coadjoint actions and* $\sharp$ *is defined as:*

$$b \sharp a := \langle a_1, {b_1}^1\rangle_B {b_1}^2 {a_2}^1 \langle {a_2}^2, b_2\rangle_A$$

The proof of this theorem will occupy the rest of this section.

4.9. In the proof of Theorem 4.3, we shall frequently use a form of the structure elements in which one tensorand is not changed:

Proposition 4.4. *We have:*

(1) $b \rightharpoonup a = \langle (S_H({a_2}^1) \to a_1) S_A(a_3), b\rangle_B {a_2}^2$
(2) $b \leftharpoonup a = \langle a, S_B(b_1)(b_3 \leftarrow S_H({b_2}^2))\rangle_A {b_2}^1$
(3) $b \sharp a = \langle a, S_B({b_1}^1){b_2}^1\rangle_A {b_1}^2 S_H({b_2}^2) = \langle {a_1}^2 S_A({a_2}^2), b\rangle_B S_H({a_1}^1){a_2}^1$

Proof. We show (1):

$$\begin{aligned}\langle b \rightharpoonup a, b'\rangle_B &= \langle a, b' \leftharpoonup b\rangle_B \\ &= \langle a, \mu_B(S_B^{-1} \otimes \mu_B)\sigma^{-1}_{B,B\otimes B}(id_B \otimes \Delta_B)(b' \otimes b)\rangle_B \\ &= \langle (id_A \otimes \mu_A)(\Delta_A^{copp} \otimes S_A)\Delta_A(a), b' \otimes b\rangle_B \\ &= \langle {a_2}^2, b'\rangle_B \langle (S_H({a_2}^1) \to a_1) S_A(a_3), b\rangle_B\end{aligned}$$

by Proposition 4.2 in Subsection 4.5. The proof of (2) is similar. We prove the first equality in (3), the proof of the second one is similar:

$$\begin{aligned}b\sharp a &= \langle a_1, {b_1}^1\rangle_B {b_1}^2 S_H({b_2}^2)\langle a_2, {b_2}^1\rangle_A \\ &= \langle a_1, S_B({b_1}^1)\rangle_A {b_1}^2 S_H({b_2}^2)\langle a_2, {b_2}^1\rangle_A \\ &= \langle a, S_B({b_1}^1){b_2}^1\rangle_A {b_1}^2 S_H({b_2}^2)\end{aligned}$$

by 2.4 (2), and equations (3) and (5) in Proposition 4.1 of Subsection 4.4. □

4.10. In order to prove Theorem 4.3, we have to verify the compatibility conditions in Definition 3.2. We begin with condition (1). By part (1) in Proposition 4.4 and the main assumption in Subsection 4.1, we have:

$$\begin{aligned}\Delta_A(b \rightharpoonup a) &= \langle (S_H({a_3}^1)S_H({a_2}^1) \to a_1) S_A(a_4), b\rangle_B {a_2}^2 \otimes {a_3}^2 \\ &= \langle S_H({a_5}^2) \to [(S_H({a_2}^1) \to a_1)S_A(a_3)], b_1\rangle_B \\ &\quad \langle (S_H({a_5}^1) \to a_4) S_A(a_6), b_2\rangle_B {a_2}^2 \otimes {a_5}^3 \\ &= \langle [(S_H({a_2}^1) \to a_1)S_A(a_3)]^2, b_1\rangle_B {a_2}^2 \otimes \\ &\quad \langle (S_H({a_5}^1) \to a_4) S_A(a_6), b_2\rangle_B S_H([(S_H({a_2}^1) \to a_1)S_A(a_3)]^1) \to {a_5}^2 \\ &= \langle (S_H({a_2}^1) \to a_1)S_A(a_3), {b_1}^1\rangle_B {a_2}^2 \otimes \langle (S_H({a_5}^1) \to a_4)S_A(a_6), b_2\rangle_B ({b_1}^2 \to {a_5}^2) \\ &= ({b_1}^1 \rightharpoonup a_1) \otimes {b_1}^2 \to (b_2 \rightharpoonup a_2),\end{aligned}$$

where the fourth equality uses Proposition 4.2 (7) in Subsection 4.5. The proof of the second equation in (1) of Definition 3.2 is strictly dual.

