New York Journal of Mathematics

New York J. Math. **2** (1996) 1–19.

Area Preserving Homeomorphisms of Open Surfaces of Genus Zero

John Franks

Abstract. We show that an area preserving homeomorphism of the open annulus which has at least one periodic point must in fact have infinitely many interior periodic points.

Contents

In this paper we investigate area preserving homeomorphisms of the open annulus and their periodic points. The main result is that an area preserving homeomorphism of the annulus which has at least one periodic point must in fact have infinitely many interior periodic points. This result was claimed in [F4] but the proof contained a gap.

1. Chain Recurrence

We briefly recall the definition of chain recurrence due to Charles Conley in [C]. In the following $f : X \to X$ will denote a homeomorphism of a metric space X.

(1.1) Definition. An *ε-chain* for f, from x to y is a sequence $x = x_1, x_2, \dots, x_n = y$, in X such that

$$d(f(x_i), x_{i+1}) < \varepsilon \qquad \text{for } 1 \le i \le n-1.$$

A point $x \in X$ is called *chain recurrent* if for every $\varepsilon > 0$ there is an ε-chain from x to itself. The set $\mathbf{R}(f)$ of chain recurrent points is called the *chain recurrent set* of f.

It is easily seen that if the metric space X is compact then the chain recurrent set $\mathbf{R}(f)$ is compact and invariant under f. Moreover it is independent of the choice

Received June 15, 1995.
Mathematics Subject Classification. primary 58F20, secondary 58F05.
Key words and phrases. periodic points, area preserving, annulus.

ISSN 1076-9803/96

of metric on X, depending only on f and the topology of X. If X is not compact then $\mathbf{R}(f)$ is closed and invariant but it depends on the metric defined on X rather than just the topology.

The following simple result is well known and is valid whether or not X is compact.

(1.2) Proposition. *If $\mathbf{R}(f) = X$ (in particular if f preserves a finite measure whose support is all of X) and if X is connected then for any $\varepsilon > 0$ and any $x,\ y \in X$ there is an ε-chain from x to y. If K is a compact subset of X then given $\varepsilon > 0$ there is an N with the property that for any $x, y \in K$ there is an ε-chain of length less than N from x to y.*

Proof. Note that if f preserves a finite measure whose support is all of X then by Poincaré recurrence almost every point of X is recurrent from which it follows that $\mathbf{R}(f) = X$.

We define a relation $\sim$ on $\mathbf{R}(f)$ by $x \sim y$ if and only if for every $\varepsilon > 0$ there is an ε-chain from x to y and another from y to x. It is clear that $\sim$ is an equivalence relation. From the definition it is easy to see that each equivalence class for the equivalence relation $\sim$ is open. Since equivalence classes are pairwise disjoint it follows that the complement of an equivalence class is open. Since X is connected there can be only one equivalence class.

If $K \subset X$ is compact and $x, y \in K$ and there is an ε-chain of length N_0 from x to y then there is an open neighborhood V of $(x, y) \in X \times X$ such that for any $(x_0, y_0) \in V$ there is a ε-chain of length N_0 from x_0 to y_0. Since $K \times K$ is compact it can be covered by finitely many such neighborhoods. The maximum of the values of N_0 for these neighborhoods is the desired N. □

By an *open surface of finite type* we mean a smooth two dimensional manifold obtained by taking a smooth compact surface without boundary and deleting finitely many points from its interior. Equivalently it is a surface obtained by taking a smooth compact surface with boundary and removing all of the boundary components. A proof of the following result can be found as Lemma (1.4) of [F5].

(1.3) Lemma. *Suppose M is an open surface of finite type and $f : M \to M$ is a homeomorphism homotopic to the identity whose canonical lift to its universal covering space is $F : \widetilde{M} \to \widetilde{M}$. If f is fixed point free, then there is a complete Riemannian metric on M which when lifted to $\widetilde{M}$ has a distance function $d(\, , \,)$ satisfying $d(F(x), x) > 1$ for all $x \in \widetilde{M}$.*

One of the the results which we will use in proving the existence of periodic points is a generalization of the Poincaré-Birkhoff Theorem. Suppose $\widetilde{A} = \mathbf{R} \times (0, 1)$ and A is the quotient of $\widetilde{A}$ under the group generated by $(x, t) \to (x + 1, t)$. If $F : \widetilde{A} \to \widetilde{A}$ is a lift of $f : A \to A$ we will say that there is a *positively returning disk* for F if there is an open disk $U \subset \widetilde{A}$ such that $F(U) \cap U = \emptyset$ and $F^n(U) \cap (U + k) \neq \emptyset$ for some integers $n, k > 0$, (here $U + k$ denotes the set $\{(x + k, t) \mid (x, t) \in U\}$). Thus, U is disjoint from its image, but under iteration by F returns to a positive translate of itself. A *negatively returning disk* is defined similarly, but with $k < 0$.

Recall that a point x is *non-wandering* if for every neighborhood U of x there is an $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. In particular if f preserves a finite measure which is positive on open sets then every point is non-wandering. A proof of the following result can be found in [F1].

(1.4) Theorem. *Let $f : A \to A$ be an orientation preserving homeomorphism of the open annulus $A = S^1 \times (0,1)$, which is homotopic to the identity and satisfies the following conditions:*

1. *Every point of A is non-wandering under f.*
2. *There is a lift of f to its universal covering space, $F : \widetilde{A} \to \widetilde{A}$, which possesses both a positively returning disk and a negatively returning disk.*

Then f has a fixed point.

We shall also need the following lemma which is Lemma (3.3) of [F4].

(1.5) Lemma. *Suppose $f : M \to M$ is a homeomorphism of a complete Riemannian manifold M and f possesses a periodic ε-chain with respect to the metric $d(,)$ induced by the Riemannian metric. Then there is an isotopy $h_t : M \to M,\ t \in [0,1]$ such that*

i) *h_t has compact support, and $h_0 = id$.*
ii) *$d(h_t(x), x) < \varepsilon$ for all $t \in [0,1]$ and all $x \in M$.*
iii) *Arbitrarily near the periodic ε-chain for f is a periodic orbit for $g = h_1 \circ f$.*

2. Homological Rotation Vectors

In this section we briefly recall the definition of homological rotation vectors for surface homeomorphisms isotopic to the identity map from [F4] and [F5]. Let M be an open surface of finite type and let $f : M \to M$ be a homeomorphism isotopic to the identity. We fix a metric on M of constant negative curvature. Even more, we assume that one can form M by taking a convex ideal geodesic polygon in hyperbolic space (vertices of which are points at infinity) and making identifications on the edges.

Pick a base point b_0 in the interior of the polygon whose sides are identified to form M. We want to define a function γ which assigns to each $x \in M$ a geodesic segment γ_x in M from b_0 to x, in such a way that the correspondence $x \to \gamma_x$ is measurable. We do this by letting γ_x be the unique geodesic segment from b_0 to x if x is in the interior of the polygon whose sides are identified to form M. For x on an identified edge we consider each pair of edges which are identified and pick one. Then choose γ_x to be the unique geodesic segment from b_0 to x which when lifted back to the polygon ends on the chosen edge.

