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Heinrich's Counterexample to Azevedo's Conjecture

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Abstract. Johannes Heinrich, a student of the University of Saarbrücken, found by computer calculation an example of a plane algebroid curve whose module of differentials has a bigger torsion than that of the canonical branch in the same equisingularity class. That contradicts a conjecture of Azevedo.

Consider an irreducible plane algebroid curve $\mathcal{C}$ over an algebraically closed field k of characteristic zero. $\mathcal{C}$ has a local ring $\mathfrak{o} = k[\![x, y]\!] = k[\![X, Y]\!]/\big(f(X, Y)\big)$ where $f(X, Y)$ is an irreducible power series with coefficients in k. Let $\Omega(\mathfrak{o}/k)$ denote the universally finite differential module of $\mathfrak{o}$ over k, $T = \tau\big(\Omega(\mathfrak{o}/k)\big)$ its torsion submodule, and $\bar{\mathfrak{o}} = k[\![t]\!]$ the integral closure of $\mathfrak{o}$.

In his paper [5] Zariski shows that for the lengths as $\mathfrak{o}$-modules one has $\ell(T) \leq 2{\cdot}\ell(\bar{\mathfrak{o}}/\mathfrak{o})$, and *equality holds if and only if*, after a suitable change of the variables, the curve $\mathcal{C}$ can be represented by an equation $f(X, Y) = Y^n - X^m$ with $\gcd(m, n) = 1$. We say in this case that "$\mathcal{C}$ has maximal torsion". (For a generalization of this notion to not necessarily plane curves see [4].) Equivalently this means, that $\mathcal{C}$ has a parametric representation of the form

$$x = t^n, \quad y = t^m \ .$$

Now any non regular plane algebroid curve $\mathcal{C}$ has a parametric representation of the form

(1)

$$\begin{aligned} x &= t^n \\ y &= \sum_{i=m}^{\infty} a_i t^i, \quad a_m \neq 0, \qquad n < m, \quad n \nmid m \ , \end{aligned}$$

the *Puiseux Expansion* of $\mathcal{C}$.
Another way of writing this is:

$$y = \sum_{i=m}^{\infty} a_i x^{\frac{i}{n}} \ . \tag{2}$$

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Among the exponents $\frac{i}{n}$ in (2) certain exponents ε_k are called the *characteristic exponents* of $\mathcal{C}$. They are defined as follows ([6]):

$\varepsilon_1 = \frac{m}{n}$.

If $(m, n) = 1$, this is the only characteristic exponent.

In any case write $\frac{m}{n} = \frac{m_1}{n_1}$ with $(m_1, n_1) = 1$.

Since $k((t)) = k((x, y))$ the greatest common divisor of n and all the i with $a_i \neq 0$ is 1. Therefore, if $n_1 < n$ there is a first i with $a_i \neq 0$ such that $\frac{i}{n}$ cannot be written as a fraction with denominator n_1.

Define:

$\varepsilon_2 = \frac{i}{n}$ is the first exponent in (2) with $a_i \neq 0$ and which cannot be written as a fraction with denominator n_1.

Write $\varepsilon_2 = \frac{m_2}{n_1 n_2}$ with $(m_2, n_2) = 1$.

As before, if $n_1 n_2 < n$ there exists a first i with $a_i \neq 0$ and such that $\frac{i}{n}$ cannot be written as a fraction with denominator $n_1 n_2$.

Define:

$\varepsilon_3 = \frac{i}{n}$ is the first exponent in (2) with $a_i \neq 0$ and which cannot be written as a fraction with denominator $n_1 n_2$.

Write $\varepsilon_3 = \frac{m_3}{n_1 n_2 n_3}$ with $(m_3, n_3) = 1$.

etc.

After finitely many steps the process stops with an exponent

$\varepsilon_g = \frac{m_g}{n_1 n_2 n_3 \cdots n_g}$ with $(m_g, n_g) = 1$ and $n_1 n_2 n_3 \cdots n_g = n$.