4.11. We now verify condition (2) in Definition 3.2:

$$\begin{aligned}&({b_1}^1\sharp a_1){a_2}^1 \otimes {b_1}^2(b_2\sharp {a_2}^2) \\ &= \langle a_1, {{b_1}^1}_1{}^1\rangle_B {{b_1}^1}_1{}^2 {a_2}^1 \langle {a_2}^2, {{b_1}^1}_2\rangle_A {a_3}^1 \otimes {b_1}^2 \langle {{a_3}^2}_1, {b_2}^1\rangle_B {b_2}^2 {{a_3}^2}_2{}^1 \langle {{a_3}^2}_2{}^2, b_3\rangle_A \\ &= \langle a_1, {b_1}^1\rangle_B {b_1}^2 {a_2}^1 {a_3}^1 {a_4}^1 \langle {a_2}^2, {b_2}^1\rangle_A \otimes \langle {a_3}^2, {b_3}^1\rangle_B {b_1}^3 {b_2}^2 {b_3}^2 {a_4}^2 \langle {a_4}^3, b_4\rangle_A \\ &= \langle a_1, {b_1}^1\rangle_B {b_1}^2 {a_2}^1 {a_3}^1 \langle {{a_2}^2}_1, {{b_2}^1}_1\rangle_A \otimes \langle {{a_2}^2}_2, {{b_2}^1}_2\rangle_B {b_1}^3 {b_2}^2 {a_3}^2 \langle {a_3}^3, b_3\rangle_A \\ &= \langle a_1, {b_1}^1\rangle_B {b_1}^2 {a_2}^1 {a_3}^1 \langle {a_2}^2, {{b_2}^1}_1 S_B({{b_2}^1}_2)\rangle_A \otimes {b_1}^3 {b_2}^2 {a_3}^2 \langle {a_3}^3, b_3\rangle_A \\ &= \langle a_1, {b_1}^1\rangle_B {b_1}^2 {a_2}^1 \otimes {b_1}^3 {a_2}^2 \langle {a_2}^3, b_2\rangle_A = \Delta_H(b\sharp a)\end{aligned}$$

4.12. Before we proceed to verify condition (3) in Definition 3.2, we record some formulas which occur several times in the course of the proof:

Proposition 4.5. *We have:*

(1) $S_A(a \rightharpoondown a') = a_1 S_A({a_2}^1 \rightarrow a') S_A({a_2}^2)$
(2) $S_B(b' \leftharpoondown b) = S_B({b_1}^1) S_B(b' \leftarrow {b_1}^2) b_2$
(3) ${a_1}^1 {a_2}^1 \otimes S_A({a_1}^2) S_A({a_2}^2 \rightharpoondown a') = a^1 \otimes S_A(a^2 \rightarrow a') S_A(a^3)$
(4) $S_B(b' \leftharpoondown {b_1}^1) S_B({b_2}^1) \otimes {b_1}^2 {b_2}^2 = S_B(b^1) S_B(b' \leftarrow b^2) \otimes b^3$
(5) ${a_1}^1 {a_2}^1 \otimes S_A({a_1}^2 \rightharpoondown ({a_2}^2 \rightarrow a')) {a_2}^3 = a^1 \otimes a^2 S_A(a')$
(6) ${b_1}^1 S_B((b' \leftarrow {b_1}^2) \leftharpoondown {b_2}^1) \otimes {b_1}^3 {b_2}^2 = S_B(b') b^1 \otimes b^2$

Proof. (1) can be written in the form:

$$S_A \mu_A(\mu_A \otimes S_A^{-1}) \sigma^{-1}_{A \otimes A, A} (\Delta_A \otimes id_A)(a \otimes a') =$$
$$\mu_A(\mu_A \otimes id_A)(id_A \otimes S_A \otimes S_A)(id_A \otimes \sigma_{A,A})(\Delta_A \otimes id_A)(a \otimes a')$$

It is a standard calculation inside monoidal categories to reduce both sides to a standard form in which all multiplications appear on the left, followed by all antipodes which are in turn followed by all braiding operators, which are in turn followed by all comultiplications on the right. A comparison of both sides in their reduced form shows that they are equal. (2) is strictly dual to (1), (3) and (5) follow from (1), (4) (resp. (6)) is dual to (3) (resp. (5)). □

4.13. We now verify condition (3) in Definition 3.2. Since the verification of the second formula is strictly dual, we only prove the first one. Using part (4) in Proposition 4.1 for the eighth equality, we have:

$$
\begin{aligned}
& b \rightharpoonup (aa') \\
& = \langle [S_H([{a_2}^2({a_3}^2 \rightarrow a'_2)]^1) \rightarrow (a_1({a_2}^1 {a_3}^1 \rightarrow a'_1))] S_A({a_3}^3 {a'}_3), b \rangle_B [{a_2}^2({a_3}^2 \rightarrow a'_2)]^2 \\
& = \langle [S_H({a_2}^2({a_3}^2 \rightarrow a'_2)^1) \rightarrow (a_1({a_2}^1 {a_3}^1 \rightarrow a'_1))] S_A({a_3}^3 a'_3), b \rangle_B {a_2}^3({a_3}^2 \rightarrow a'_2)^2 \\
& = \langle [S_H({a_2}^2({a_3}^2 {a'_2}^1 S_H({a_3}^4))) \rightarrow (a_1({a_2}^1 {a_3}^1 \rightarrow a'_1))] S_A({a_3}^5 a'_3), b \rangle_B {a_2}^3({a_3}^3 \rightarrow {a'_2}^2) \\
& = \langle [S_H({a_2}^3 {a_3}^3 {a'_2}^2 S_H({a_3}^5)) \rightarrow a_1] \\
& \quad [S_H({a_2}^2 {a_3}^2 {a'_2}^1 S_H({a_3}^6)) {a_2}^1 {a_3}^1 \rightarrow a'_1] S_A({a_3}^7 a'_3), b \rangle_B {a_2}^4({a_3}^4 \rightarrow {a'_2}^3) \\
& = \langle [S_H({a_2}^1 {a_3}^1 {a'_2}^2 S_H({a_3}^3)) \rightarrow a_1][{a_3}^4 S_H({a'_2}^1) \rightarrow a'_1] S_A({a_3}^5 a'_3), b \rangle_B {a_2}^2({a_3}^2 \rightarrow {a'_2}^3) \\
& = \langle (S_H({a_2}^1({a_3}^1 \rightarrow {a'_2}^2)^1) \rightarrow a_1)({a_3}^2 S_H({a'_2}^1) \rightarrow a'_1) S_A({a_3}^3 \rightarrow a'_3) S_A({a_3}^4), b \rangle_B \\
& \quad {a_2}^2({a_3}^1 \rightarrow {a'_2}^2)^2 \\
& = \langle S_H({a_2}^1({a_3}^1 \rightarrow {a'_2}^2)^1) \rightarrow a_1, b_1 \rangle_B \\
& \quad \langle {a_3}^2 \rightarrow [(S_H({a'_2}^1) \rightarrow a'_1) S_A(a'_3)] \otimes S_A({a_3}^3), b_2 \otimes b_3 \rangle_B {a_2}^2({a_3}^1 \rightarrow {a'_2}^2)^2 \\
& = \langle S_H(({a_3}^1 \rightarrow {a'_2}^2)^1) S_H({a_2}^1) \rightarrow a_1, b_1 \rangle_B \\
& \quad \langle S_A({a_3}^2) \otimes (S_H({a'_2}^1) \rightarrow a'_1) S_A(a'_3), (b_3 \leftarrow S_H({b_2}^2)) \otimes {b_2}^1 \rangle_B {a_2}^2({a_3}^1 \rightarrow {a'_2}^2)^2 \\
& = \langle [S_H({a_2}^1) \rightarrow a_1]^2, b_1 \rangle_B \langle {a_3}^2, b_3 \leftarrow S_H({b_2}^2) \rangle_A \langle (S_H({a'_2}^1) \rightarrow a'_1) S_A(a'_3), {b_2}^1 \rangle_B \\
& \quad {a_2}^2(S_H([S_H({a_2}^1) \rightarrow a_1]^1) {a_3}^1 \rightarrow {a'_2}^2) \\
& = \langle S_H({a_2}^1) \rightarrow a_1, {b_1}^1 \rangle_B \langle {a_3}^2, b_3 \leftarrow S_H({b_2}^2) \rangle_A \langle (S_H({a'_2}^1) \rightarrow a'_1) S_A(a'_3), {b_2}^1 \rangle_B \\
& \quad {a_2}^2({b_1}^2 {a_3}^1 \rightarrow {a'_2}^2)
\end{aligned}
$$

$$\begin{aligned}
&= \langle S_H({a_2}^1) \to a_1, {b_1}^1\rangle_B \langle {a_3}^2, b_3 \leftarrow S_H({b_2}^2)\rangle_A {a_2}^2({b_1}^2{a_3}^1 \to ({b_2}^1 \rightharpoonup a'))\\
&= \langle (S_H({a_2}^1) \to a_1)S_A(a_3)a_4, {b_1}^1\rangle_B \langle {a_5}^2, b_2 S_B(b_3)(b_5 \leftarrow S_H({b_4}^2))\rangle_A\\
&\quad {a_2}^2({b_1}^2{a_5}^1 \to ({b_4}^1 \rightharpoonup a'))\\
&= \langle (S_H({a_2}^1) \to a_1)S_A(a_3), {{b_1}^1}_1\rangle_B \langle a_4, {{b_1}^1}_2\rangle_B \langle {{a_5}^2}_1, b_2\rangle_A\\
&\quad \langle {{a_5}^2}_2, S_B(b_3)(b_5 \leftarrow S_H({b_4}^2))\rangle_A {a_2}^2({b_1}^2{a_5}^1 \to ({b_4}^1 \rightharpoonup a'))\\
&= \langle (S_H({a_2}^1) \to a_1)S_A(a_3), {b_1}^1\rangle_B {a_2}^2\\
&\quad ({b_1}^2\langle a_4, {b_2}^1\rangle_B {b_2}^2{a_5}^1\langle {a_5}^2, b_3\rangle_A {a_6}^1 \to [\langle {a_6}^2, S_B(b_4)(b_6 \leftarrow S_H({b_5}^2))\rangle_A {b_5}^1 \rightharpoonup a'])\\
&= ({b_1}^1 \rightharpoonup a_1)({b_1}^2(b_2\sharp a_2){a_3}^1 \to [(b_3 \leftharpoonup {a_3}^2) \rightharpoonup a'])
\end{aligned}$$

4.14. We now verify condition (4) in Definition 3.2. We only prove the second formula, the proof of the first one being strictly dual. We observe first that the right adjoint action $b' \otimes b \mapsto (b' \leftharpoonup b)$ is colinear since it was written in Subsection 4.6 as the composition of colinear mappings. This implies the following formula for the left coadjoint action:

$$(b \rightharpoonup a)^1 \otimes (b \rightharpoonup a)^2 = b^2 a^1 \otimes (b^1 \rightharpoonup a^2)$$