Let $f_t(x)$ be a homotopy from $f_0 = id : M \to M$ to $f_1 = f$. Because the Euler characteristic of M is negative, f_t is unique up to homotopy. This means that if g_t is another homotopy with $g_0 = id$ and $g_1 = f$, then there is a homotopy from f_t to g_t, i.e., a map $H : M \times [0,1] \times [0,1] \to M$ such that $H(x,t,0) = f_t(x)$ and $H(x,t,1) = g_t(x)$.

For any point $x \in M$ we want to construct a path in M from x to $f^n(x)$ and then form a loop with the segments γ_x and $\gamma_{f^n(x)}$. To do this we observe that if $\pi : \widetilde{M} \to M$ is the universal covering space of M there is a canonical lift of

f to a homeomorphism $F : \widetilde{M} \to \widetilde{M}$; namely, F is that lift obtained by lifting the homotopy f_t from the identity to f to form a homotopy on $\widetilde{M}$ starting at the identity on $\widetilde{M}$. The other end of this homotopy is then defined to be F. The uniqueness of f_t up to homotopy implies that F does not depend on the choice of homotopy from the identity to f. Alternatively, F is the unique lift whose extension to the ideal points at infinity of $\widetilde{M}$ has all those points as fixed points.

Consider the path $\alpha(n, x)$ from x to $f^n(x)$ in M which is given by

$$\alpha(n, x)(t) = f_t^n(x).$$

Again the homotopy class of this path relative to its endpoints is independent of the choice of the homotopy f_t because of the uniqueness (up to homotopy) of this homotopy.

For each $x \in M$ let $h_n(x, f)$ be the closed loop based at b_0 formed by the concatenation of γ_x, the path $\alpha(n, x)$ in M from x to $f^n(x)$ and $\gamma_{f^n(x)}$ traversed backwards. If the homeomorphism f is clear from the context we will abbreviate $h_n(x, f)$ to $h_n(x)$.

Let $*$ denote concatenation of based loops and observe that $h_n(x) * h_m(f^n(x))$ is homotopic to $h_{n+m}(x)$. Let $[h_n(x)]$ denote the homology class in $H_1(M, \mathbb{R})$ of the loop $h_n(x)$. Note that $[h_{n+m}(x)] = [h_n(x)] + [h_m(f^n(x))]$. We can now formulate the definition of homology rotation vector.

(2.1) Definition. Let M be an open surface of finite type with negative Euler characteristic. Suppose $f : M \to M$ is a homeomorphism which is isotopic to the identity map. The *homological rotation vector* of $x \in M$, is an element of $H_1(M, \mathbb{R})$ denoted $\mathcal{R}(x, f)$, and is defined by

$$\mathcal{R}(x, f) = \lim_{n \to \infty} \frac{[h_n(x)]}{n}$$

if this limit exists.

If the limit in Definition (2.1) above does not exist then $\mathcal{R}(x, f)$ is undefined and we say x has no homological rotation vector.

Let μ be an f invariant measure on M which is homeomorphic to Lebesgue measure. By this we mean there is a homeomorphism $h : M \to M$ such that for any Borel set A in M we have $\mu(A) = m(h(A))$ where $m(\,)$ denotes Lebesgue measure associated with a hyperbolic metric on M.

The homology classes $[h_1(x)] \in H_1(M, \mathbb{R})$ depend measurably on x. In fact there is a closed set of measure zero in M (consisting of the "edges" of the polygon and their inverse images under f) on the complement of which the function $[h_1(x)]$ is locally constant.

We observe that since M is not compact, the function $[h_1(x)]$ might not be bounded. In fact it is easy to construct a diffeomorphism of the disk punctured at its center which has the property that as one approaches the central puncture the diffeomorphism rotates around that puncture an arbitrarily large amount. One can construct such a diffeomorphism for which $[h_1(x)]$ is not integrable.

Much of this article deals with the case when $[h_1(x)]$ is not integrable. In the case that it is integrable we can apply the Birkhoff ergodic theorem. Since $[h_{n+m}(x)] = [h_n(x)] + [h_m(f^n(x))]$,

$$\sum_{i=0}^{n-1} [h_1(f^i(x))] = [h_n(x)].$$

Hence by the Birkhoff ergodic theorem, for $\mu-$almost all x the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} [h_1(f^i(x))] = \lim_{n\to\infty} \frac{[h_n(x)]}{n} = \mathcal{R}(x, f)$$

exists. Thus in this case the homological rotation vector exists for $\mu-$almost all x.

The Birkhoff ergodic theorem also asserts that $\mathcal{R}(x, f)$ is a $\mu-$measurable function of x and that

$$\int \mathcal{R}(x, f) d\mu = \int [h_1(x, f)] d\mu.$$

(2.3) Definition. Let M be an open surface of finite type with negative Euler characteristic. Suppose $f : M \to M$ is a homeomorphism of the surface M which is isotopic to the identity map and preserves a finite measure μ homeomorphic to Lebesgue measure. The mean rotation vector of f, if it exists, is an element of $H_1(M, \mathbb{R})$ denoted $\mathcal{R}_\mu(f)$, and is defined by

$$\mathcal{R}_\mu(f) = \int \mathcal{R}(x, f) d\mu,$$

when this integral exists. If the integral does not exist then the mean rotation vector is undefined.

(2.4) Proposition. *Suppose f and g are homeomorphisms of M which are isotopic to the identity and preserve a finite measure μ homeomorphic to Lebesgue measure. Then*

$$\mathcal{R}_\mu(f \circ g) = \mathcal{R}_\mu(f) + \mathcal{R}_\mu(g),$$

if all these integrals exist.

Proof. The loop $h_1(x, f \circ g)$ is homotopic to the concatenation of the loops $h_1(x, f)$ and $h_1(f(x), g)$ so $[h_1(x, f \circ g)] = [h_1(x, f)] + [h_1(f(x), g)]$. Thus

$$\int [h_1(x, f \circ g)] d\mu = \int [h_1(x, f)] d\mu + \int [h_1(f(x), g)] d\mu.$$

Since f preserves μ, we have that $\int [h_1(f(x), g)] d\mu = \int [h_1(x, g)] d\mu$. Hence

$$\mathcal{R}_\mu(f \circ g) = \mathcal{R}_\mu(f) + \mathcal{R}_\mu(g).$$

□

The following result is due to Bestvina and Handel. A proof can be found in (2.7) of [F4]. It is based on an important fixed point theorem of Handel [H1].

(2.5) Proposition [BH]**.** *Suppose f is a homeomorphism of M, an oriented surface of finite type with genus 0 and Euler characteristic ≤ 0. If f is isotopic to the identity and f has no interior fixed points then every periodic point x in the interior of M has a non-zero homological rotation vector $\mathcal{R}(x, f) \in H_1(M, \mathbb{R})$. The same conclusion is valid if the canonical lift $F : \widetilde{M} \to \widetilde{M}$ has no interior fixed points.*

3. The Mean Rotation Vector Relative to a Subset

In this section we consider the mean rotation vector of f restricted to a subset of an open surface. In particular we consider the implications of its non-existence. Suppose M is an open surface of finite type and $f : M \to M$ is a homeomorphism leaving invariant a measure μ which is homeomorphic to Lebesgue measure.