Putting in evidence the characteristic exponents one can rewrite the expansion (2) in the form

$$\begin{aligned} y &= b_1 x^{\frac{m_1}{n_1}} + \cdots + b_2 x^{\frac{m_2}{n_1 n_2}} + \cdots + b_3 x^{\frac{m_3}{n_1 n_2 n_3}} + \cdots + b_g x^{\frac{m_g}{n_1 n_2 n_3 \cdots n_g}} + \ldots, \\ & b_k \neq 0, \quad k = 1, \ldots, g \ . \end{aligned}$$

Note that the characteristic exponents ε_k uniquely determine the pairs (m_k, n_k) and vice versa.

The pairs (m_k, n_k), $k = 1 \ldots g$ are called the *characteristic pairs* of the curve $\mathcal{C}$.

Two plane algebroid curves over k are called *equisingular* if and only if they have the same characteristic exponents (or characteristic pairs). Equisingularity obviously is an equivalence relation.

Of all curves in an equisingularity class (i.e. with a given set of characteristic exponents $\varepsilon_1 \ldots \varepsilon_g$) there is one called the *canonical branch* of this class:

$$y = x^{\varepsilon_1} + x^{\varepsilon_2} + \cdots + x^{\varepsilon_g}$$

or in parametric expansion:

$$\begin{aligned} x &= t^n \\ y &= t^{\beta_1} + t^{\beta_2} + t^{\beta_3} + \cdots + t^{\beta_{g-1}} + t^{\beta_g} \end{aligned}$$

with $\beta_i := m_i n_{i+1} \cdots n_g$ for $i = 1, \ldots, g$. Now the theorem of Zariski cited above can be reformulated as follows

An irreducible singular algebroid curve $\mathcal{C}$ has maximal torsion (i.e. $\ell(T) = 2\ell(\bar{o}/o)$) *if and only if it is a canonical branch with exactly one characteristic pair.*

In his thesis [1] Azevedo *conjectures* the following generalization of this result:

Of all irreducible plane algebroid curves in an equisingularity class the canonical *branch has the biggest torsion*, i.e. if $\mathcal{C}$ is an arbitrary irreducible plane algebroid curve and $\mathcal{C}'$ is the canonical branch in the equisingularity class of $\mathcal{C}$ then for the torsions T and T' of the differential modules of $\mathcal{C}$ and $\mathcal{C}'$ respectively holds $\ell(T) \leq \ell(T')$.

In [1] this conjecture is backed up by examples and also proved for certain curves with two characteristic pairs. There is also an example in [1] that, contrary to the case of only one characteristic pair, other curves in the same equisingularity class can have the same length of the torsion of the differential module as the canonical branch.

Now Heinrich shows in his Diplomarbeit [3] that **the above conjecture is false** by giving a counterexample:

Example: *Let $\mathcal{C}$ be the curve :*

$$\begin{aligned} x &:= t^6 \\ y &:= t^9 + 2t^{10} - 2t^{11}. \end{aligned}$$

Then the canonical branch $\mathcal{C}'$ in the class of $\mathcal{C}$ is

$$\begin{aligned} x &:= t^6 \\ y &:= t^9 + t^{10}. \end{aligned}$$

But for the torsions T and T' of $\mathcal{C}$ and $\mathcal{C}'$ respectively one has:

$$\ell(T) = 36 > 35 = \ell(T').$$

Proof ([3])**:** Let us denote by:
ν the natural valuation of $\mathrm{Quot}(\mathfrak{o})$,
$\nu(\mathfrak{o})$ the value semigroup of $\mathfrak{o}$,
$\mathfrak{c}$ the conductor of $\bar{\mathfrak{o}}$ into $\mathfrak{o}$,
$c := \nu(\mathfrak{c})$ the value of the conductor,
D the universally finite derivation of $\bar{\mathfrak{o}}$ over k.
In the usual manner we extend the valuation ν to a valuation of $\bar{\mathfrak{o}}\ \mathrm{D}\, \bar{\mathfrak{o}} = \left\{ \sum_{i=0}^{\infty} a_i \cdot t^i\ \mathrm{D}\, t \mid a_i \in k \right\}$ by

$$\nu\left(\sum_{i=0}^{\infty} a_i \cdot t^i\ \mathrm{D}\, t\right) := \nu\left(\sum_{i=0}^{\infty} a_i \cdot t^i\right).$$

Then obviously for every $z \in \bar{\mathfrak{m}} := \bar{\mathfrak{o}} \cdot t$ we have $\nu(\mathrm{D}\, z) = \nu(z) - 1$.