Using condition (1) of Definition 3.2 and Proposition 4.4 (3) from Subsection 4.9, we now calculate:

$$\begin{aligned}
&(b\sharp({b'_1}^1 \rightharpoonup a_1)){b'_1}^2(b'_2\sharp a_2)\\
&\quad = \langle ({b'_1}^1 \rightharpoonup a_1)_1, {b_1}^1\rangle_B {b_1}^2{({b'_1}^1 \rightharpoonup a_1)_2}^1\langle {({b'_1}^1 \rightharpoonup a_1)_2}^2, b_2\rangle_A {b'_1}^2\\
&\qquad \langle a_2, S_B({b'_2}^1){b'_3}^1\rangle_A {b'_2}^2 S_H({b'_3}^2)\\
&\quad = \langle {b'_1}^1 \rightharpoonup a_1, {b_1}^1\rangle_B \langle {b'_1}^2 \to ({b'_2}^1 \rightharpoonup {a_2}^2), S_B^{-1}(b_2)\rangle_B \langle a_3, S_B({b'_3}^1){b'_4}^1\rangle_A\\
&\qquad {b_1}^2{b'_2}^2{a_2}^1{b'_1}^3{b'_2}^3{b'_3}^2 S_H({b'_4}^2)\\
&\quad = \langle a_1, S_B({b_1}^1 \leftharpoonup {b'_1}^1)\rangle_A \langle {a_2}^2, (S_B^{-1}(b_2) \leftarrow {b'_1}^2) \leftharpoonup {b'_2}^1\rangle_B\\
&\qquad \langle a_3, S_B({b'_3}^1){b'_4}^1\rangle_A {b_1}^2{b'_2}^2{a_2}^1{b'_1}^3{b'_2}^3{b'_3}^2 S_H({b'_4}^2)\\
&\quad = \langle a_1, S_B({b_1}^1 \leftharpoonup {b'_1}^1)\rangle_A \langle a_2, S_B((S_B^{-1}({b_2}^1) \leftarrow {b'_1}^2) \leftharpoonup {b'_2}^1)\rangle_A\\
&\qquad \langle a_3, S_B({b'_3}^1){b'_4}^1\rangle_A {b_1}^2{b'_2}^3 S_H({b_2}^2{b'_2}^2){b'_1}^3{b'_2}^4{b'_3}^2 S_H({b'_4}^2)\\
&\quad = \langle a, S_B({b_1}^1 \leftharpoonup {b'_1}^1)S_B((S_B^{-1}({b_2}^1) \leftarrow {b'_1}^2) \leftharpoonup {b'_2}^1)S_B({b'_3}^1){b'_4}^1\rangle_A\\
&\qquad {b_1}^2 S_H({b_2}^2){b'_1}^3{b'_2}^2{b'_3}^2 S_H({b'_4}^2)\\
&\quad = \langle a, S_B({b_1}^1 \leftharpoonup {b'_1}^1)S_B({b'_2}^1)S_B(S_B^{-1}({b_2}^1) \leftarrow {b'_1}^2{b'_2}^2){b'_3}^1\rangle_A {b_1}^2 S_H({b_2}^2){b'_1}^3{b'_2}^3 S_H({b'_3}^2)\\
&\quad = \langle a, S_B({b'_1}^1)S_B({b_1}^1 \leftarrow {b'_1}^2)({b_2}^1 \leftarrow {b'_1}^3){b'_2}^1\rangle_A {b_1}^2 S_H({b_2}^2){b'_1}^4 S_H({b'_2}^2)\\
&\quad = \langle a, S_B({b_1}^1{b'_1}^1)({b_2}^1 \leftarrow {b'_1}^2){b'_2}^1\rangle_A {b_1}^2 S_H({b_2}^2){b'_1}^3 S_H({b'_2}^2)\\
&\quad = \langle a, S_B((b_1{b'_1}^1)^1)((b_2 \leftarrow {b'_1}^2)b'_2)^1\rangle_A (b_1{b'_1}^1)^2 S_H(((b_2 \leftarrow {b'_1}^2)b'_2)^2)\\
&\quad = \langle a, S_B({(bb')_1}^1){(bb')_2}^1\rangle_A {(bb')_1}^2 S_H({(bb')_2}^2) = (bb')\sharp a
\end{aligned}$$

Here the sixth and the seventh equality follow from Proposition 4.5 (4) of Subsection 4.12, whereas the last one holds by Proposition 4.4 (3) in Subsection 4.9.

4.15. We omit the proofs of the conditions (5), (6) and (7) in Definition 3.2 and continue with the proof of the first formula in condition (8):

