

ON A THEOREM OF KY-FAN TYPE

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Abstract. We prove a statement of the Ky-Fan type which makes it possible in some situations to find the second term of the singular values asymptotics of a sum of two operators.

Finding the first term in the asymptotic expansion of singular values of some classes of operators [1] is very convenient using the following theorem of Ky-Fan [2; Theorem 2.3, p. 52]:

THEOREM 1. *If A and B are compact operators on a Hilbert space \mathcal{H} such that for some $a > 0$, $r > 0$*

$$\lim_{n \rightarrow \infty} n^r s_n(A) = a, \quad \lim_{n \rightarrow \infty} n^r s_n(B) = 0,$$

then $\lim_{n \rightarrow \infty} n^r s_n(A + B) = a$.

Here, $s_n(T) = \lambda_n((T^*T)^{1/2})$. $(\lambda_n((T^*T)^{1/2}))$ denote the non increasing sequence of the eigenvalues of $(T^*T)^{1/2}$.

Finding the higher terms in the singular values asymptotics of a sum of two operators cannot be done using the previous result. The following statement of the Ky-Fan type makes it possible in some situations to find the second term of the singular values asymptotics of a sum of two operators.

THEOREM 2. *If A and B are compact operators on a Hilbert space \mathcal{H} such that*

$$\begin{aligned} s_n(A) &= a_1 n^{-\alpha_1} + a_2 n^{-\alpha_2} + o(n^{-\alpha_2}) & (a_1 > 0, a_2 \in \mathbf{R}) \\ s_n(B) &= o(n^{-\beta_1}) \end{aligned} \tag{1}$$

and $0 < \alpha_1 < \alpha_2 < \alpha_1 + 1$, $\beta_1 > \alpha_2 / (1 + \alpha_1 - \alpha_2)$, then

$$s_n(A + B) = a_1 n^{-\alpha_1} + a_2 n^{-\alpha_2} + o(n^{-\alpha_2}). \tag{2}$$

Proof. Since $0 < \alpha_1 < \alpha_2 < \alpha_1 + 1$, from $\beta_1 > \alpha_2 / (1 + \alpha_1 - \alpha_2)$ it follows

$$\frac{\beta_1 - \alpha_2}{\alpha_2} > \frac{\alpha_2 - \alpha_1}{\alpha_1 + 1 - \alpha_2}.$$

Let θ be some number such that

$$\frac{\beta_1 - \alpha_2}{\alpha_2} > \theta > \frac{\alpha_2 - \alpha_1}{\alpha_1 + 1 - \alpha_2} \quad (3)$$

and let $k = k(n) = [n^{\frac{\theta}{1+\theta}}] - 1$, $m = m(n) = [n^{\frac{1}{1+\theta}}]$ ($[x]$ is the integer part of x). Since every integer n can be represented in the form

$$n = (k+1)m + j, \quad j = j(n) \in \mathbf{N}, \quad 0 \leq j \leq n^{\frac{1}{1+\theta}} + n^{\frac{\theta}{1+\theta}},$$

then from the properties of the singular values of a sum of two operators [2] it follows

$$s_n(A+B) \leq s_{km+j}(A) + s_{m+1}(B)$$

and consequently

$$\begin{aligned} & n^{\alpha_2} \left(s_n(A+B) - \frac{a_1}{n^{\alpha_2}} - \frac{a_2}{n^{\alpha_2}} \right) \\ & \leq \left(\frac{n}{km+j} \right)^{\alpha_2} (km+j)^{\alpha_2} \left[s_{km+j}(A) - \frac{a_1}{(km+j)^{\alpha_1}} - \frac{a_2}{(km+j)^{\alpha_2}} \right] + n^{\alpha_2} s_{m+1}(B) \\ & \quad + a_1 n^{\alpha_2} \left[\frac{1}{(km+j)^{\alpha_1}} - \frac{1}{n^{\alpha_1}} \right] + a_2 n^{\alpha_2} \left[\frac{1}{(km+j)^{\alpha_2}} - \frac{1}{n^{\alpha_2}} \right]. \end{aligned}$$

Now, we prove that

$$\overline{\lim}_{n \rightarrow \infty} n^{\alpha_2} \left[s_n(A+B) - \frac{a_1}{n^{\alpha_1}} - \frac{a_2}{n^{\alpha_2}} \right] \leq 0. \quad (4)$$

We observe that $k = k(n) \sim m^\theta = m(n)^\theta$ and $\lim_{n \rightarrow \infty} \frac{j(n)}{n} = 0$ ($= \lim_{n \rightarrow \infty} \frac{j}{km}$). Since $n = (k+1)m + j \rightarrow \infty$ (when $m = m(n) \rightarrow \infty$) then from (1) (because $(n/(km+j))^{\alpha_2} \leq 2^{\alpha_2}$) it follows

$$\lim_{n \rightarrow \infty} \left(\frac{n}{km+j} \right)^{\alpha_2} (km+j)^{\alpha_2} \left[s_{km+j}(A) - \frac{a_1}{(km+j)^{\alpha_1}} - \frac{a_2}{(km+j)^{\alpha_2}} \right] = 0.$$