If $K \subset M$ is a set of positive measure then the *first return map* $g : K \to K$ is defined by $g(x) = f^n(x)$ where n is the smallest positive integer such that $f^n(x) \in K$. By the Poincaré recurrence theorem it is well defined for x in a subset of K of full measure. We will use g to define the mean homological rotation vector of f relative to K. Note that we do not assume that K is invariant. For a subset of full measure of K we can define the loop $h_K(x) = h_n(x)$ where n is the smallest positive integer such that $f^n(x) \in K$. As before let $[h_K(x)]$ denote the homology class of $h_K(x)$.

(3.1) Definition. Let M be an open surface of finite type with non-positive Euler characteristic. Suppose $f : M \to M$ is a homeomorphism of the surface M which is isotopic to the identity map and preserves a measure μ. The mean rotation vector of f, relative to a Borel subset $K \subset M$, if it exists, is an element of $H_1(M, \mathbb{R})$ denoted $\mathcal{R}_\mu(f, K)$, and is defined by

$$\mathcal{R}_\mu(f, K) = \int_K [h_K(x)] d\mu,$$

when this integral exists. If the integral does not exist then the mean rotation vector relative to K is undefined.

In the case that M is an annulus it is necessary for a lift F of f to be chosen in order for this mean rotation vector to be well defined. In this case will will write $\mathcal{R}_\mu(F, K) = \int_K [h_K(x, F)] d\mu$, with the F added to $[h_K(x, F)]$ to indicate the dependence on this lift.

It is clear from the comments before (2.3) that if $K = M$ this reduces to the usual definition since $h_K(x) = h_1(x)$ in this case. However, the following proposition better illustrates the connection between this definition and (2.3).

(3.2) Proposition. *Suppose f, K, and μ are as in* (3.1) *and $\mathcal{R}_\mu(f)$ exists. If $B = \cup_{n=0}^{\infty} f^n(K)$, then*

$$\mathcal{R}_\mu(f, K) = \int_B \mathcal{R}(x, f) d\mu.$$

Proof. Let $U_n \subset K$ be the set of $x \in K$ such that n is the smallest positive integer with $f^n(x) \in K$. Define a partition of B by

$$V_n = \cup_{j=0}^{n-1} f^j(U_n).$$

The set B is invariant under f modulo a set of measure zero, by the Poincaré recurrence theorem, and $B = \cup_{n=0}^{\infty} V_n$, up to sets of measure zero. We can apply the Birkhoff ergodic theorem to f restricted to B and obtain

$$\int_B \mathcal{R}(x, f) d\mu = \mathcal{R}_\mu(f|B) = \int_B [h_1(x, f)] d\mu.$$

Since the sets V_n are pairwise disjoint and their union is B (up to measure zero), we have

$$\int_B [h_1(x,f)]d\mu = \sum_{n=0}^{\infty} \int_{V_n} [h_1(x,f)]d\mu.$$

Since

$$\sum_{j=0}^{n-1} [h_1(f^j(x))] = [h_n(x)],$$

we conclude

$$\begin{aligned}\int_{V_n} [h_1(x,f)]d\mu &= \sum_{j=0}^{n-1} \int_{f^j(U_n)} [h_1(x,f)]d\mu \\ &= \sum_{j=0}^{n-1} \int_{U_n} [h_1(f^j(x),f)]d\mu \\ &= \int_{U_n} [h_n(x,f)]d\mu.\end{aligned}$$

It follows that

$$\begin{aligned}\int_B \mathcal{R}(x,f)d\mu &= \sum_{n=0}^{\infty} \int_{U_n} [h_n(x,f)]d\mu \\ &= \int_K [h_K(x)]d\mu \\ &= \mathcal{R}_\mu(f,K).\end{aligned}$$

□

The next result shows the existence of many fixed points if there is a *compact* subset K for which $\mathcal{R}_\mu(f,K)$ does not exist.

(3.3) Proposition. *Let M be an open surface of finite type with negative Euler characteristic and genus zero. Suppose $f : M \to M$ is a homeomorphism of the surface M which is isotopic to the identity map and preserves a finite measure μ homeomorphic to Lebesgue measure. If there is a compact subset $K \subset M$ with $\mu(K) > 0$ such that $\mathcal{R}_\mu(f,K)$ fails to exist then f has infinitely many fixed points. In fact f has fixed points in infinitely many Nielsen classes.*

Proof. We first observe that if $\mathcal{R}_\mu(f,K_0)$ fails to exist then $\int_K [h_K(x)]d\mu$ must fail to exist. M is a sphere with at least three punctures. Choose a basis for $H_1(M)$ represented by loops around all but one of the punctures. Since the vector function $[h_K(x)]$ is not integrable it must be the case that one of its component functions with respect to the chosen basis is not integrable. Form an annulus A by filling in all the punctures of M except two – the one without a loop around it representing a basis element, and the one whose loop corresponds to a non-integrable component function of $[h_K(x)]$.

There is a natural extension of f to $f : A \to A$ which fixes the points added at punctures and it is clear that if we consider μ to be a measure on A then this extended f preserves μ. Let p be one of the fixed points added at a puncture and let $F : \widetilde{A} \to \widetilde{A}$ be the lift of f to its universal covering space which fixes $\pi^{-1}(p)$ pointwise.

Note that for the annulus A we can define the loop $h_1(x, F)$ to be the image under the inclusion $i : M \to A$ of the loop $h_1(x)$ in M. We indicate the dependence on F because when the mean rotation number exists for an annulus homeomorphism, it depends on a choice of lift to the universal covering space. In our setting this mean rotation number $\mathcal{R}_\mu(F)$ is equal to $\int [h_1(x, F)] d\mu$ if this integral exists, and we identify $H_1(A, \mathbb{R})$ with $\mathbb{R}$.

Hence we have the measurable function $[h_1(x, F)] \in H_1(A)$ and, as before, we can define $[h_n(x, F)]$ and $[h_K(x, F)]$. We then have $[h_K(x, F)] = i_*([h_K(x)]) \in H_1(A)$, where $i : M \to A$ is the inclusion, so that $\int_K [h_K(x, F)] d\mu$ fails to exist, because i_* is essentially projection onto the component of $H_1(M)$ for which $\int_K [h_K(x)] d\mu$ fails to exist. Since $K \subset M \subset A$ has finite μ measure this implies that $[h_K(x, F)]$ is essentially unbounded.

