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69, 2 (2017), 75–88
June 2017

research paper
оригинални научни рад

FIXED POINTS OF GENERALIZED TAC-CONTRACTIVE MAPPINGS IN b-METRIC SPACES

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Abstract. We introduce generalized TAC-contractive mappings in b-metric spaces and we prove some new fixed point results for this class of mappings. We provide examples in support of our results. Our results extend the results of [S. Chandok, K. Tas and A. H. Ansari, Some fixed point results for TAC-type contractive mappings, J. Function Spaces, Vol. 2016, Article ID 1907676, 6 pages] from the metric space setting to b-metric spaces and generalize a result of [D. Đorić, Common fixed point for generalized (ψ, φ)-weak contractions, Appl. Math. Lett. 22 (2009) 1896–1900].

1. Introduction

Banach contraction principle has been extended by various authors based on the generalization of contraction conditions and/or generalization of ambient space. In 1997, Alber and Guerre-Delabriere [1] introduced weakly contractive maps which are extensions of contraction maps and obtained fixed point results in the setting of Hilbert spaces. Rhoades [14] extended this concept to metric spaces. In 2008, Dutta and Choudhury [8] introduced (ψ, φ)-weakly contractive maps and proved the existence of fixed points in complete metric spaces. In continuation to the extensions of contraction maps, Đorić [7] studied (ψ, φ)-weakly contractive maps and proved the existence of fixed points in complete metric spaces. On the other hand, in the direction of generalizing metric spaces, in 1993, Czerwik [6] introduced the concept of b-metric spaces and proved the Banach contraction mapping principle in this setting. Afterwards, several research papers appeared on the existence of fixed points for single-valued and multi-valued mappings in b-metric spaces [4, 13, 15–18].

Very recently, Chandok, Tas and Ansari [5] introduced the concept of TAC-contractive mappings and proved some fixed point results in the setting of complete metric spaces.

2010 Mathematics Subject Classification: 47H10, 54H25

Keywords and phrases: b-metric space; cyclic (α, β)-admissible mapping; generalized TAC-contractive mapping; fixed point.

Definition 1.1. [5] Let (X, d) be a metric space and let $\alpha, \beta : X \to [0, \infty)$ be two given mappings. We say that $T : X \to X$ is a TAC-contractive mapping if

$$x, y \in X \text{ with } \alpha(x)\beta(y) \geq 1 \implies \psi(d(Tx, Ty)) \leq f(\psi(d(x, y)), \phi(d(x, y))),$$

where:

(i) ψ is continuous and nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$;

(ii) $\phi : [0, \infty) \to [0, \infty)$ is continuous with $\lim_{n\to\infty} \phi(t_n) = 0 \implies \lim_{n\to\infty} t_n = 0$; and

(iii) $f : [0, \infty)^2 \to \mathbb{R}$ is continuous, $f(s, t) \leq s$ and $f(s, t) = s$ implies that either $s = 0$ or $t = 0$, for all $s, t \in [0, \infty)$.

Theorem 1.2. [5] *Let (X, d) be a complete metric space, $\alpha, \beta : X \to [0, \infty)$ be two mappings and let $T : X \to X$ be a cyclic (α, β)-admissible mapping. Assume that T is a TAC-contractive mapping. Suppose that there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$ and either of the following conditions hold:*

(a) *T is continuous;*

(b) *If $\{x_n\}$ is a sequence in X such that $x_n \to z$ and $\beta(x_n) \geq 1$ for all n, then $\beta(z) \geq 1$.*

Then T has a fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in Fix(T)$ where $Fix(T)$ is the set of all fixed points of T, then T has a unique fixed point.

Motivated by this work, we introduce generalized TAC-contractive mappings in b-metric spaces and extend Theorem 1.2 to b-metric spaces. In Section 2, we present preliminaries. In Section 3, we prove our main results in which we study the existence of fixed points of generalized TAC-contractive mappings in b-metric spaces. We provide corollaries and examples in support of our results in Section 4.