It is enough to prove that

$$\begin{aligned} & \lim_{n \rightarrow \infty} ((k+1)m+j)^{\alpha_2} s_m(B) = 0, \\ & \lim_{n \rightarrow \infty} ((k+1)m+j)^{\alpha_2} \left[\frac{1}{(km+j)^{\alpha_1}} - \frac{1}{n^{\alpha_1}} \right] = 0, \\ & \lim_{n \rightarrow \infty} n^{\alpha_2} \left[\frac{1}{(km+j)^{\alpha_2}} - \frac{1}{n^{\alpha_2}} \right] = 0. \end{aligned} \quad (5)$$

Since

$$\begin{aligned} & ((k+1)m+j)^{\alpha_2} s_m(B) \sim \frac{m^{\theta\alpha_2}}{m^{\beta_1-\alpha_2}} m^{\beta_1} s_m(B), \\ & ((k+1)m+j)^{\alpha_2} \left[\frac{1}{(km+j)^{\alpha_1}} - \frac{1}{((k+1)m+j)^{\alpha_1}} \right] \sim \alpha_1 \frac{m^{\alpha_2-\alpha_1}}{m^{\theta(\alpha_1+1-\alpha_2)}}, \\ & ((k+1)m+j)^{\alpha_2} \left[\frac{1}{(km+j)^{\alpha_2}} - \frac{1}{((k+1)m+j)^{\alpha_2}} \right] \sim \frac{\alpha_2}{m^\theta} \quad (n \rightarrow \infty) \end{aligned}$$

then (5) follows from (1) and (3).

From $A = A + B - B$ it follows

$$s_{km+j}(A+B) \geq s_{(k+1)m+j}(A) - s_{m+1}(B).$$

Similarly, starting from the previous inequality, we get

$$\underline{\lim}_{n \rightarrow \infty} n^{\alpha_2} (s_n(A+B) - a_1 n^{-\alpha_1} - a_2 n^{-\alpha_2}) \geq 0. \quad (6)$$

From (4) and (6) the statement of Theorem 2 follows. ■

REMARK. The proof is carried out without any change if we start from the more general assumptions

$$\begin{aligned} s_n(A) &= \frac{L_1(n)}{n^{\alpha_1}} + \frac{L_2(n)}{n^{\alpha_2}} + o\left(\frac{L_2(n)}{n^{\alpha_2}}\right), \\ s_n(B) &= o\left(\frac{L_3(n)}{n^{\beta_1}}\right), \end{aligned}$$

where $0 < \alpha_1 < \alpha_2 < \alpha_1 + 1$, $\beta_1 > \alpha_2 / (1 + \alpha_1 - \alpha_2)$ and L_i ($i = 1, 2, 3$) are slowly varying functions. Then

$$s_n(A+B) = \frac{L_1(n)}{n^{\alpha_1}} + \frac{L_2(n)}{n^{\alpha_2}} + o\left(\frac{L_2(n)}{n^{\alpha_2}}\right).$$

We give an example how Theorem 2 can be applied. By $\int_{-\pi}^{\pi} R(x, y) \cdot dy$ we denote operator acting on $L^2(-\pi, \pi)$ with the kernel $R(\cdot, \cdot)$.

Let $H(x, y) = k(x-y) + K(x, y)$, where the function k is even, periodic with the period 2π , such that

$$\left| \int_{-\pi}^{\pi} k(t) \cos nt \, dt \right| = a_1 n^{-\alpha_1} + a_2 n^{-\alpha_2} + o(n^{-\alpha_2}) \quad (\alpha_1 < \alpha_2 < \alpha_1 + 1, a_1 > 0). \quad (7)$$

THEOREM 3. *If for some $r > \left[\frac{\alpha_2}{1 + \alpha_1 - \alpha_2} \right]$ the function $\partial^r K / \partial y^r$ is bounded on $[-\pi, \pi]^2$, then*

$$s_n \left(\int_{-\pi}^{\pi} H(x, y) \cdot dy \right) = a_1 n^{-\alpha_1} + a_2 n^{-\alpha_2} + o(n^{-\alpha_2}). \quad (8)$$

Proof. It is well known that the even periodic function k with period 2π generates an integral operator on $L^2(-\pi, \pi)$ whose singular values are $|\int_{-\pi}^{\pi} k(t) \cos nt \, dt|$. Since $\partial^r K / \partial y^r$ is a bounded function on $[-\pi, \pi]^2$, then by Krein theorem [2; p. 157] we have

$$s_n \left(\int_{-\pi}^{\pi} K(x, y) \cdot dy \right) = o\left(n^{-r-\frac{1}{2}}\right).$$

From Theorem 2, previous asymptotic formula and (7) we get (8). ■

REFERENCES

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