МАТЕМАТИЧКИ ВЕСНИК
46 (1994), 77–79

UDC 515.126.4
оригинални научни рад
research paper

# A FIXED POINT THEOREM IN BANACH SPACES OVER TOPOLOGICAL SEMIFIELDS

**Slobodan Č. Nešić**

**Abstract.** Let $X$ be a Banach space over a topological semifield and $T_1$, $T_2: X \to X$ two maps satisfying the condition (1). Then $T_1$ and $T_2$ have a common fixed point.

## 1. Introduction

The notion of topological semifield has been introduced by M. Antonovskiĭ, V. Boltyanskiĭ and T. Sarymsakov in [1]. Let $E$ be a topological semifield and $K$ the set of all its positive elements. Take any two elements $x$, $y$ in $E$. If $y - x$ is in $\overline{K}$ (in $K$), this is denoted by $x \ll y$ ($x < y$). As proved in [1], every topological semifield $E$ contains a subsemifield, so called the axis of $E$, isomorphic to the field $\mathbf{R}$ of real numbers. Consequently by identifying the axis and $\mathbf{R}$, each topological semifield can be regarded as a topological linear space over the field $\mathbf{R}$.

The ordered triple $(X, d, E)$ is called a metric space over the topological semifield if there exists a mapping $d: X \times X \to E$ satisfying the usual axioms for a metric (see [1], [2] and [4]).

Linear spaces considered in this paper are defined on the field $\mathbf{R}$. Let $X$ be a linear space. The ordered triple $(X, \|\ \|, E)$ is called a feeble normed space over the topological semifield if there exists a mapping $\|\ \|: X \to E$ satisfying the usual axioms for a norm (see [1] and [3]).

## 2. Main result

We shall use the following definition.

DEFINITION 1. Let $(X, \|\ \|, E)$ be a feeble normed space over a topological semifield $E$ and let $d(x, y) = \|x - y\|$ for all $x$, $y$ in $X$. A space $(X, \|\ \|, E)$ is said to be a Banach space over the topological semifield $E$ if $(X, d, E)$ is sequentially complete metric space over the topological semifield $E$.

*Key words*: Banach space over a topological semifield, common fixed point, Cauchy sequence, sequentially complete metric space.

Now we shall prove the following result.

THEOREM 1. *Let $X$ be a Banach space over a topological semifield $E$ and $T_1$, $T_2 \colon X \to X$ two maps satisfying the condition*

$$\|x - T_1x\|^m + \|y - T_2y\|^m \ll p\|x - y\|^m \tag{1}$$

*for all $x$, $y$ in $X$, where $p$, $t$ are in $\mathbf{R}$, $0 < t < 1$, $1 \leqslant pt^m < 2$ and $m = 1, 2, \ldots$ . Then the sequence $\{x_n\}_{n=0}^{\infty}$, the members of which are*

$$x_{2n+1} = (1-t)x_{2n} + tT_1x_{2n}, \quad x_{2n+2} = (1-t)x_{2n+1} + tT_2x_{2n+1}, \quad x_0 \in X, \tag{2}$$

*converges to the common fixed point of $T_1$ and $T_2$ in $X$.*

*Proof.* Let $x_0$ in $X$ be an arbitrary point . From (2) we get

$$\|x_{2n+1} - x_{2n}\| = t\|T_1x_{2n} - x_{2n}\|, \quad \|x_{2n+2} - x_{2n+1}\| = t\|T_2x_{2n+1} - x_{2n+1}\|. \tag{3}$$

If in (1) we put $x = x_{2n}$ and $y = x_{2n+1}$, then by (3) we have

$$t^{-m}(\|x_{2n+1} - x_{2n}\|^m + \|x_{2n+2} - x_{2n+1}\|^m) \ll p\|x_{2n} - x_{2n+1}\|^m$$

and hence

$$\|x_{2n+2} - x_{2n+1}\| \ll (pt^m - 1)^{1/m}\|x_{2n} - x_{2n+1}\| \tag{4}$$

for all $n$. Now, if we put in (1) $x = x_{2n+2}$ and $y = x_{2n+1}$, and use (3), we get

$$t^{-m}(\|x_{2n+3} - x_{2n+2}\|^m + \|x_{2n+2} - x_{2n+1}\|^m) \ll p\|x_{2n+2} - x_{2n+1}\|^m$$

and hence

$$\|x_{2n+3} - x_{2n+2}\| \ll (pt^m - 1)^{1/m}\|x_{2n+2} - x_{2n+1}\| \tag{5}$$

for all $n$. From (4) and (5) we then obtain

$$\|x_n - x_{n+1}\| \ll (pt^m - 1)^{1/m}\|x_{n-1} - x_n\|$$

which implies

$$\|x_n - x_{n+1}\| \ll (pt^m - 1)^{1/m}\|x_0 - x_1\|.$$

Since $0 \leqslant pt^m - 1 < 1$, it follows that $\{x_n\}$ is a Cauchy sequence in $X$. As $X$ is a Banach space over the topological semifield $E$, we deduce that $\{x_n\}$ converges to a point $u$ in $X$.

Now putting $x = u$ and $y = x_{2n+1}$ in (1) we have

$$\|u - T_1u\|^m + \|x_{2n+1} - T_2x_{2n+1}\|^m \ll p\|u - x_{2n+1}\|^m,$$

i.e.

$$\|u - T_1u\|^m + t^{-m}\|x_{2n+2} - x_{2n+1}\|^m \ll p\|u - x_{2n+1}\|^m.$$

If now $n$ tends to infinity one has $\|u - T_1u\|^m \ll 0$, which implies $T_1u = u$. Hence, $u$ is a fixed point for $T_1$. Similarly, $T_2u = u$. So $u$ is a common fixed point of $T_1$ and $T_2$. This completes the proof. ■

REFERENCES

[**1**] Antonovskiĭ, M., Boltyanskiĭ, V. and Sarymsakov, T., *Topological semifields*, Tashkent 1960.

[**2**] Antonovskiĭ, M., Boltyanskiĭ, V. and Sarymsakov, T., *Metric spaces over semifields*, Tashkent 1961.

[**3**] Kasahara, S., *On formulations of topological linear spaces by topological semifields*, Math. Seminar Notes, Vol. 1 (1973), 11–29

[**4**] Mamuzić, Z., *Some remarks on abstract distance in general topology*, $E\Lambda EY\Theta EPIA$ 2 (1979), 433–446, Athens, Greece

(received 14.06.1994.)

Studentska 14, 11000 Belgrade, Yugoslavia