2. Preliminaries

Definition 2.1. [11] A function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties hold:

(i) ψ is continuous and nondecreasing function,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

We denote the set of all altering distance functions by Ψ.

Definition 2.2. [6] Let X be a non-empty set. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric if the following conditions are satisfied;

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) there exists $s \geq 1$ such that $d(x, z) \leq s\big[d(x, y) + d(y, z)\big]$ for all $x, y, z \in X$.

In this case, the pair (X, d) is called a b-metric space with coefficient s.

Every metric space is a b-metric space with $s = 1$. In general, not every b-metric space is a metric space. Throughout this paper, $\mathbb{R}$ denotes the real line, and $\mathbb{N}$ is the set of all natural numbers.

EXAMPLE 2.3. Let $X = \mathbb{R}$, and let the mapping $d : X \times X \to [0, \infty)$ be defined by $d(x, y) = |x - y|^2$ for all $x, y \in X$. Then (X, d) is a b-metric space with coefficient $s = 2$, but it is not a metric space.

DEFINITION 2.4. [4] Let (X, d) be a b-metric space.

(i) A sequence $\{x_n\}$ in X is called b-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}$ in X is called b-Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

(iii) The b-metric space (X, d) is said to be b-complete if every b-Cauchy sequence in X is b-convergent.

(iv) A set $B \subset X$ is said to be b-closed if for any sequence $\{x_n\}$ in B such that $\{x_n\}$ is b-convergent to $z \in X$, it is $z \in B$.

REMARK 2.5. A b-metric need not be a continuous function. For more details, we refer to [10].

LEMMA 2.6. [9] *Let (X, d) be a b-metric space with $s \geq 1$.*

(i) *If a sequence $\{x_n\} \subset X$ is b-convergent, then it admits a unique limit.*

(ii) *Every b-convergent sequence in X is b-Cauchy.*

DEFINITION 2.7. Let (X, d) and (M, d') be two b-metric spaces. A function $f : X \to M$ is b-continuous at $x \in X$ if it is b-sequentially continuous at X. That is, whenever $\{x_n\}$ is b-convergent to x, $\{fx_n\}$ is b-convergent to fx.

DEFINITION 2.8. [11] Let A and B be nonempty subsets of X. A mapping $f : A \cup B \to A \cup B$ is said to be cyclic if $f(A) \subset B$ and $f(B) \subset A$.

DEFINITION 2.9. [2] Let X be a nonempty set, f be s selfmap on X and $\alpha, \beta : X \to [0, \infty)$ be two mappings. We say that f is a cyclic (α, β)-admissible mapping if

(i) for any $x \in X$ with $\alpha(x) \geq 1 \implies \beta(fx) \geq 1$, and

(ii) for any $y \in X$ with $\beta(y) \geq 1 \implies \alpha(fy) \geq 1$.

We denote:

$\Phi = \{\phi : [0, \infty) \to [0, \infty)$ with $\lim_{n\to\infty} \phi(t_n) = 0 \implies \lim_{n\to\infty} t_n = 0\}$, and

$\mathcal{C} = \{f : [0, \infty)^2 \to \mathbb{R} \mid$ (i) f is continuous, (ii) $f(a, t) \leq a$, (iii) $f(a, t) = a \implies$ either $a = 0$ or $t = 0$ and (iv) $f(a, t) \leq f(b, t)$ whenever $a \leq b\}$.

We observe that:

(i) if $f \in \mathcal{C}$ then $f(0, 0) = 0$;

(ii) if $\phi \in \Phi$ then $\phi(t) = 0 \implies t = 0$.

(iii) if $\phi \in \Phi$ then $\limsup_{n\to\infty} \phi(t_n) = 0 \implies \limsup_{n\to\infty} t_n = 0$.