We can consider the other lifts of $f : A \to A$ to the universal covering space. These are defined by $F_m = T^m \circ F$ where $T : \widetilde{A} \to \widetilde{A}$ is a generator of the group of covering transformations. Recall that the loop $h_n(x, F)$ in A is formed by the paths γ_x and $\gamma_{f^n(x)}$ together with an arc from x to $f^n(x)$ which is the image under $\pi : \widetilde{A} \to A$ of an arc from a point $x_0 \in \pi^{-1}(x)$ to $F^n(x_0)$. We can define a similar loop using the lift F_m instead of F. More precisely let $h_n(x, F_m)$ denote the closed loop in A obtained by concatenating the arc from the basepoint to x followed by an arc from x to $f^n(x)$ which lifts to an arc in $\widetilde{A}$ from y to $F_m^n(y)$ followed by the arc $\gamma_{f^n(x)}$ traced backwards from $f^n(x)$ to the basepoint.

For a full measure subset of the set K we can define $h_K(x, F_m)$ to be $h_n(x, F_m)$, where n is the least positive integer such that $f^n(x) \in K$. The function $[h_K(x, F_m)]$ is then measurable and we will need the fact that it is integrable over K if and only if $h_K(x, F)$ is. To establish this fact we consider the sets $U_n \subset K$ consisting of all x in K with the property that n is the smallest positive integer with $f^n(x) \in K$. Since they form a partition of K up to measure zero, to check integrability over K it suffices to consider it over each U_n and show that the sum over n of the integrals converges absolutely.

We note that

$$\begin{aligned}\int_{U_n} [h_K(x, F_m)] d\mu &= \int_{U_n} [h_n(x, F_m)] d\mu \\ &= \sum_{j=0}^{n-1} \int_{f^j(U_n)} [h_1(x, F_m)] d\mu,\end{aligned}$$

since

$$\sum_{j=0}^{n-1} [h_1(f^j(x), F_m)] = [h_n(x, F_m)].$$

Hence

$$\begin{aligned}
\int_{U_n}[h_K(x,F_m)]d\mu &= \sum_{j=0}^{n-1}\int_{f^j(U_n)}[h_1(x,F_m)]d\mu \\
&= \sum_{j=0}^{n-1}\int_{U_n}[h_1(f^j(x),F_m)]d\mu \\
&= \sum_{j=0}^{n-1}\int_{U_n}([h_1(f^j(x),F)]+m)d\mu \\
&= nm\mu(U_n) + \sum_{j=0}^{n-1}\int_{U_n}[h_1(f^j(x),F)]d\mu \\
&= nm\mu(U_n) + \int_{U_n}[h_n(x,F)]d\mu \\
&= nm\mu(U_n) + \int_{U_n}[h_K(x,F)]d\mu.
\end{aligned}$$

Since the sets $f^j(U_n)$, for $0 \le j \le n-1$ are pairwise disjoint we conclude that the sum $\sum_n n\mu(U_n)$ converges (in fact to the measure of $\cup_{i\ge 0} f^i(K)$). Hence it follows that $\int_K [h_K(x,F)]d\mu$ exists if and only if $\int_K [h_K(x,F_m)]d\mu$ exists. In the instance at hand our hypothesis is that the first of these integrals fails to exist from which we conclude that $\int_K [h_K(x,F_m)]d\mu$ fails to exist for all m. In particular the functions $[h_K(x,F_m)]$ are unbounded. For definiteness we suppose they are unbounded above. The other case can be treated similarly.

We will use this fact to conclude that for infinitely many m the lift $F_m : \widetilde{A} \to \widetilde{A}$ possesses a fixed point. The image of these points in A are all fixed and are all in different Nielsen classes for $f : A \to A$.

Suppose to the contrary that F_m has a fixed point for only finitely many m. This implies that some lift $f_1 : A_1 \to A_1$ of the homeomorphism f to a finite cover is fixed point free. This is because the finite cover A_1 is the quotient of $\widetilde{A}$ by some covering translation T^k and if f_1 is the lift to A_1 corresponding to F_p then a fixed point for f_1 would correspond to a point $z \in \widetilde{A}$ such that $F(z) = T^{ik-p}(z)$ for some i and hence would be a fixed point of F_{-ik+p}. Clearly no such z exists if p is chosen so that F_p has no fixed points and k is chosen so large that F_{jk+p} has no fixed points for any j.

Choose a negative value m_0 less than any m for which F_m has a fixed point and so that m_0 is a multiple of k. Let $G = F_{m_0}$. By Lemma (1.3) (applied to $f_1 : A_1 \to A_1$ which is fixed point free) there is a complete Riemannian metric on A_1 which lifts to a metric on $\widetilde{A}$ with the property that $d(x, G(x)) > 1$ for all $x \in \widetilde{A}$. Let $\varepsilon > 0$ be less than $1/2$. Since K is compact by Proposition (1.2), there is an $N > 0$ with the property that for each $x \in A$ there is a ε-chain of length less than N from the basepoint b_0 to x and a ε-chain of length less than N from x to b_0. Let p be a fixed point of $f : A \to A$, for example one of those which were inserted at a puncture. Then perhaps enlarging N we can assume there is a ε-chain of length less than N from the basepoint b_0 to p and a ε-chain of length less than N from p to b_0.

Choose $b \in \pi^{-1}(b_0)$. If ε is sufficiently small each ε-chain in A will lift to a unique ε-chain in $\widetilde{A}$ for G. For any compact set J in A there is a uniform upper bound, say D, for the distance between the start and end of such a lift of an ε-chain for f of length at most N which begins in J. We will consider the case that $J = K \cup \{p, b_0\}$.

We wish to construct two ε-chains for G: one from b to $T^r(b)$ for $r > 0$ and one from b to $T^s(b)$ for $s < 0$. Let D_0 be an upper bound for the length of the arcs γ_x for $x \in J$. There is a constant $C > 0$ such that every loop in A of length $D + D_0$ represents a homology class with size less than C (if we identify $H_1(A, \mathbb{R})$ with $\mathbb{R}$).

To construct the first of the two desired ε-chains choose a point $x \in K$ for which $[h_K(x, G)] > 3C$, Lift an ε-chain from b_0 to x to one from b to a point $y \in \pi^{-1}(x)$. Now $h_K(x, G) = \pi(G^n(y))$ for some n and we extend the ε-chain by letting it be the G orbit of y from y to $G^n(y)$. Finally we extend the ε-chain further by lifting an ε-chain from $\pi(G^n(y))$ to b_0 to one starting at $G^n(y)$ and ending at a point in $\pi^{-1}(b_0)$, i.e., $T^r(b)$ for some r. Since $[h_K(x, G)] > 3C$ and the two segments of ε-chain on the two ends of the G orbit can each alter this homology class by at most C we can conclude that $r > C > 0$.

The second desired ε-chain is easier to construct. Recall that $G = F_{m_0}$ and $m_0 < 0$ is less than any m for which F_m has a fixed point. Since p is a fixed point for f it follows that if $q \in \pi^{-1}(p)$ then $G(q) = T^a(q)$ for some a. If m_0 is sufficiently negative, $a < 0$. Hence $G^n(q) = T^{an}(q)$ for all n. For n sufficiently large that the integer an is less than $-3C$ we can use the G orbit of a point in $\pi^{-1}(p)$ for an iterates together with ε-chains for G lifted from ε-chains from b_0 to p and from p to b_0, to create a ε-chain from b to $T^s(b)$ where $s < 0$.