$$
\begin{aligned}
&(b_1{}^1 \rightharpoonup a_1)^1 b_1{}^2 (b_2 \sharp a_2) \langle (b_1{}^1 \rightharpoonup a_1)^2, b' \rangle_B \\
&\quad = b_1{}^2 a_1{}^1 b_1{}^3 \langle a_2, S_B(b_2{}^1) b_3{}^1 \rangle_A b_2{}^2 S_H(b_3{}^2) \langle b_1{}^1 \rightharpoonup a_1{}^2, b' \rangle_B \\
&\quad = \langle a_2, S_B(b_2{}^1) b_3{}^1 \rangle_A \langle a_1, b'^1 \leftharpoondown b_1{}^1 \rangle_B b_1{}^3 S_H(b'^2 b_1{}^2) b_1{}^4 b_2{}^2 S_H(b_3{}^2) \\
&\quad = \langle a, S_B(b'^1 \leftharpoondown b_1{}^1) S_B(b_2{}^1) b_3{}^1 \rangle_A S_H(b'^2) b_1{}^2 b_2{}^2 S_H(b_3{}^2) \\
&\quad = \langle a, S_B(b_1{}^1) S_B(b'^1 \leftarrow b_1{}^2) b_2{}^1 \rangle_A S_H(b'^2) b_1{}^3 S_H(b_2{}^2) \\
&\quad = \langle a, S_B(b_1{}^1) b_2{}^1 S_B((b'^1 \leftarrow b_1{}^2 b_2{}^2) \leftharpoondown b_3{}^1) \rangle_A S_H(b'^2) b_1{}^3 S_H(b_2{}^3 b_3{}^2) \\
&\quad = \langle a_1, S_B(b_1{}^1) b_2{}^1 \rangle_A \langle a_2, S_B((b'^1 \leftarrow b_1{}^3 b_2{}^3) \leftharpoondown b_3{}^1 \rangle_A S_H(b'^2) b_1{}^2 S_H(b_2{}^2 b_3{}^2) \\
&\quad = \langle a_1, S_B(b_1{}^1{}_1{}^1) b_1{}^1{}_2{}^1 \rangle_A \langle a_2, (b'^1 \leftarrow b_1{}^2) \leftharpoondown b_2{}^1 \rangle_B S_H(b'^2) b_1{}^1{}_1{}^2 S_H(b_1{}^1{}_2{}^2 b_2{}^2) \\
&\quad = (b_1{}^1 \sharp a_1) \langle a_2{}^2, (b' \leftarrow b_1{}^2) \leftharpoondown b_2 \rangle_B a_2{}^1 \\
&\quad = (b_1{}^1 \sharp a_1) a_2{}^1 \langle b_1{}^2 \rightarrow (b_2 \rightharpoonup a_2{}^2), b' \rangle_B
\end{aligned}
$$

Here the first and the eighth equality follow from Proposition 4.4 (3) in Subsection 4.9 whereas the fourth and the fifth one follow from part (4) resp. (6) of Proposition 4.5.

4.16. Condition (9) in Definition 3.2 is the dualization of the H-linearity of the adjoint actions. We now verify condition (10). Since H is commutative and cocommutative, we have:

$$
\begin{aligned}
(b \sharp (h_1 \rightarrow a)) h_2 &= \langle (h_1 \rightarrow a)_1, b_1{}^1 \rangle_B b_1{}^2 (h_1 \rightarrow a)_2{}^1 h_2 \langle (h_1 \rightarrow a)_2{}^2, b_2 \rangle_A \\
&= \langle h_1 \rightarrow a_1, b_1{}^1 \rangle_B b_1{}^2 a_2{}^1 h_3 \langle h_2 \rightarrow a_2{}^2, b_2 \rangle_A \\
&= \langle a_1, b_1{}^1 \leftarrow h_1 \rangle_B b_1{}^2 a_2{}^1 h_3 \langle a_2{}^2, b_2 \leftarrow h_2 \rangle_A \\
&= \langle a_1, b_1{}^1 \leftarrow h_2 \rangle_B h_1 b_1{}^2 a_2{}^1 \langle a_2{}^2, b_2 \leftarrow h_3 \rangle_A \\
&= \langle a_1, (b \leftarrow h_2)_1{}^1 \rangle_B h_1 (b \leftarrow h_2)_1{}^2 a_2{}^1 \langle a_2{}^2, (b \leftarrow h_2)_2 \rangle_A \\
&= h_1((b \leftarrow h_2) \sharp a)
\end{aligned}
$$

4.17. Finally, we have to verify condition (11). We have by Proposition 4.4 (2) in Subsection 4.9:

$$
\begin{aligned}
&\langle b_1{}^2 \rightarrow (b_2 \rightharpoonup a_2{}^2), b' \rangle_B \langle a', (b_1{}^1 \leftharpoonup a_1) \leftarrow a_2{}^1 \rangle_A \\
&\quad = \langle a_2{}^2, (b' \leftarrow b_1{}^2) \leftharpoondown b_2 \rangle_B \langle a_1, S_B(b_1{}^1{}_1)(b_1{}^1{}_3 \leftarrow S_H(b_1{}^1{}_2{}^2)) \rangle_A \\
&\qquad \langle a', b_1{}^1{}_2{}^1 \leftarrow a_2{}^1 \rangle_A \\
&\quad = \langle a_2{}^2, S_B((b' \leftarrow b_1{}^2 b_2{}^3 b_3{}^2) \leftharpoondown b_4) \rangle_A \langle a_1, S_B(b_1{}^1)(b_3{}^1 \leftarrow S_H(b_2{}^2)) \rangle_A \\
&\qquad \langle a_2{}^1 \rightarrow a', b_2{}^1 \rangle_A \\
&\quad = \langle a_1 \otimes (a_2{}^1 \rightarrow a') \otimes a_2{}^2, \\
&\qquad S_B(b_1{}^1)(b_3{}^1 \leftarrow S_H(b_2{}^2)) \otimes b_2{}^1 \otimes S_B((b' \leftarrow b_1{}^2 b_2{}^3 b_3{}^2) \leftharpoondown b_4) \rangle_A \\
&\quad = \langle a_1 \otimes a_2 \otimes a', \\
&\qquad S_B(b_1{}^1)(b_3{}^1 \leftarrow S_H(b_2{}^3)) \otimes S_B((b' \leftarrow b_1{}^2 b_2{}^4 b_3{}^2) \leftharpoondown b_4) \leftarrow S_H(b_2{}^2) \otimes b_2{}^1 \rangle_A \\
&\quad = \langle a \otimes a', S_B(b_1{}^1)[(b_3{}^1 S_B((b' \leftarrow b_1{}^2 b_2{}^3 b_3{}^2) \leftharpoondown b_4)) \leftarrow S_H(b_2{}^2)] \otimes b_2{}^1 \rangle_A
\end{aligned}
$$