EXAMPLE 2.10. The following functions $f : [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$: (i) $f(a,t) = a - t$, (ii) $f(a,t) = \frac{a-t}{1+t}$, (iii) $f(a,t) = \frac{a}{1+t}$, and (iv) $f(a,t) = \frac{a}{1+t+a}$, for $a, t \in [0,\infty)$.

We denote $\Phi_1 = \{\varphi : [0,\infty) \to [0,\infty) \mid \varphi$ is lower semicontinuous with $\varphi(t) = 0$ if and only if $t = 0\}$. We observe that $\Phi_1 \subset \Phi$.

Đorić proved the following theorem by using $\varphi \in \Phi_1$ in complete metric spaces.

THEOREM 2.11. [7] *Let (X,d) be a complete metric space and let $T : X \to X$ be a selfmap of X. If there exist $\psi \in \Psi$ and $\varphi \in \Phi_1$ such that*

$$\psi(d(Tx,Ty)) \leq \psi(M(x,y)) - \varphi(M(x,y)), \tag{2.1}$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$ for all $x, y \in X$, then T has a unique fixed point in X.

The following lemma is useful in proving our main results.

LEMMA 2.12. [3] *Suppose (X,d) is a b-metric space with coefficient s and $\{x_n\}$ is a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$, $d(x_{m_k}, x_{n_k - 1}) < \epsilon$ and*

(i) $\epsilon \leq \limsup_{k\to\infty} d(x_{m_k}, x_{n_k}) \leq s\epsilon$ (iii) $\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k}) \leq s^2\epsilon$

(ii) $\frac{\epsilon}{s} \leq \limsup_{k\to\infty} d(x_{m_k}, x_{n_k+1}) \leq s^2\epsilon$ (iv) $\frac{\epsilon}{s^2} \leq \limsup_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \leq s^3\epsilon$.

3. Main results

In this section, we introduce the notion of a generalized TAC-contractive map in b-metric spaces and prove fixed point results for such mapping in b-complete metric spaces.

DEFINITION 3.1. Let (X,d) be a b-metric space and let $\alpha, \beta : X \to [0,\infty)$ be two given mappings. Let $T : X \to X$ be a selfmap of X. If there exist $\psi \in \Psi$, $\phi \in \Phi$ and $f \in \mathcal{C}$ such that

$$\text{for all } x, y \in X,\ \ \alpha(x)\beta(y) \geq 1 \ \Rightarrow\ \psi(s^3 d(Tx,Ty)) \leq f(\psi(M_s(x,y)), \phi(M_s(x,y))), \tag{3.1}$$

where $M_s(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\}$, then we say that T is a generalized TAC-contractive map in b-metric spaces.

THEOREM 3.2. *Let (X,d) be a b-complete metric space with coefficient $s \geq 1$. Let $T : X \to X$ be a selfmap of X. Assume that there exist two mappings $\alpha, \beta : X \to [0,\infty)$ and $\psi \in \Psi$, $\phi \in \Phi$ and $f \in \mathcal{C}$ such that T is a generalized TAC-contractive mapping. Further, suppose that there exists $x_0 \in X$ such that*

$\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$, T is a cyclic (α, β)-admissible mapping and either of the following conditions hold:

(i) *T is continuous,*

(ii) *if $\{x_n\}$ is a sequence in X such that $x_n \to z$ and $\beta(x_n) \geq 1$ for all n, then $\beta(z) \geq 1$.*

Then T has a fixed point.

Proof. By the hypotheses we have $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$. Now, we define an iterative sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \dots$. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbb{N} \cup \{0\}$, we have $Tx_{n_0} = x_{n_0+1} = x_{n_0}$, so that x_{n_0} is a fixed point of T and we are through.