Concatenating $|s|$ translates of copies of the first we can construct an ε-chain from b to $T^{r|s|}(b)$. Similarly using r translates of copies of the second we can construct an ε-chain from $T^{r|s|}(b)$ to b. This is then a periodic ε chain for $G : \widetilde{A} \to \widetilde{A}$. It then follows from Lemma (1.5) that we can alter f_1 by an isotopy which can be lifted to $\widetilde{A}$ giving a new map $G' : \widetilde{A} \to \widetilde{A}$ which has a periodic point. Since G' has a periodic point and $\widetilde{A}$ is homeomorphic to $\mathbb{R}^2$ it follows that G' has a fixed point by the Brouwer plane translation theorem ([F5] for example). But this gives rise to a contradiction since (also by Lemma (1.5)) $d(G(x), G'(x)) < 1/2$ for all x so a fixed point z for G' would satisfy

$$d(G(z), z) \le d(G(z), G'(z)) + d(G'(z), z) < 1/2.$$

And this clearly contradicts the property that $d(G(x), x) > 1$ for all x established above.

Thus we have contradicted the assumption that $F_m : \widetilde{A} \to \widetilde{A}$ has a fixed point for only finitely many m. Hence there is an infinite collection of fixed points for these lifts. These fixed points project under $\pi : \widetilde{A} \to A$ to fixed points for f. These fixed points are all in different Nielsen classes for $f : A \to A$ as different lifts of f fix their inverse image under π. Hence they are distinct points. Infinitely many of them must be in $M \subset A$ as $A \setminus M$ is finite. These will be fixed points in distinct Nielsen classes for $f : M \to M$.

□

(3.4) Theorem. *Let M be an open surface of finite type with negative Euler characteristic and genus zero. Suppose $f : M \to M$ is a homeomorphism of the surface M which is isotopic to the identity map and preserves a finite measure μ homeomorphic to Lebesgue measure. If there is a compact subset $K \subset M$ with $\mu(K) > 0$ such that $\mathcal{R}_\mu(f, K)$ is non-zero then f has infinitely many periodic points.*

Proof. The surface M is topologically a sphere with at least three punctures. We want to construct a homeomorphism of an annulus A by compactifying all but two of the punctures, i.e., to remove all but two of the punctures by adding a point at each puncture to be removed. The homeomorphism f can be extended to a homeomorphism g of A by making each of the added points be a fixed point. If P is the set of these added fixed points then $g : A \to A$ is isotopic to the identity relative to the set P. Clearly g preserves a measure μ on A homeomorphic to Lebesgue measure. It suffices to find infinitely many periodic points for the homeomorphism $g : A \to A$.

If K is a compact set with $\mathcal{R}_\mu(f, K)$ non-zero then there is a choice of the set of punctures to fill in with the property that $i_*(\mathcal{R}_\mu(f, K)) \neq 0 \in H_1(A, \mathbb{R})$ where $i : M \to A$ is the inclusion. Moreover $i_*(\mathcal{R}_\mu(f, K)) = \mathcal{R}_\mu(g, K)$. We need to make a comment about the definition of $\mathcal{R}_\mu(g, K)$, since g is defined on the annulus A and there is not a canonical isotopy from g to the identity (which was used in the definition of rotation vector). Clearly the one to use is the extension to A of the isotopy on M from f to the identity, or, in fact, any isotopy relative to P of g to the identity.

Assume now that $\mathcal{R}_\mu(g, K)$ is positive (we identify $H_1(A, \mathbb{R})$ with $\mathbb{R}$). Then since $\mathcal{R}_\mu(g, K) = \int_K [h_K(x)] d\mu > 0$ it follows from the ergodic decomposition theorem that there is a Borel measure ν on A which is ergodic with respect to g and with $\int_K [h_K(x)] d\nu > 0$.

Let $T : K \to K$ be the first return map under g. Then by the Birkhoff ergodic theorem

$$\rho_\nu = \lim_{m \to \infty} \frac{1}{m} \sum_{i=0}^{m-1} [h_K(T^i(x))] \tag{1}$$

exists and is independent of x for ν almost all $x \in K$. Also this theorem asserts that

$$0 < \rho_\nu = \int_K [h_K(x)] d\nu = [h_K(x)],$$

for ν almost all $x \in K$.

Now for each $x \in K$ for which the limit above exists and equals ρ_ν we can consider the integer valued function $n(m)$ defined by $g^{n(m)}(x) = T^m(x)$. It depends on x. Note that if E_K is the characteristic function of K then

$$\sum_{i=0}^{n(m)} E_K(g^i(x)) = \sum_{j=0}^{m} E_K(T^j(x)) = m.$$

Hence

$$\frac{1}{n(m)} \sum_{i=0}^{n(m)} E_K(g^i(x)) = \frac{m}{n(m)}.$$

Another application of the Birkhoff ergodic theorem tells us that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} E_K(g^i(x)) = \nu(K)$$

for ν almost all x and consequently

$$\lim_{m\to\infty} \frac{m}{n(m)} = \nu(K) \tag{2}$$

for ν almost all x. We now define $[h_K(y)]$ to be 0 for $y \notin K$ (it is already defined for $y \in K$) and observe that for a fixed x for which equations (1) and (2) above are valid, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} [h_K(g^i(x))] &= \lim_{m\to\infty} \frac{1}{n(m)} \sum_{i=0}^{m-1} [h_K(T^i(x))] \\
&= \lim_{m\to\infty} \frac{m}{n(m)} \frac{1}{m} \sum_{i=0}^{m-1} [h_K(T^i(x))] \\
&= \nu(K)\rho_\nu > 0
\end{aligned}$$

for ν almost all $x \in A$.

Note that whenever $y, g^j(y) \in K$ and $g^i(y) \notin K$ for $0 < i < j$ we have

$$[h_K(y)] = \sum_{i=0}^{j-1} [h_1(g^i(x))].$$

From this it follows that whenever $x, g^n(x) \in K$ we have

$$\sum_{i=0}^{n-1} [h_K(g^i(x))] = \sum_{i=0}^{n-1} [h_1(g^i(x))].$$

Thus if we define

$$\Phi(x, n, g) = \sum_{i=0}^{n-1} [h_1(g^i(x), g)]$$

and consider any subsequence n_j satisfying $g^{n_j}(x) \in K$ we observe that for ν almost all $x \in A$

$$\lim_{j\to\infty} \frac{1}{n_j} \Phi(x, n_j, g) = \nu(K)\rho_\nu.$$

Of course the subsequences $\{n_j\}$ would be different for different values of x, but such subsequences exist for ν almost all values of x.

There is a geometric interpretation of the quantity $\Phi(x, n, g)$. Let $G : \widetilde{A} \to \widetilde{A}$ be the lift of G to its universal covering space obtained by lifting our chosen isotopy from g to the identity, i.e., G is the lift which fixes the lifts of the puncture points we added to M to get A. Then $\Phi(x, n, g)$ is equal to $(G^n(x_0) - x_0)_1$ where the subscript 1 indicates the first component in $\widetilde{A} = \mathbb{R} \times (0,1)$ and x_0 is a lift of the point x.