By Korollar 2 to Satz 8 of [2], which is valid also in the algebroid case, one has for every plane algebroid curve:

$$\ell(T) = \ell(\bar{\mathfrak{o}}\ \mathrm{D}\, \bar{\mathfrak{o}} / \mathfrak{o}\ \mathrm{D}\, \mathfrak{o}) + \ell(\bar{\mathfrak{o}}/\mathfrak{o}).$$

Now for any $\mathfrak{o}$-module M of finite length we have $\ell(M) = \dim_k(M)$, since k is algebraically closed. Therefore $\ell(\bar{\mathfrak{o}}/\mathfrak{o}) = \frac{1}{2} \cdot \ell(\bar{\mathfrak{o}}/\mathfrak{c}) = \frac{1}{2} \cdot \dim_k(\bar{\mathfrak{o}}/\mathfrak{c}) = \frac{1}{2} \cdot c$.
So we get:

$$\ell(T) = \dim_k\left(\bar{\mathfrak{o}}\ \mathrm{D}\, \bar{\mathfrak{o}} / \mathfrak{o}\ \mathrm{D}\, \mathfrak{o}\right) + \frac{c}{2}\ .$$

Because of $\mathfrak{o} \supseteq \mathfrak{c}$ we have the inclusions $\bar{\mathfrak{o}}\,\mathrm{D}\,\bar{\mathfrak{o}} \supseteq \mathfrak{o}\,\mathrm{D}\,\mathfrak{o} \supseteq \mathrm{D}\,\mathfrak{c}$ and therefore

$$\bar{\mathfrak{o}}\,\mathrm{D}\,\bar{\mathfrak{o}}/\mathfrak{o}\,\mathrm{D}\,\mathfrak{o} \cong (\bar{\mathfrak{o}}\,\mathrm{D}\,\bar{\mathfrak{o}}/\,\mathrm{D}\,\mathfrak{c}) \Big/ (\mathfrak{o}\,\mathrm{D}\,\mathfrak{o}/\,\mathrm{D}\,\mathfrak{c}) \ .$$

From $\mathfrak{c} = \{z \mid z \in \bar{\mathfrak{o}}, \nu(z) \geq c\}$ we obtain

$$\mathrm{D}\,\mathfrak{c} = \left\{ \Big(\sum_{i=c-1}^{\infty} a_i \cdot t^i \Big) \cdot \mathrm{D}\,t \;\Big|\; a_i \in k \right\} = \bar{\mathfrak{o}} \cdot t^{c-1}\,\mathrm{D}\,t \ .$$

It follows that the residue classes modulo $\bar{\mathfrak{o}} \cdot t^{c-1}\,\mathrm{D}\,t$

$$\left\{ \overline{\mathrm{D}\,t}, \overline{t\,\mathrm{D}\,t}, \overline{t^2\,\mathrm{D}\,t} \ldots, \overline{t^{c-2}\,\mathrm{D}\,t} \right\}$$

are a basis of $\bar{\mathfrak{o}}\,\mathrm{D}\,\bar{\mathfrak{o}}/\,\mathrm{D}\,\mathfrak{c}$ as a k-vector space, and therefore

$$\dim_k (\bar{\mathfrak{o}}\,\mathrm{D}\,\bar{\mathfrak{o}}/\,\mathrm{D}\,\mathfrak{c}) = c - 1 \ .$$