Hence, without loss of generality, we assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\alpha(x_0) \geq 1$ and T is a cyclic (α, β)-admissible mapping, we have $\beta(x_1) = \beta(Tx_0) \geq 1$, and this implies that $\alpha(x_2) = \alpha(Tx_1) \geq 1$. On continuing this process, we obtain

$$\alpha(x_{2k}) \geq 1 \text{ and } \beta(x_{2k+1}) \geq 1 \text{ for all } k \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

Since $\beta(x_0) \geq 1$ and T is a cyclic (α, β)-admissible mapping, we have $\alpha(x_1) = \alpha(Tx_0) \geq 1$ and this implies that $\beta(x_2) = \beta(Tx_1) \geq 1$. In general, on continuing this process, we obtain

$$\beta(x_{2k}) \geq 1 \text{ and } \alpha(x_{2k+1}) \geq 1 \text{ for all } k \in \mathbb{N} \cup \{0\}. \tag{3.3}$$

Therefore from (3.2) and (3.3) we have $\alpha(x_n) \geq 1$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

First we show that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Since $\alpha(x_n)\beta(x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, from (3.1), we have

$$\psi(s^3 d(Tx_n, Tx_{n+1})) \leq f(\psi(M_s(x_n, x_{n+1})), \phi(M_s(x_n, x_{n+1}))) \tag{3.4}$$

where

$$\begin{aligned} &M_s(x_n, x_{n+1}) \\ &= \max\left\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \frac{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)}{2s}\right\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}. \end{aligned}$$

Now, if $d(x_n, x_{n+1}) < d(x_{n+1}, x_{n+2})$ for some $n \in \mathbb{N} \cup \{0\}$, it follows from (3.4) that

$$\begin{aligned} \psi(d(x_{n+1}, x_{n+2})) &= \psi(d(Tx_n, Tx_{n+1})) \leq \psi(s^3 d(Tx_n, Tx_{n+1})) \\ &\leq f(\psi(d(x_{n+1}, x_{n+2})), \phi(d(x_{n+1}, x_{n+2}))) \leq \psi(d(x_{n+1}, x_{n+2})), \end{aligned}$$

so that $f(\psi(d(x_{n+1}, x_{n+2})), \phi(d(x_{n+1}, x_{n+2}))) = \psi(d(x_{n+1}, x_{n+2}))$. Hence by (ii) of the definition of f, we have either $\psi(d(x_{n+1}, x_{n+2})) = 0$ or $\phi(d(x_{n+1}, x_{n+2})) = 0$, a contradiction since $x_n \neq x_{n+1}$.

Hence $d(x_n, x_{n+1}) \geq d(x_{n+1}, x_{n+2})$ for all $n \in \mathbb{N} \cup \{0\}$. Therefore, the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and bounded from below. Thus there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. Suppose that $r > 0$. Then we have

$$\begin{aligned}\psi(d(x_{n+1}, x_{n+2})) &= \psi(d(Tx_n, Tx_{n+1})) \leq \psi(s^3 d(Tx_n, Tx_{n+1})) \\ &\leq f(\psi(d(x_n, x_{n+1})), \phi(d(x_n, x_{n+1}))) \leq \psi(d(x_n, x_{n+1})).\end{aligned} \tag{3.5}$$

On letting $n \to \infty$ in (3.5) and using the continuity of ψ and f, we have $\psi(r) \leq f(\psi(r)$, and $\lim_{n\to\infty} \phi(d(x_n, x_{n+1}))) \leq \psi(r)$, so that $f(\psi(r), \lim_{n\to\infty} \phi(d(x_n, x_{n+1}))) = \psi(r)$. Hence, either $\psi(r) = 0$ or $\lim_{n\to\infty} \phi(d(x_n, x_{n+1})) = 0$. In any case it is a contradiction. Hence, $r = 0$, i.e., $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

We now prove that $\{x_n\}$ is a b-Cauchy sequence. If $\{x_n\}$ is not b-Cauchy, then by Lemma 2.12, there exist $\epsilon > 0$ and sequences of positive integers $\{n_k\}$ and $\{m_k\}$ with $n_k > m_k > k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$, $d(x_{m_k}, x_{n_k-1}) < \epsilon$ and (i)–(iv) of Lemma 2.12 hold. Since $\alpha(x_{m_k}) \geq 1$ and $\beta(x_{n_k}) \geq 1$ we have that $\alpha(x_{m_k})\beta(x_{n_k}) \geq 1$.