Suppose now that p is a positive integer such that $p\nu(K)\rho_\nu > 1$ and $\{n_j\}$ is a sequence satisfying $g^{pn_j}(x) \in K$. Then

$$\lim_{j\to\infty} \frac{1}{n_j}\Phi(x, n_j, g^p) = \lim_{j\to\infty} p\frac{1}{pn_j}\Phi(x, pn_j, g) = p\nu(K)\rho_\nu > 1.$$

Hence

$$\lim_{j\to\infty} \frac{1}{n_j}(G^{pn_j}(x_0) - x_0)_1 = p\nu(K)\rho_\nu > 1.$$

If we define $H : \widetilde{A} \to \widetilde{A}$ to be the lift of g^p given by $H(t,s) = G^p(t,s) - (1,0)$ for $(t,s) \in \mathbb{R} \times (0,1) = \widetilde{A}$ then

$$\lim_{j\to\infty} \frac{1}{n_j}(H^{n_j}(x_0) - x_0)_1 = p\nu(K)\rho_\nu - 1 > 0. \tag{3}$$

If $z_0 = (t_0, s_0)$ is one of the fixed points of G in $\widetilde{A}$ then $H(t_0, s_0) = (t_0 - 1, s_0)$. Thus any sufficiently small disk around z_0 will be a negatively returning disk for H. Also if x was chosen to be a recurrent point for g^p and the sequence n_j chosen so that $\lim g^{pn_j}(x) = x$ then it follows from equation (3) that any sufficiently small disk containing x_0 will be a positively returning disk for H.

It then follows from Theorem (1.4) that H has a fixed point and hence that g has a periodic point. Clearly the rotation number with respect to g of this periodic point is $1/p$. Since p was any sufficiently large positive integer we conclude that g has infinitely many distinct periodic points.

□

The mean rotation relative to a compact set K satisfies a result analogous to Proposition (2.4), if we assume that one of the homeomorphisms is supported on K.

(3.5) Proposition. *Suppose f and ϕ are homeomorphisms of M which are isotopic to the identity and preserve a finite measure μ homeomorphic to Lebesgue measure. Suppose that K is a compact subset of M with $\mu(K) > 0$ and $\phi(x) = x$ for all $x \in M \setminus K$. Then*

$$\mathcal{R}_\mu(\phi \circ f, K) = \mathcal{R}_\mu(\phi, K) + \mathcal{R}_\mu(f, K),$$

if the two integrals in the right hand side of this equation exist.

Proof. Let $g : K \to K$ be the first return map with respect to f. It is defined for a subset of K of full measure. Then the first return map for $\phi \circ f$ is $\phi \circ g$ since ϕ is the identity outside K and $\phi(K) = K$.

Note that if $x \in K$ and n is the smallest positive integer such that $f^n(x) \in K$, (i.e., if $f^n(x) = g(x)$) then

$$(\phi \circ f)^n(x) = \phi \circ f^n(x) = \phi \circ g(x).$$

The loop $h_K(x, \phi \circ f)$ is equal to $h_n(x, \phi \circ f)$. It is homotopic to the concatenation of the loops $h_n(x, f)$ and $h_1(f^n(x), \phi)$, which is the same as the concatenation of the loops $h_K(x, f)$ and $h_1(g(x), \phi)$.

Thus $[h_K(x, \phi \circ f)] = [h_K(x, f)] + [h_1(g(x), \phi)]$, and

$$\int_K [h_K(x, \phi \circ f)] d\mu = \int_K [h_K(x, f)] d\mu + \int_K [h_1(g(x), \phi)] d\mu.$$

Since g preserves μ, we can conclude that $\int_K [h_1(g(x), \phi)] d\mu = \int_K [h_1(x, \phi)] d\mu$. Also $[h_1(x, \phi)] = [h_K(x, \phi)]$, since the first return map for K under ϕ is ϕ itself. Hence

$$\int_K [h_K(x, \phi \circ f)] d\mu = \int_K [h_K(x, f)] d\mu + \int_K [h_K(x, \phi)] d\mu,$$

or

$$\mathcal{R}_\mu(\phi \circ f, K) = \mathcal{R}_\mu(\phi, K) + \mathcal{R}_\mu(f, K).$$

$\square$

4. The Open Annulus

In this section we prove our main result, Theorem (4.4) below, that an area preserving homeomorphism of the open annulus which is isotopic to the identity and has at least one fixed point must have infinitely many.

We will consider the vector space $H_1(M, \mathbb{R})$ and arbitrarily choose a norm $\| \ \|$ on it. If $\widetilde{M}$ is the universal covering space of M we denote by $F : \widetilde{M} \to \widetilde{M}$ the canonical lift of f which fixes the "ends at infinity" of $\widetilde{M}$. We will identify $\Pi_1(M)$ with the group of covering transformations for the universal cover. For any element $\alpha \in \Pi_1(M)$ we will denote the homology class it determines by $[\alpha]$.

(4.1) Lemma. *Let M be a surface obtained by deleting k points from a sphere where $k > 2$. Let μ be a finite measure homeomorphic to Lebesgue measure on M. Suppose $f : M \to M$ is a homeomorphism which is isotopic to the identity and preserves the measure μ. Suppose γ is an oriented embedded simple closed curve in M. Let $\pi : \widetilde{M} \to M$ be the universal covering of M and let $x_0 \in \widetilde{M}$. Given $\varepsilon, \delta > 0$ then either*

i) *f has a periodic point, or*
ii) *there exists a positive integer P, an element $\alpha \in \Pi_1(M)$, and an ε-chain for $F : \widetilde{M} \to \widetilde{M}$ from x_0 to $\alpha(x_0)$ satisfying*

$$\left\| [\gamma] - \frac{[\alpha]}{P} \right\| < \delta.$$

The ε-chain can be with respect to any metric on $\widetilde{M}$ obtained by lifting a complete Riemannian metric on M.

Proof. Let K be an annulus which is a tubular neighborhood of the embedded curve γ. Construct a μ preserving flow ϕ_t on M supported in the interior of K and with each non-trivial orbit a circle "parallel" to γ and oriented the same as γ. The existence of such a flow is a consequence of the fact that up to homeomorphism μ on K is equivalent to Lebesgue measure on a standard annulus of the same area. It is then clear from the definition that $\mathcal{R}_\mu(\phi_t) = \mathcal{R}_\mu(\phi_t, K) \in H_1(M, \mathbb{R})$ is equal to $tk[\gamma]$ for some positive constant k.

We note that by Proposition (3.3), if $\mathcal{R}_\mu(f, K)$ does not exist, then f has infinitely many fixed points so option i) of our conclusion holds. Hence we may assume that $\mathcal{R}_\mu(f, K)$ exists. Then by Theorem (3.4), if $\mathcal{R}_\mu(f, K) \neq 0$ it follows that f has infinitely many periodic points so again option i) holds. Hence without loss of generality we may assume that $\mathcal{R}_\mu(f, K) = 0$.