$$= \langle a \otimes a', S_B(b_1{}^1)[(b_3{}^1 S_B(b_4{}^1) S_B(b' \leftarrow b_1{}^2 b_2{}^3 b_3{}^2 b_4{}^2) b_5) \leftarrow S_H(b_2{}^2)] \otimes b_2{}^1 \rangle_A$$
$$= \langle a \otimes a', S_B(b_1{}^1)[(b_3{}^1{}_1 S_B(b_3{}^1{}_2) S_B(b' \leftarrow b_1{}^2 b_2{}^3 b_3{}^2) b_4) \leftarrow S_H(b_2{}^2)] \otimes b_2{}^1 \rangle_A$$
$$= \langle a \otimes a', S_B(b_1{}^1)[(S_B(b' \leftarrow b_1{}^2 b_2{}^3) b_3) \leftarrow S_H(b_2{}^2)] \otimes b_2{}^1 \rangle_A$$
$$= \langle a \otimes a', S_B(b_1{}^1) S_B(b' \leftarrow b_1{}^2)(b_3 \leftarrow S_H(b_2{}^2)) \otimes b_2{}^1 \rangle_A$$
$$= \langle a \otimes a', S_B(b_1{}^1) S_B(b' \leftarrow b_1{}^2) b_2 S_B(b_3)(b_5 \leftarrow S_H(b_4{}^2)) \otimes b_4{}^1 \rangle_A$$
$$= \langle a_1 \otimes a_2 \otimes a', S_B(b' \leftharpoondown b_1) \otimes S_B(b_2)(b_4 \leftarrow S_H(b_3{}^2)) \otimes b_3{}^1 \rangle_A$$
$$= \langle a_1, b' \leftharpoondown b_1 \rangle_B \langle a_2, S_B(b_2)(b_4 \leftarrow S_H(b_3{}^2)) \rangle_A \langle a', b_3{}^1 \rangle_A$$
$$= \langle b_1 \rightharpoonup a_1, b' \rangle_B \langle a', b_2 \leftharpoondown a_2 \rangle_A,$$

where we have used part (4) and (5) of Proposition 4.1 in the fourth resp. eleventh equality, part (2) of Proposition 4.5 in the sixth and the eleventh equality and Proposition 4.4 (2) in the last one. This finishes the proof of the theorem.

4.18. The Drinfel'd Double construction is contained in this construction as a special case, as we now indicate. As in Subsection 3.9, we set $H = K$ and assume that the Hopf algebra A is finite dimensional. We set $C = A^*$ and obtain $B = A^{*cop}$. Identifying $A \otimes K \otimes B$ with $A \otimes B$, we want to rewrite the multiplication in Theorem 4.3 in a more familiar way. We calculate, using Proposition 4.4 (1) in the first and Proposition 4.5 (1) in the third equality:

$$\begin{aligned} b_1 \rightharpoonup a'_1 \langle a, b_2 \leftharpoondown a'_2 \rangle_A &= \langle a'_1 S_A(a'_3), b_1 \rangle_B a'_2 \langle a'_4 \rightharpoondown a, b_2 \rangle_A \\ &= \langle a'_1 S_A(a'_3), b_1 \rangle_B a'_2 \langle S_A(a'_4 \rightharpoondown a), b_2 \rangle_B \\ &= \langle a'_1 S_A(a'_3), b_1 \rangle_B a'_2 \langle a'_4 S_A(a) S_A(a'_5), b_2 \rangle_B \\ &= \langle a'_1 S_A(a'_3) a'_4 S_A(a) S_A(a'_5), b \rangle_B a'_2 = \langle a'_3 a S_A^{-1}(a'_1), b \rangle_A a'_2 \end{aligned}$$

We therefore have the following form of the multiplication:

$$(a \otimes b)(a' \otimes b') = a a'_2 \otimes \langle a'_3 \cdot S_A^{-1}(a'_1), b \rangle_A b',$$

where $f(\cdot)$ is the mapping $x \mapsto f(x)$. Passing to the opposite and coopposite Hopf algebra and reversing the ordering of the tensorands, we obtain a Hopf algebra structure with multiplication

$$\mu' : (B \otimes A) \otimes (B \otimes A) \to B \otimes A$$
$$(b \otimes a) \otimes (b' \otimes a') \mapsto \langle a_3 \cdot S_A^{-1}(a_1), b' \rangle_A b \otimes a' a_2$$

and comultiplication

$$\Delta : B \otimes A \to (B \otimes A) \otimes (B \otimes A)$$
$$b \otimes a \mapsto (b_2 \otimes a_2) \otimes (b_1 \otimes a_1)$$

This is the Drinfel'd Double of $A^{op\ cop}$ (cf. [9, p. 299], [7, Definition 10.3.5], [2, §13, p. 816]).