Now, from (3.1) we have

$$\begin{aligned}\psi(d(x_{m_k+1}, fx_{n_k+1})) &= \psi(d(Tx_{m_k}, Tx_{n_k})) \leq \psi(s^3 d(Tx_{m_k}, Tx_{n_k})) \\ &\leq f(\psi(M_s(x_{m_k}, x_{n_k}), \phi(M_s(x_{m_k}, x_{n_k}))),\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}&M_s(x_{m_k}, x_{n_k}) \\ &= \max\left\{ d(x_{m_k}, x_{n_k}), d(x_{m_k}, fx_{m_k}), d(x_{n_k}, fx_{n_k}), \frac{d(fx_{m_k}, x_{n_k}) + d(x_{m_k}, fx_{n_k})}{2s} \right\}.\end{aligned} \tag{3.7}$$

Letting $n \to \infty$ in (3.7) and using (i)–(iv) of Lemma 2.12, we have

$$\epsilon \leq \limsup_{k\to\infty} M_s(x_{m_k}, x_{n_k}) \leq \max\{s\epsilon, 0, \frac{s^2\epsilon + s^2\epsilon}{2s}\} = s\epsilon. \tag{3.8}$$

Now, from (3.6) and using (3.8) we have

$$\begin{aligned}\psi(s\epsilon) = \psi(s^3 \frac{\epsilon}{s^2}) &\leq \psi(s^3 \limsup_{k\to\infty}(d(x_{m_k+1}, x_{n_k+1}))) = \psi(s^3 \limsup_{k\to\infty} d(Tx_{m_k}, Tx_{n_k}) \\ &= \limsup_{k\to\infty} \psi(s^3 d(Tx_{m_k}, Tx_{n_k}) \\ &\leq f(\psi(\limsup_{k\to\infty} M_s(x_{m_k}, x_{n_k}), \limsup_{k\to\infty} \phi(M_s(x_{m_k}, x_{n_k}))) \\ &\leq f(\psi(s\epsilon), \limsup_{k\to\infty} \phi(M_s(x_{m_k}, x_{n_k}))) \leq \psi(s\epsilon),\end{aligned}$$

which implies that $f(\psi(s\epsilon), \limsup_{k\to\infty} \phi(M_s(x_{m_k}, x_{n_k}))) = \psi(s\epsilon)$. Hence, by the property (ii) of f, we have either $\psi(s\epsilon) = 0$ or $\limsup_{k\to\infty} \phi(M_s(x_{m_k}, x_{n_k})) = 0$, in either case it is a contradiction. So we conclude that $\{x_n\}$ is a b-Cauchy sequence in (X, d). Since (X, d) is b-complete, it follows that there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

First, we assume that T is continuous. Then we have $\lim_{n\to\infty} Tx_n = Tz$, so that $Tz = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z$.

Now we assume that (ii) holds, that is $\beta(x_n) \geq 1$ for all n. Then we have $\beta(z) \geq 1$. We assume that $Tz \neq z$. From the triangular inequality, we have $d(z, Tz) \leq s[d(z, Tx_n) + d(Tx_n, Tz)]$. On taking the upper limit as $n \to \infty$, we have

$$\frac{1}{s} d(z, Tz) \leq \limsup_{n\to\infty} d(Tx_n, Tz). \tag{3.9}$$

Also we have $d(Tx_n, Tz) \leq s[d(Tx_n, z) + d(z, Tz)]$. On taking the upper limit as $n \to \infty$, we obtain

$$\limsup_{n\to\infty} d(Tx_n, Tz) \leq sd(z, Tz). \tag{3.10}$$

From (3.9) and (3.10), we have

$$\frac{1}{s} d(z, Tz) \leq \limsup_{n\to\infty} d(Tx_n, Tz) \leq sd(z, Tz). \tag{3.11}$$