On the other hand the residue classes modulo $\bar{\mathfrak{o}} \cdot t^{c-1}\,\mathrm{D}\,t$

$$\left\{ \overline{x^i y^j\,\mathrm{D}\,x} \mid i,j \in \mathbb{N}_0,\ \nu(x^i y^j\,\mathrm{D}\,x) < c-1 \right\} \cup \left\{ \overline{x^i y^j\,\mathrm{D}\,y} \mid i,j \in \mathbb{N}_0,\ \nu(x^i y^j\,\mathrm{D}\,y) < c-1 \right\}$$

form a set of generators of $\mathfrak{o}\,\mathrm{D}\,\mathfrak{o}/\,\mathrm{D}\,\mathfrak{c}$ as a k-vector space.
Denote these generators by $\omega_1, \ldots, \omega_s$ in any order, represent them by the above basis of $\bar{\mathfrak{o}}\,\mathrm{D}\,\bar{\mathfrak{o}}/\,\mathrm{D}\,\mathfrak{c}$:

$$\omega_i = \sum_{j=0}^{c-2} a_{ji} \cdot \overline{t^j\,\mathrm{D}\,t}, \qquad a_{ji} \in k, \qquad i = 1, \ldots, s \ ,$$

and define

$$\mathcal{A} = \Big(a_{ji} \Big)_{\left\{ {j=0,\ldots,c-2 \atop i=1,\ldots,s} \right\}} \ .$$

Then

$$\dim_k (\bar{\mathfrak{o}}\,\mathrm{D}\,\bar{\mathfrak{o}}/\mathfrak{o}\,\mathrm{D}\,\mathfrak{o}) = \dim_k (\bar{\mathfrak{o}}\,\mathrm{D}\,\bar{\mathfrak{o}}/\,\mathrm{D}\,\mathfrak{c}) - \dim_k (\mathfrak{o}\,\mathrm{D}\,\mathfrak{o}/\,\mathrm{D}\,\mathfrak{c}) = c - 1 - \operatorname{rank} \mathcal{A},$$

and consequently

$$\ell(T) = \frac{3c}{2} - 1 - \operatorname{rank} \mathcal{A} \ . \qquad (3)$$

We will now compute $\ell(T)$ and $\ell(T')$ for the example using formula (3):
Both curves have the same characteristic pairs

$$(m_1, n_1) = (3, 2) \quad (m_2, n_2) = (10, 3)$$

and consequently the same value group. We obtain $\beta_1 = m = 9$, $\beta_2 = m_2 = 10$. In order to compute c we use formula (3.14) from Chapter II of [7]:

$$c = \beta_2(e_1 - 1) + \beta_1(n - e_1) - n + 1 \ ,$$

where $e_1 := \gcd(n, m) = 3$. (Loc. cit. definition 3.2.) So we get:

$$c = 10(3-1) + 9(6-3) - 6 + 1 = 42 \ .$$

For both curves one has $\nu(x) = 6$, $\nu(y) = 9$, $\nu(\mathrm{D}\,x) = 6$, $\nu(\mathrm{D}\,y) = 9$. Therefore the set $\{\omega_1, \ldots, \omega_s\}$ for both curves consists of the following elements:

$$
\begin{array}{llll}
\omega_1 = \overline{\mathrm{D}\,x}, & \omega_2 = \overline{y\,\mathrm{D}\,x}, & \omega_3 = \overline{y^2\,\mathrm{D}\,x}, & \omega_4 = \overline{y^3\,\mathrm{D}\,x}, \\
\omega_5 = \overline{x\,\mathrm{D}\,x}, & \omega_6 = \overline{xy\,\mathrm{D}\,x}, & \omega_7 = \overline{xy^2\,\mathrm{D}\,x}, & \omega_8 = \overline{xy^3\,\mathrm{D}\,x}, \\
\omega_9 = \overline{x^2\,\mathrm{D}\,x}, & \omega_{10} = \overline{x^2y\,\mathrm{D}\,x}, & \omega_{11} = \overline{x^2y^2\,\mathrm{D}\,x}, & \\
\omega_{12} = \overline{x^3\,\mathrm{D}\,x}, & \omega_{13} = \overline{x^3y\,\mathrm{D}\,x}, & & \\
\omega_{14} = \overline{x^4\,\mathrm{D}\,x}, & \omega_{15} = \overline{x^4y\,\mathrm{D}\,x}, & & \\
\omega_{16} = \overline{x^5\,\mathrm{D}\,x}, & & & \\
\omega_{17} = \overline{\mathrm{D}\,y}, & \omega_{18} = \overline{y\,\mathrm{D}\,y}, & \omega_{19} = \overline{y^2\,\mathrm{D}\,y}, & \omega_{20} = \overline{y^3\,\mathrm{D}\,y}, \\
\omega_{21} = \overline{x\,\mathrm{D}\,y}, & \omega_{22} = \overline{xy\,\mathrm{D}\,y}, & \omega_{23} = \overline{xy^2\,\mathrm{D}\,y}, & \\
\omega_{24} = \overline{x^2\,\mathrm{D}\,y}, & \omega_{25} = \overline{x^2y\,\mathrm{D}\,y}, & \omega_{26} = \overline{x^2y^2\,\mathrm{D}\,y}, & \\
\omega_{27} = \overline{x^3\,\mathrm{D}\,y}, & \omega_{28} = \overline{x^3y\,\mathrm{D}\,y}, & & \\
\omega_{29} = \overline{x^4\,\mathrm{D}\,y}, & & & \\
\omega_{30} = \overline{x^5\,\mathrm{D}\,y}. & & &
\end{array}
$$