5. Deformed enveloping algebras

5.1. In this section, we explain how the second construction provides a method to construct the deformed enveloping algebras of semisimple Lie algebras. We work in Lusztig's setting, which is reproduced from his book [4] in the next paragraphs.

In this section, the base field K is the field $\mathbf{Q}(v)$ of rational functions of one indeterminate v over $\mathbf{Q}$.

5.2. A Cartan datum is a pair $(I, \cdot)$ consisting of a finite set I and a symmetric bilinear form $\nu, \nu' \mapsto \nu \cdot \nu'$ on the free abelian group $\mathbf{Z}[I]$, with values in $\mathbf{Z}$. It is assumed that:

(1) $i \cdot i \in \{2, 4, 6, \dots\}$ for any $i \in I$;
(2) $2\frac{i \cdot j}{i \cdot i} \in \{0, -1, -2, \dots\}$ for any $i \neq j$ in I.

5.3. We define a group homomorphism

$$\mathbf{Z}[I] \to \mathbf{Q}(v)\backslash\{0\}, \quad \nu \mapsto v_\nu$$

which takes the value $v^{i \cdot i/2}$ for a basis element $i \in I$. We also shall use the notation $tr\ \nu = \sum_i \nu_i \in \mathbf{Z}$ for $\nu = \sum_i \nu_i i \in \mathbf{Z}[I]$. In analogy to [4, 3.1.1], we shall also use the group homomorphism

$$\mathbf{Z}[I] \to \mathbf{Z}[I], \quad \nu \mapsto \tilde{\nu}$$

which takes the value $\frac{i \cdot i}{2} i$ on the basis element i.

5.4. A root datum of type $(I, \cdot)$ consists, by definition, of

(1) two finitely generated free abelian groups Y, X and a bilinear pairing

$$\langle\, ,\rangle : Y \times X \to \mathbf{Z}$$

(We do not require the pairing to be perfect, cf. [1, p. 281]);
(2) an embedding $I \subset X$ $(i \mapsto i')$ and an embedding $I \subset Y$ $(i \mapsto i)$ such that
(3) $\langle i, j' \rangle = 2\frac{i \cdot j}{i \cdot i}$ for all $i, j \in I$.

The embeddings (2) induce homomorphisms $\mathbf{Z}[I] \to Y$, $\mathbf{Z}[I] \to X$; we shall often denote, again by ν, the image of $\nu \in \mathbf{Z}[I]$ under either of these homomorphisms.

5.5. We denote by $'\mathbf{f}$ the free associative $\mathbf{Q}(v)$-algebra with 1 with generators θ_i $(i \in I)$. Let $\mathbf{N}[I]$ be the submonoid of $\mathbf{Z}[I]$ consisting of all linear combinations of elements of I with coefficients in $\mathbf{N}$. For any $\nu = \sum_i \nu_i i \in \mathbf{N}[I]$, we denote by $'\mathbf{f}_\nu$ the $\mathbf{Q}(v)$-subspace of $'\mathbf{f}$ spanned by the monomials $\theta_{i_1}\theta_{i_2}\dots\theta_{i_r}$ such that for any $i \in I$, the number of occurrences of i in the sequence $i_1, i_2, \dots, i_r$ is equal to ν_i. Then each $'\mathbf{f}_\nu$ is a finite dimensional $\mathbf{Q}(v)$-vector space and we have a direct sum decomposition $'\mathbf{f} = \oplus_\nu {'\mathbf{f}_\nu}$ where ν runs over $\mathbf{N}[I]$. An element of $'\mathbf{f}$ is said to be homogeneous if it belongs to $'\mathbf{f}_\nu$ for some ν. We then set $|x| = \nu$.

5.6. We take our Hopf algebra H to be the group ring $K[Y]$. H is obviously commutative and cocommutative. Following [13], we turn $'\mathbf{f}$ into a left Yetter-Drinfel'd module over H by defining for a homogeneous element $x \in {'\mathbf{f}}$:

$$K'_\mu \to x := v^{-\langle \mu, |x| \rangle} x, \quad \delta(x) = \tilde{K}'_{-|x|} \otimes x$$

where K'_μ is the basis element of the group ring corresponding to $\mu \in Y$, and $\tilde{K}'_\nu$ for $\nu \in \mathbf{Z}[I]$ is defined as in [4, 3.1.1] to be $K'_{\tilde{\nu}}$. It is obvious that $'\mathbf{f}$ becomes a Yetter-Drinfel'd module in this way, and it is also an algebra in that category. We therefore can form the tensor product algebra inside that category. Since $'\mathbf{f}$ is free, there is a unique algebra morphism $r : {'\mathbf{f}} \to {'\mathbf{f}} \otimes {'\mathbf{f}}$ such that

$$r(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i.$$

Using this comultiplication and the unique algebra morphism from $'\mathbf{f}$ to $\mathbf{Q}(v)$ annihilating the θ_i's as a counit, $'\mathbf{f}$ becomes a Yetter-Drinfel'd bialgebra.