Since $\alpha(x_n)\beta(z) \geq 1$, from (3.1), we get

$$\begin{aligned} \psi(d(z, Tz)) &\leq \psi(s^2 d(z, Tz)) = \psi(s^3[\frac{1}{s} d(z, Tz)]) \leq \psi(s^3[\limsup_{n\to\infty} d(Tx_n, Tz)]) \\ &= \limsup_{n\to\infty} \psi(s^3[d(Tx_n, Tz)]) \leq \limsup_{n\to\infty} f(\psi(M_s(x_n, z)), \phi(M_s(x_n, z))) \\ &\leq f(\limsup_{n\to\infty} \psi(M_s(x_n, z)), \limsup_{n\to\infty} \phi(M_s(x_n, z))), \end{aligned} \tag{3.12}$$

where $M_s(x_n, z) = \max\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), \frac{d(x_n, Tz) + d(z, Tx_n)}{2s}\}$.

On taking the upper limit and using (3.11), we have $\limsup_{n\to\infty} M_s(x_n, z) = \max\{0, 0, d(z, Tz), \limsup_{n\to\infty} \frac{d(x_n, Tz)}{2s}\} = d(z, Tz)$. Now, from (3.12) we obtain

$$\begin{aligned} \psi(d(z, Tz)) &\leq f(\psi(\limsup_{n\to\infty} M_s(x_n, z)), \limsup_{n\to\infty} \phi(M_s(x_n, z))) \\ &\leq f(\psi(d(z, Tz)), \limsup_{n\to\infty} \phi(M_s(x_n, z))) \leq \psi(d(z, Tz)), \end{aligned}$$

so that $f(\psi(d(z, Tz)), \limsup_{n\to\infty} \phi(M_s(x_n, z))) = \psi(d(z, Tz))$. Hence, either $\psi(d(z, Tz)) = 0$ or $\limsup_{n\to\infty} \phi(M_s(x_n, z)) = 0$. In either case it is a contradiction. Hence $Tz = z$. ∎

Theorem 3.3. *In addition to the hypotheses of Theorem* 3.2*, suppose that* $\alpha(u) \geq 1$ *and* $\beta(u) \geq 1$ *whenever* $Tu = u$*. Then* T *has a unique fixed point.*

Proof. Let u and v be fixed points of T; by hypothesis $\alpha(u) \geq 1$ and $\beta(v) \geq 1$. Hence, from (3.1) we have

$$\psi(d(u, v)) = \psi(d(Tu, Tv)) \leq \psi(s^3 d(Tu, Tv)) \leq f(\psi(M_s(u, v)), \phi(M_s(u, v))), \tag{3.13}$$

where

$$\begin{aligned} M_s(u, v) &= \max\left\{d(u, v), d(u, Tu), d(v, Tv), \frac{d(u, Tv) + d(v, Tu)}{2s}\right\} \\ &= \max\left\{d(u, v), 0, \frac{d(u, v)}{s}\right\} = d(u, v). \end{aligned}$$

By using inequality (3.13), we get

$$\begin{aligned}\psi(d(u,v)) = \psi(d(Tu,Tv)) &\leq \psi(s^3 d(Tu,Tv)) \leq f(\psi(M_s(u,v)), \phi(M_s(u,v)))\\ &= f(\psi(d(u,v)), \phi(d(u,v))) \leq \psi(d(u,v)),\end{aligned}$$

so that $f(\psi(M_s(u,v)), \phi((M_s(u,v))) = \psi(d(u,v))$. Hence, either $\psi(d(u,v)) = 0$ or $\phi(d(u,v)) = 0$. In any case it implies that $d(u,v) = 0$. Thus, $u = v$. Therefore f has a unique fixed point. ∎

REMARK 3.4. Theorem 3.2 and Theorem 3.3 extend Theorem 1.2 to b-metric spaces.