Choose $s > 0$ sufficiently small that $d(\phi_s(x), x) < \varepsilon/3$ for all $x \in M$. Note $\mathcal{R}_\mu(\phi_s) = \mathcal{R}_\mu(\phi_s, K) = sk[\gamma]$.

By Proposition (3.5), if $g = \phi_s \circ f$, then $\mathcal{R}_\mu(g, K) = sk[\gamma] + \mathcal{R}_\mu(f, K) = sk[\gamma]$. Also $d(g(x), f(x)) = d(\phi_s(f(x)), f(x)) < \varepsilon/3$. In other words any orbit segment for g is a ε-chain for f.

The remainder of the argument is almost the same as the proof of (3.2) of [F4]. We give it for completeness and because the minor but frequent changes in the proof would be difficult to enumerate. If we let T denote the first return map for K under g and define

$$\mathcal{R}_K(x, g) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} [h_K(T^i(x))]$$

then the Birkhoff ergodic theorem asserts that $\mathcal{R}_K(x, g)$ exists for μ almost all $x \in K$ and

$$\int_K \frac{\mathcal{R}_K(x, g)}{sk\mu(K)} d\mu = [\gamma].$$

This integral (like any integral) can be approximated by a weighted average of values of the integrand. That is, there exist m points $x_1, x_2, \ldots, x_m \in M$ and positive constants b_i such that

$$\left\| [\gamma] - \sum_{i=1}^{m} b_i \mathcal{R}_K(x_i, g) \right\| < \frac{\delta}{2}. \tag{1}$$

We can, of course, assume that each b_i is rational. Moreover, we can assume that each x_i is recurrent under T (and hence g) since by the Poincaré recurrence theorem this is true for a set of full measure in K.

It follows then that for any $y_i \in \pi^{-1}(x_i)$ there is an arbitrarily long ε-chain for F from y_i to some translate $\alpha_i(y_i)$, $\alpha_i \in \Pi_1(M)$. This ε-chain is obtained by lifting g to $G : \widetilde{M} \to \widetilde{M}$ and defining the ε-chain to be $y_i, G(y_i), G^2(y_i), \ldots, G^{k-1}(y_i), \alpha_i(y_i)$, where k is chosen so that $d(g^k(x_i), x_i) < \varepsilon/6$ and the last point on the chain, $\alpha_i(y_i)$, is within $\varepsilon/6$ of $G^k(y_i)$ and so that $g^k(x_i) \in K$ so $g^k(x_i) = T^j(x_i)$ for some j. We denote this ε-chain by $\mathcal{D}_i$.

If j (and hence k, the length of $\mathcal{D}_i$) is sufficiently large then

$$\left\| \frac{[\alpha_i]}{j} - \mathcal{R}_K(x_i, g) \right\| < \frac{\delta}{2mb_i}.$$

If we concatenate $\mathcal{D}_i$ with an appropriate translate of itself n times we obtain an ε-chain for F, denoted $\mathcal{D}_i(n)$, from y_i to $\alpha_i^n(y_i)$, with length nk and such that

$$\left\| \frac{[\alpha_i^n]}{jn} - \mathcal{R}_K(x_i, g) \right\| < \frac{\delta}{2mb_i}$$

or,

$$\left\| \frac{q_i[\alpha_i^n]}{len(D_i(n))} - \mathcal{R}_K(x_i, g) \right\| < \frac{\delta}{2mb_i}, \tag{2}$$

where $q_i = k/j$.

If x_0 is any point in K then by (1.2) there is an ε-chain for f from x_0 to x_i and one from x_i to x_0. Choosing lifts of these to ε-chains for F in $\widetilde{M}$ we can obtain an ε-chain from some point $z_i \in \pi^{-1}(x_0)$ to y_i and from $\alpha_i^n(y_i)$ to some covering translate of z_i. Since the choice of the starting point $y_i \in \pi^{-1}(x_i)$ of $\mathcal{D}_i$ and $\mathcal{D}_i(n)$ was arbitrary, we can assume they were chosen so that $z_i = y_0$ for some $y_0 \in \pi^{-1}(x_0)$ independent of i. If we concatenate the ε-chain from y_0 to y_i with $\mathcal{D}_i(n)$ and then with the ε-chain which is a lift of the one for f from x_i to x_0 we obtain an ε-chain for F which we denote $\mathcal{C}_i(n)$, from y_0 to a translate of y_0 (say by the element $\beta_i(n) \in \Pi_1(M)$), with several desirable properties. First $len(\mathcal{C}_i(n)) - len(\mathcal{D}_i(n))$ is bounded above by a constant independent of n and i, since this difference is just the sum of the lengths of the ε-chain for f from x_0 to x_i and the one from x_i to x_0. Thus from the definition of $\beta_i(n)$ it follows that $\|[\beta_i(n)] - [\alpha_i^n]\|$ has an upper bound independent of n and i. From these two facts it follows that

$$\lim_{n\to\infty} \frac{[\beta_i(n)]}{len(C_i(n))} = \lim_{n\to\infty} \frac{[\alpha_i^n]}{len(D_i(n))}.$$

It then follows from (2) that if we choose n_0 sufficiently large, then

$$\left\| \frac{[q_i\beta_i(n_0)]}{len(\mathcal{C}_i(n_0))} - \mathcal{R}_K(x_i, g) \right\| < \frac{\delta}{2mb_i}, \tag{3}$$

for all $1 \le i \le m$.

Let P be a large integer chosen so that $B_i = Pb_iq_i/len(\mathcal{C}_i(n_0))$ is an integer for $1 \le i \le m$. Then multiplying the inequality (3) by Pb_i gives

$$\|B_i[\beta_i(n_0)] - Pb_i\mathcal{R}_K(x_i, g)\| < \frac{P\delta}{2m},$$

for all $1 \le i \le m$.

Hence

$$\left\| \sum_{i=1}^m B_i[\beta_i(n_0)] - P\sum_{i=1}^m b_i\mathcal{R}_K(x_i, g) \right\| < \frac{P\delta}{2}. \tag{4}$$

If we concatenate each of the ε-chains $\mathcal{C}_i(n_0)$ with themselves B_i times and concatenate the resulting ε-chains in order as indexed by i, we obtain an ε-chain for F from y_0 to $\alpha(y_0)$ where

$$\alpha = \Pi_{i=1}^m \beta_i(n_0)^{B_i},$$

so $[\alpha] = \sum_{i=1}^{m} B_i[\beta_i(n_0)]$. Dividing equation (4) by P we get

$$\left\| \frac{[\alpha]}{P} - \sum_{i=1}^{m} b_i \mathcal{R}_K(x_i, g) \right\| < \frac{\delta}{2},$$

which together with equation (1) implies

$$\left\| [\gamma] - \frac{[\alpha]}{P} \right\| < \delta.$$

□

(4.2) Lemma. *Let M be a surface obtained by deleting k points from a sphere where $k > 2$. Let μ be a finite measure homeomorphic to Lebesgue measure on M. Suppose $f : M \to M$ is a homeomorphism which is isotopic to the identity and preserves the measure μ. Let $\widetilde{M}$ be the universal covering space of M and let $x_0 \in \widetilde{M}$. Given $\varepsilon, \delta > 0$, then either*

i) *f has a periodic point, or*
ii) *there exists an element $\alpha \in \Pi_1(M)$, and an ε-chain for $F : \widetilde{M} \to \widetilde{M}$ from x_0 to $\alpha(x_0)$ satisfying $[\alpha] = 0 \in H_1(M)$. The ε-chain can be with respect to any metric on $\widetilde{M}$ obtained by lifting a complete Riemannian metric on M.*