Now one can compute the matrices $\mathcal{A}'$ and $\mathcal{A}$ for the curves $\mathcal{C}'$ and $\mathcal{C}$ respectively. The matrix $\mathcal{A}'$ for the curve $x = t^6$, $y = t^9 + t^{10}$ is (dots representing zeros):

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9	ω_{10}	ω_{11}	ω_{12}	ω_{13}	ω_{14}	ω_{15}	ω_{16}	ω_{17}	ω_{18}	ω_{19}	ω_{20}	ω_{21}	ω_{22}	ω_{23}	ω_{24}	ω_{25}	ω_{26}	ω_{27}	ω_{28}	ω_{29}	ω_{30}
$\overline{t^0\,\mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^1\,\mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^2\,\mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^3\,\mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^4\,\mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^5\,\mathrm{D}\,t}$	6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^6\,\mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^7\,\mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^8\,\mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	9	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^9\,\mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	10	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{10}\,\mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{11}\,\mathrm{D}\,t}$	.	.	.	.	6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{12}\,\mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{13}\,\mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{14}\,\mathrm{D}\,t}$	.	6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	9	.	.	.	.	.	.	.	.	.
$\overline{t^{15}\,\mathrm{D}\,t}$	.	6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	10	.	.	.	.	.	.	.	.	.
$\overline{t^{16}\,\mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{17}\,\mathrm{D}\,t}$	.	.	.	.	.	.	.	.	6	.	.	.	.	.	.	.	.	9	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{18}\,\mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	19	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{19}\,\mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	10	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{20}\,\mathrm{D}\,t}$	.	.	.	.	.	6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	9	.	.	.	.	.	.
$\overline{t^{21}\,\mathrm{D}\,t}$	.	.	.	.	.	6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	10	.	.	.	.	.	.
$\overline{t^{22}\,\mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{23}\,\mathrm{D}\,t}$	.	.	6	.	.	.	.	.	.	.	.	6	.	.	.	.	.	.	.	.	.	9	.	.	.	.	.	.	.	.
$\overline{t^{24}\,\mathrm{D}\,t}$	.	.	12	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	19	.	.	.	.	.	.	.	.