DEFINITION 3.5. Let (X,d) be a b-metric space with coefficient $s \geq 1$, and A and B be two closed subsets of X such that $A \cap B \neq \emptyset$. Let $T : A \cup B \to A \cup B$ be a cyclic mapping. If there exist $\psi \in \Psi$, $\phi \in \Phi$ and $f \in \mathcal{C}$ such that

$$\psi(s^3 d(Tx,Ty)) \leq f(\psi(M_s(x,y)), \phi(M_s(x,y))), \tag{3.14}$$

for all $x \in A$ and $y \in B$. Then we say that T is a generalized TAC-cyclic contractive mapping.

THEOREM 3.6. *Let A and B be two nonempty closed subsets of a b-complete b-metric space (X,d) such that $A \cap B \neq \emptyset$, and let $T : A \cup B \to A \cup B$ be a cyclic mapping. If T is a generalized TAC-cyclic contractive mapping, then T has a unique fixed point in $A \cap B$.*

Proof. We define $\alpha, \beta : A \cup B \to [0,\infty)$ by

$$\alpha(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \beta(x) = \begin{cases} 1, & \text{if } x \in B \\ 0, & \text{otherwise.} \end{cases}$$

For any $x, y \in A \cup B$ with $\alpha(x)\beta(y) \geq 1$, we have $x \in A$ and $y \in B$. Hence, by the hypotheses, the inequality (3.14) holds, which in turn means that the inequality (3.1) holds. Therefore T is a generalized TAC-contractive mapping on $A \cup B$.

Since $A \cap B \neq \emptyset$, there exists $x_0 \in A \cap B$ and hence $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$. Let $\{x_n\}$ be a sequence in X such that $\beta(x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ so that $x_n \in B$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$. Since B is b-closed we have $x \in B$ and hence $\beta(x) \geq 1$. Therefore all the hypotheses of Theorem 3.2 hold and hence T has a fixed point.

Let u (say) be a fixed point of T. If $u \in A$, then $u = Tu \in B$. Similarly, if $u \in B$, then $u = Tu \in A$, hence $u \in A \cap B$. This implies that $\alpha(u) \geq 1$ and $\beta(u) \geq 1$. Therefore, by Theorem 3.3, T has a unique fixed point. ∎

4. Corollaries and examples

COROLLARY 4.1. *Let (X,d) be a b-complete metric space with coefficient $s \geq 1$ and $T : X \to X$ be a selfmap of X. If there exist $\psi \in \Psi$, $\phi \in \Phi$ and $f \in \mathcal{C}$ such that*

$$\psi(s^3 d(Tx,Ty)) \leq f(\psi(M_s(x,y)), \phi(M_s(x,y))) \text{ for all } x, y \in X, \tag{4.1}$$

then T has a unique fixed point.

Proof. By choosing $\alpha(x) = \beta(x) = 1$ for all $x \in X$, clearly the inequality (4.1) implies the inequality (3.1) and hence by Theorem 3.3, the conclusion of corollary follows. ∎

COROLLARY 4.2. *Let (X, d) be a complete metric space. Let $T : X \to X$ be a selfmap of X. Assume that there exist two mappings $\alpha, \beta : X \to [0, \infty)$ and $\psi \in \Psi$, $\phi \in \Phi$ and $f \in \mathcal{C}$ such that $\alpha(x)\beta(y) \geq 1$ implies $\psi(d(Tx, Ty)) \leq f(\psi(M(x, y)), \phi(M(x, y)))$ for all x, y in X, where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$. Further, suppose that there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$, T is a cyclic (α, β)-admissible mapping and either of the following conditions hold:*

(i) *T is continuous,*

(ii) *if $\{x_n\}$ is a sequence in X such that $x_n \to z$ and $\beta(x_n) \geq 1$ for all n, then $\beta(z) \geq 1$.*

Then T has a fixed point.

Proof. The result follows from Theorem 3.2 by taking $s = 1$. ∎

From Theorem 3.3 by taking $s = 1$ and $\alpha(x) = \beta(x) = 1$ we deduce the following corollary.