5.7. In contrast to the previous sections, we here follow [4] and denote by σ the unique algebra antiautomorphism of $'\mathbf{f}$ such that $\sigma(\theta_i) = \theta_i$.

Proposition 5.1. *$'\mathbf{f}$ is a left Yetter-Drinfel'd Hopf algebra with antipode:*

$$S_{'\mathrm{f}}(x) = (-1)^{tr|x|} v^{|x|\cdot|x|/2} v_{-|x|} \sigma(x)$$

for a homogeneous element $x \in {}'\mathbf{f}$.

The proof is based on a direct computation and is omitted.

5.8. If $'\mathbf{f}$ is considered as a left Yetter-Drinfel'd module as in Subsection 5.6, it is denoted by A'. We now also introduce the structure of a right Yetter-Drinfel'd module on $'\mathbf{f}$ by defining: $x \leftarrow K'_\mu := v^{-\langle \mu, |x| \rangle} x$, $\delta(x) := x \otimes \tilde{K}'_{-|x|}$. $'\mathbf{f}$ is then a right Yetter-Drinfel'd Hopf algebra with the same multiplication, comultiplication, unit, counit and antipode, which is denoted by C'. (This is true in this particular case, not in general, even if H is commutative and cocommutative, cf. Subsection 4.1.)

5.9. We now introduce the following bilinear form $\langle \cdot, \cdot \rangle_{A'}$ on $A' \times C'$: For $i \in I$ suppose that $\zeta_i \in A'^*$ is the linear form which satisfies:

$$\zeta_i(\theta_i) = \frac{1}{(v_i^{-1} - v_i)}$$

and vanishes on $x \in {}'\mathbf{f}_\nu$ if $\nu \neq i$. Since A' is a coalgebra, A'^* is an algebra. Consider the algebra homomorphism $\phi : C' \to A'^*$ satisfying $\phi(\theta_i) = \zeta_i$. We set:

$$\langle x, y \rangle_{A'} := \phi(y)(x)$$

This is a bialgebra form by Lemma 2.3 since it satisfies 2.6 (2) by definition and 2.4 (1), 2.4 (2), 2.6 (1) and 2.6 (3) on the generators. The form $\langle \cdot, \cdot \rangle_{A'}$ is not equal to the form $(\cdot, \cdot)$ of [4], but it has the same radical, since both forms are related via $\langle x, y \rangle_{A'} = (-1)^{tr\,|x|} v_{-|x|}(x, y)$ for homogeneous elements $x, y \in {}'\mathbf{f}$.

5.10. We now use the method from Subsection 2.7 to obtain a nondegenerate bialgebra form $\langle \cdot, \cdot \rangle_A$ on $A \times C$ where $A := A'/R_{A'}$ and $C := C'/R_{C'}$. We denote the equivalence class of $a \in A'$ in A by $\bar{a}$, and similarly for C. We now apply the second construction to A and C. The main assumption in Subsection 4.1 is satisfied since the form $(\nu, \nu') \mapsto \nu \cdot \nu'$ is symmetric. Defining

$$\mathbf{V} := A \otimes H \otimes B,$$

we shall see now that $\mathbf{V}^{cop}$ is isomorphic to the algebra $\mathbf{U}$ defined in [4]. We set:

$$F_i := \bar{\theta}_i \otimes 1 \otimes 1, \quad E_i := 1 \otimes 1 \otimes \bar{\theta}_i, \quad K_\mu := 1 \otimes K'_\mu \otimes 1$$

It is easy to verify that these elements satisfy the defining relations of the algebra $\mathbf{U}$, which will be carried out in one case only. A short calculation shows that $\bar{\theta}_i \rightharpoonup \bar{\theta}_j$

and $\bar\theta_i \leftharpoondown \bar\theta_j$ vanish. We therefore have:

$$\begin{aligned} E_iF_j &= (1\otimes 1\otimes \bar\theta_i)(\bar\theta_j\otimes 1\otimes 1)\\ &= 1\otimes(\bar\theta_i\sharp\bar\theta_j)\otimes + \bar\theta_j\otimes 1\otimes\bar\theta_i\\ &= \langle\bar\theta_j, S_B(\bar\theta_i)\rangle_A 1\otimes \tilde K_i'\otimes 1 + \langle\bar\theta_j,\bar\theta_i\rangle_A 1\otimes\tilde K_{-j}'\otimes 1 + \bar\theta_j\otimes 1\otimes\bar\theta_i\\ &= \frac{-\delta_{ij}}{v_i^{-1}-v_i}\tilde K_i + \frac{\delta_{ij}}{v_i^{-1}-v_i}\tilde K_{-j} + F_jE_i \end{aligned}$$

Therefore we have:

$$E_iF_j - F_jE_i = \delta_{ij}\frac{\tilde K_i - \tilde K_{-i}}{v_i - v_i^{-1}}$$

We therefore get an algebra map from $\mathbf{U}$ to $\mathbf{V}^{cop}$ which is in fact a Hopf algebra map. By the triangular decomposition theorem [4, corollary 3.2.4], this map must be an isomorphism.

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Added in Proof. S. Majid has recently announced a construction of deformed enveloping algebras based on the notion of a weakly quasitriangular pair (cf. [6]). A revised version of P. Schauenburg's article [13] has been accepted for publication [14].

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