Proof. Choose a set of oriented embedded simple closed curves $\{\gamma_i\}$ such that 0 is in the interior of the convex hull of $\{[\gamma_i]\}$ in $H_1(M, \mathbb{R})$. By Lemma (4.1), applied to each γ_i, either f has a periodic point or there exist P_i and α_i with an ε-chain for F from x_0 to $\alpha_i(x_0)$ such that 0 is in the interior of the convex hull of $\{[\alpha_i]/P_i\}$ in $H_1(M, \mathbb{R})$. From this it follows that 0 is in the interior of the convex hull of $\{[\alpha_i]\}$ or equivalently there are positive integers n_i such that

$$\sum_i n_i[\alpha_i] = 0.$$

We now concatenate n_1 translates of the ε-chain corresponding to α_1 to give an ε-chain from x_0 to $\alpha_1^{n_1}(x_0)$. We follow this with n_2 translates of the ε-chain corresponding to α_2 (starting at $\alpha_1^{n_1}(x_0)$ and ending at $\alpha_2^{n_2}\alpha_1^{n_1}(x_0)$) etc. When this is all done we have a ε-chain from x_0 to $\alpha(x_0)$ where

$$\alpha = \Pi_i \alpha_i^{n_i}$$

and hence

$$[\alpha] = \sum_i n_i[\alpha_i] = 0.$$

□

We can now state and prove our main result.

(4.4) Theorem. *Let μ be a finite measure homeomorphic to Lebesgue measure on the open annulus $A = S^1 \times (0,1)$. Suppose $f : A \to A$ is a homeomorphism which preserves the measure μ. If f has one periodic point then in fact it has infinitely many periodic points.*

Proof. Since some power of f will be isotopic to the identity and proving this result for f is the same as proving it for some positive iterate, we can assume without loss of generality that f is isotopic to the identity map on A.

We will argue by contradiction, assuming f has finitely many periodic points and then proving the existence of at least one more. If f has finitely many periodic points then there is a positive integer n such that f^n has finitely many fixed points (at least one) and no other periodic points.

Since proving our theorem is equivalent to proving it for f^n we can, without loss of generality, assume that f has a finite, non-empty fixed point set and no other periodic points.

Let M denote A with all these fixed points deleted. M is topologically a sphere with finitely many (at least three) punctures. Both the measure μ and the homeomorphism f restrict to M and will be denoted in the same way they were for A. Clearly any periodic points of $f : M \to M$ are periodic points for $f : A \to A$.

It follows from Proposition (3.1) of [F4] that there is an m so that $f^m : M \to M$ is isotopic to the identity. So, again replacing f with f^m, we may assume without loss of generality that f is isotopic to the identity on M.

Using Lemma (1.3), choose a Riemannian metric so that $d(F(x), x) > 1$ for all x and choose $\varepsilon < 1$, where $F : \widetilde{M} \to \widetilde{M}$ is the canonical lift of f to its universal cover.

By Lemma (4.2) either f has a periodic point (a contradiction) or for this Riemannian metric on M and this ε there exists an element $\alpha \in \Pi_1(M)$, and an ε-chain for $F : \widetilde{M} \to \widetilde{M}$ from x_0 to $\alpha(x_0)$ satisfying $[\alpha] = 0 \in H_1(M)$. We will show that this also leads to a contradiction.

Clearly the projection of the ε-chain for F into M via the covering map gives an ε-chain for $f : M \to M$. Applying Lemma (1.5) to this, we obtain a new map $g = h_1 \circ \tilde{f}$ with the property that the canonical lift $G : \widetilde{M} \to \widetilde{M}$ satisfies $G^n(y_0) = \alpha(y_0)$ for some $n > 0$ and $y_0 \in \widetilde{M}$ near to x_0.

This implies that y, the projection of y_0 in M, is a periodic point of of g of period n. The homeomorphism G has no fixed points since by the triangle inequality

$$d(G(z), z) \geq d(F(z), z) - d(G(z), F(z)) \geq 1 - \varepsilon > 0$$

for all $z \in \widetilde{M}$. Also since $[\alpha] = 0 \in H_1(M)$ it follows that the homological rotation vector $\mathcal{R}(y, 0) = 0 \in H_1(M, \mathbb{R})$. But this contradicts Proposition (2.5). $\square$

Theorem (4.4) was claimed in [F4] for both the open and closed annulus. But the proof given there contained a gap in the case of the open annulus. It was valid for the closed annulus. The error in that proof was the assumption that the mean rotation vector for an annulus homeomorphism always exists (this assumption is true for the closed annulus). The author also wishes to take this opportunity to point out that the same error occurs in Corollary (2.3) of [F2] which claims that given any area preserving homeomorphism of the open annulus isotopic to the identity there is a dense set of rigid rotations such that the given composition of one of these rotations followed by the given homeomorphism will have a periodic point. The proof given is valid only if the mean rotation vector of the given homeomorphism exists and the validity of the general case is unknown to the author.

References

[C] C. Conley *Isolated Invariant Sets and the Morse index*, C.B.M.S. Regional Conference Series in Math. **38** (1978 Amer. Math. Soc. Providence, RI).

[BH] M. Bestvina and Michael Handel, personal communication.

[F1] J. Franks, *Generalizations of the Poincaré–Birkhoff Theorem*, Annals of Math. **128** (1988), 139–151.

[F2] J. Franks, *Rotation numbers for area preserving homeomorphisms of the open annulus*, Proceedings of the International Conference Dynamical Systems and Related Topics (K Shiraiwa, ed.), World Scientific, Singapore, 1995, pp. 123–128.

[F3] J. Franks, *A new proof of the Brouwer Plane Translation Theorem*, Ergodic Theory and Dynamical Systems **12** (1992), 217–226.

[F4] J. Franks, *Geodesics on S^2 and periodic points of annulus homeomorphisms*, Invent. Math. **108** (1992), 403–418. See http://math.nwu.edu/preprints/franks/geodesic/index.html.

[F5] J. Franks, *Rotation vectors and fixed points of area preserving surface diffeomorphisms,* http://math.nwu.edu/preprints/franks/rotvec/index.html, to appear in Trans. Amer. Math. Soc.

[H1] Michael Handel, *A fixed point theorem for planar homeomorphisms*, Preprint.

Department of Mathematics, Northwestern University, Evanston, IL 60208-2730
john@math.nwu.edu

Typeset by $\mathcal{AMS}$-TEX