$\overline{t^{25}\ \mathrm{D}\,t}$	.	.	6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	10	.	.	.	.	.	.	.	.
$\overline{t^{26}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	6	.	.	.	.	.	.	.	.	9	.	.	.	.	.	.	.	9	.	.	.
$\overline{t^{27}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	6	.	.	.	.	.	.	.	.	28	.	.	.	.	.	.	.	10	.	.	.
$\overline{t^{28}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	29	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{29}\ \mathrm{D}\,t}$	.	.	.	.	.	.	6	.	.	.	.	.	.	6	.	.	.	.	10	.	.	.	.	.	9	.	.	.	.	.
$\overline{t^{30}\ \mathrm{D}\,t}$	.	.	.	.	.	.	12	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	19	.	.	.	.	.
$\overline{t^{31}\ \mathrm{D}\,t}$	.	.	.	.	.	.	6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	10	.	.	.	.	.
$\overline{t^{32}\ \mathrm{D}\,t}$	.	.	.	6	.	.	.	.	.	.	.	.	6	.	.	.	.	.	.	.	.	.	9	.	.	.	.	.	9	.
$\overline{t^{33}\ \mathrm{D}\,t}$	.	.	.	18	.	.	.	.	.	.	.	.	6	.	.	.	.	.	.	.	.	.	28	.	.	.	.	.	10	.
$\overline{t^{34}\ \mathrm{D}\,t}$	.	.	.	18	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	29	.	.	.	.	.	.	.
$\overline{t^{35}\ \mathrm{D}\,t}$	.	.	.	6	.	.	.	.	.	.	6	.	.	.	.	6	.	.	.	9	.	.	10	.	.	.	.	9	.	.
$\overline{t^{36}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	12	.	.	.	.	.	.	.	.	37	.	.	.	.	.	.	.	19	.	.
$\overline{t^{37}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	6	.	.	.	.	.	.	.	.	57	.	.	.	.	.	.	.	10	.	.
$\overline{t^{38}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	6	.	.	.	.	.	.	6	.	.	.	.	39	.	.	.	.	.	9	.	.	.	9
$\overline{t^{39}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	18	.	.	.	.	.	.	6	.	.	.	.	10	.	.	.	.	.	28	.	.	.	10
$\overline{t^{40}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	18	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	29	.	.	.	.

After deleting all rows that consist of zeros alone, deleting row $\overline{t^9\ \mathrm{D}\,t}$ because it depends of row $\overline{t^8\ \mathrm{D}\,t}$, deleting row $\overline{t^{18}\ \mathrm{D}\,t}$ because it depends of row $\overline{t^{19}\ \mathrm{D}\,t}$, and some permutations of the columns one obtains the following matrix with the same rank:

	ω_1	ω_{17}	ω_5	ω_2	ω_{21}	ω_9	ω_{18}	ω_6	ω_{24}	ω_{12}	ω_3	ω_{22}	ω_{27}	ω_{10}	ω_{19}	ω_{14}	ω_7	ω_{25}	ω_{13}	ω_{29}	ω_4	ω_{23}	ω_{11}	ω_{28}	ω_{20}	ω_{15}	ω_8	ω_{26}	ω_{16}	ω_{30}
$\overline{t^{5}\ \mathrm{D}\,t}$	6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{8}\ \mathrm{D}\,t}$	.	9	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{11}\ \mathrm{D}\,t}$	.	.	6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{14}\ \mathrm{D}\,t}$	.	.	.	6	9	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{15}\ \mathrm{D}\,t}$	.	.	.	6	10	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{17}\ \mathrm{D}\,t}$	.	.	.	.	.	6	9	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{19}\ \mathrm{D}\,t}$	.	.	.	.	.	.	10	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{20}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	6	9	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{21}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	6	10	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{23}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	6	6	9	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{24}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	12	19	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{25}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	6	10	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{26}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	9	6	9	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{27}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	10	6	28	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{28}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	29	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{29}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	10	6	6	9	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{30}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	12	19	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{31}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	6	10	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{32}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	6	9	6	9	.	.	.	.	.	.	.	.
$\overline{t^{33}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	6	10	18	28	.	.	.	.	.	.	.	.
$\overline{t^{34}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	18	29	.	.	.	.	.	.	.	.
$\overline{t^{35}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	6	10	6	9	9	.	.	.	6	.
$\overline{t^{36}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	12	19	37	.	.	.	.	.
$\overline{t^{37}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	6	10	57	.	.	.	.	.
$\overline{t^{38}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	39	6	6	9	.	9
$\overline{t^{39}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	10	6	18	28	.	10
$\overline{t^{40}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	18	29	.	.

One can easily see that all the 27 rows are linearly independent and so

$$\begin{aligned} \operatorname{rank}\mathcal{A}' &= 27\ , \\ \ell(T') &= \frac{3\cdot 42}{2} - 1 - 27 = 35\ . \end{aligned}$$

Now we compute the matrix $\mathcal{A}$ for the curve $x = t^6$, $y = t^9 + 2t^{10} - 2t^{11}$ (we omit the $\overline{t^0\ \mathrm{D}\,t}, \ldots, \overline{t^{40}\ \mathrm{D}\,t}$ in front of the rows in order to save space):

ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9	ω_{10}	ω_{11}	ω_{12}	ω_{13}	ω_{14}	ω_{15}	ω_{16}	ω_{17}	ω_{18}	ω_{19}	ω_{20}	ω_{21}	ω_{22}	ω_{23}	ω_{24}	ω_{25}	ω_{26}	ω_{27}	ω_{28}	ω_{29}	ω_{30}
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
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.	−12	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	−22	.	.	.	.	.	.	.	.	.
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.	.	24	.	.	.	.	.	.	12	.	.	.	.	.	.	.	.	56	.	.	44	.	.	.	.	20	.	.	.
.	.	.	.	.	.	.	.	.	−12	.	.	.	.	.	.	.	.	58	.	.	.	.	.	.	.	−22	.	.	.
.	.	.	.	.	.	6	.	.	.	.	.	.	6	.	.	.	.	−160	.	.	.	.	.	9	.	.	.	.	.
.	.	.	.	.	.	24	.	.	.	.	.	.	.	.	.	.	.	−124	.	.	.	.	.	38	.	.	.	.	.
.	.	.	.	.	.	0	.	.	.	.	.	.	.	.	.	.	.	256	.	.	.	.	.	0	.	.	.	.	.
.	.	.	6	.	.	−48	.	.	.	.	.	6	.	.	.	.	.	−88	.	.	.	9	.	−84	.	.	.	9	.
.	.	.	36	.	.	24	.	.	.	.	.	12	.	.	.	.	.	.	.	.	.	56	.	44	.	.	.	20	.
.	.	.	36	.	.	.	.	.	.	.	.	−12	.	.	.	.	.	.	.	.	.	58	.	.	.	.	.	−22	.
.	.	.	−96	.	.	.	.	.	.	6	.	.	.	.	6	.	.	.	9	.	.	−160	.	.	.	.	9	.	.
.	.	.	−72	.	.	.	.	.	.	24	.	.	.	.	.	.	.	.	74	.	.	−124	.	.	.	.	38	.	.
.	.	.	144	.	.	.	.	.	.	0	.	.	.	.	.	.	.	.	152	.	.	256	.	.	.	.	0	.	.
.	.	.	−48	.	.	.	6	.	.	−48	.	.	.	6	.	.	.	.	−156	.	.	−88	.	.	9	.	−84	.	9
.	.	.	.	.	.	.	36	.	.	24	.	.	.	12	.	.	.	.	−560	.	.	.	.	.	56	.	44	.	20
.	.	.	.	.	.	.	36	.	.	.	.	.	.	−12	.	.	.	.	328	.	.	.	.	.	58	.	.	.	−22

After deleting all rows that consist of zeros alone, deleting rows $\overline{t^9\ \mathrm{D}\,t}$ and $\overline{t^{10}\ \mathrm{D}\,t}$ because they depend of row $\overline{t^8\ \mathrm{D}\,t}$, and some permutations of the columns one obtains the following matrix with the same rank:

$$\begin{pmatrix} \mathcal{B} & \mathcal{O}_1 \\ \mathcal{O}_2 & \mathcal{D} \end{pmatrix},$$

where the matrices $\mathcal{O}_1, \mathcal{O}_2$ contain zeros only, and:

$\mathcal{B}$:

	ω_1	ω_{17}	ω_5	ω_2	ω_{21}	ω_9	ω_{18}	ω_6	ω_{24}
$\overline{t^5\ \mathrm{D}\,t}$	6	.	.	.	.	.	.	.	.
$\overline{t^8\ \mathrm{D}\,t}$	.	9	.	.	.	.	.	.	.
$\overline{t^{11}\ \mathrm{D}\,t}$	.	.	6	.	.	.	.	.	.
$\overline{t^{14}\ \mathrm{D}\,t}$	.	.	.	6	9	.	.	.	.
$\overline{t^{15}\ \mathrm{D}\,t}$	.	.	.	12	20	.	.	.	.
$\overline{t^{16}\ \mathrm{D}\,t}$	.	.	.	−12	−22	.	.	.	.
$\overline{t^{17}\ \mathrm{D}\,t}$	.	.	.	.	.	6	9	.	.
$\overline{t^{18}\ \mathrm{D}\,t}$	.	.	.	.	.	.	38	.	.
$\overline{t^{20}\ \mathrm{D}\,t}$	.	.	.	.	.	.	−84	6	9
$\overline{t^{21}\ \mathrm{D}\,t}$	.	.	.	.	.	.	44	12	20
$\overline{t^{22}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	−12	−22

$\mathcal{D}$:

	ω_{12}	ω_3	ω_{22}	ω_{10}	ω_{27}	ω_{19}	ω_{14}	ω_7	ω_{25}	ω_{13}	ω_{29}	ω_4	ω_{23}	ω_{16}	ω_{11}	ω_{28}	ω_{20}	ω_8	ω_{15}	ω_{26}	ω_{30}
$\overline{t^{23}\ \mathrm{D}\,t}$	6	6	9	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{24}\ \mathrm{D}\,t}$	.	24	38	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{26}\ \mathrm{D}\,t}$	.	−48	−84	6	9	9	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{27}\ \mathrm{D}\,t}$	.	24	44	12	20	56	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{28}\ \mathrm{D}\,t}$	.	.	.	−12	−22	58	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{29}\ \mathrm{D}\,t}$	.	.	.	.	.	−160	6	6	9	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{30}\ \mathrm{D}\,t}$	.	.	.	.	.	−124	.	24	38	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{31}\ \mathrm{D}\,t}$	.	.	.	.	.	256	.	0	0	.	.	.	.	.	.	.	.	.	.	.	.
$\overline{t^{32}\ \mathrm{D}\,t}$	.	.	.	.	.	−88	.	−48	−84	6	9	6	9	.	.	.	.	.	.	.	.
$\overline{t^{33}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	24	44	12	20	36	56	.	.	.	.	.	.	.	.
$\overline{t^{34}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	−12	−22	36	58	.	.	.	.	.	.	.	.
$\overline{t^{35}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	−96	−160	6	6	9	9	.	.	.	.
$\overline{t^{36}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	−72	−124	.	24	38	74	.	.	.	.
$\overline{t^{37}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	144	256	.	0	0	152	.	.	.	.
$\overline{t^{38}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	−48	−88	.	−48	−84	−156	6	6	9	9
$\overline{t^{39}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	24	44	−560	36	12	56	20
$\overline{t^{40}\ \mathrm{D}\,t}$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	328	36	−12	58	−22

It can easily be seen that the matrix $\mathcal{B}$ has rank 9 (rows $\overline{t^{16}\ \mathrm{D}\,t}$ and $\overline{t^{22}\ \mathrm{D}\,t}$ can be eliminated, the rest is linearly independent), while the the matrix $\mathcal{D}$ consists of only 17 rows, so that its rank is ≤ 17. Therefore the rank of the entire matrix is ≤ 26, giving

$$\ell(T) \geq \frac{3 \cdot 42}{2} - 1 - 26 = 36 > \ell(T'),$$

which is enough to prove the counterexample.
A more detailed analysis (e.g. by using a computer or computing the rank modulo 5) shows that the rank of the matrix $\mathcal{D}$ is equal to 17, so that indeed

$$\ell(T) = 36 \ .$$

Final Remarks: Heinrich found this counterexample by computing the matrix $\mathcal{A}$

for a "generic" curve with the characteristic pairs $(3, 2)$ and $(10, 3)$

$$
\begin{aligned}
x &= t^6 \, , \\
y &= t^9 + a_{10}t^{10} + a_{11}t^{11} + a_{13}t^{13} + a_{14}t^{14} + a_{16}t^{16} + a_{17}t^{17} + \\
&\quad a_{20}t^{20} + a_{22}t^{22} + a_{23}t^{23} + a_{26}t^{26} + a_{29}t^{29} + a_{32}t^{32} + a_{35}t^{35} + a_{41}t^{41} \, ,
\end{aligned}
$$

and indeterminate coefficients a_i (where one can even omit the two terms of degree 22 and 41 by [1], Chapter 3, Proposition 1) and transforming it to a staircase form with the help of MAPLE. From the result he got conditions on the coefficients to lower the rank of the matrix.

For this general computation, but also for a little bit more complicated special curves, the use of a computer algebra system seems to be indispensable.

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