COROLLARY 4.3. *Let (X, d) be a complete metric space and $T : X \to X$ be a selfmap of X. If there exist $\psi \in \Psi$, $\phi \in \Phi$ and $f \in \mathcal{C}$ such that $\psi(d(Tx, Ty)) \leq f(\psi(M(x, y)), \phi(M(x, y)))$ for all $x, y \in X$, where $M(x, y)$ is defined as in Corollary 4.2. Then T has a unique fixed point.*

COROLLARY 4.4. *Let (X, d) be a b-complete metric space with coefficient $s \geq 1$. Let $T : X \to X$ be a selfmap of X. Assume that there exist two mappings $\alpha, \beta : X \to [0, \infty)$ and $\psi \in \Psi$, $\phi \in \Phi$ such that $\alpha(x)\beta(y) \geq 1$ implies $\psi(s^3 d(Tx, Ty)) \leq \psi(M_s(x, y)) - \phi(M_s(x, y))$. Further, suppose that there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$, T is a cyclic (α, β)-admissible mapping and either of the following conditions hold:*

(i) *T is continuous,*

(ii) *if $\{x_n\}$ is a sequence in X such that $x_n \to z$ and $\beta(x_n) \geq 1$ for all n, then $\beta(z) \geq 1$.*

Then T has a fixed point.

Proof. Follows from Theorem 3.2 by taking $f(a, t) = a - t$. ∎

REMARK 4.5. Theorem 2.11 follows as a corollary to Corollary 4.4 by taking $s = 1$ and $\alpha(x) = \beta(x) = 1$ for all $x \in X$, since $\Phi_1 \subset \Phi$.

COROLLARY 4.6. *Let A and B be two nonempty closed subsets of a b-complete metric space (X, d) such that $A \cap B \neq \emptyset$, and let $T : A \cup B \to A \cup B$ be a cyclic mapping. If there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that $\psi(s^3 d(Tx, Ty)) \leq \psi(M_s(x, y)) - \phi(M_s(x, y))$, for all $x \in A$ and $y \in B$, then T has a unique fixed point in $A \cap B$.*

Proof. The result follows from Theorem 3.6 by taking $f(a, t) = a - t$. ∎

The following is an example in support of Theorem 3.2.

EXAMPLE 4.7. Let $X = [0,\infty)$ and let $d : X \times X \to [0,\infty)$ be defined by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ \frac{7}{2} + x + y, & \text{if } x, y \in [0,1),\ x \neq y \\ 5 + \frac{1}{x+y}, & \text{if } x, y \in (1,\infty),\ x \neq y \\ \frac{5}{2}, & \text{otherwise.} \end{cases}$$

Clearly (X, d) is a b-metric space with coefficient $s = \frac{11}{10}$. Define $T : X \to X$ by $Tx = \begin{cases} 2 - x, & \text{if } x \in [0,2] \\ x, & \text{if } x \in (2,\infty) \end{cases}$ and $\alpha, \beta : X \to [0,\infty)$ by

$$\alpha(x) = \begin{cases} 1, & \text{if } x \in [1,2] \\ 0, & \text{if } x \in [0,1) \cup (2,\infty), \end{cases} \quad \text{and} \quad \beta(x) = \begin{cases} 1, & \text{if } x \in [0,1] \\ 0, & \text{if } x \in (1,\infty). \end{cases}$$

Since for any $x \in X$, $\alpha(x) \geq 1 \Leftrightarrow x \in [1,2]$, where $Tx = 2 - x \in [0,1]$, hence $\beta(Tx) \geq 1$. Also for $x \in X, \beta(x) \geq 1 \Leftrightarrow x \in [0,1]$, where $Tx = 2 - x \in [1,2]$, hence $\alpha(Tx) \geq 1$. Therefore T is a cyclic (α,β)-admissible mapping.

Next we show that T is a generalized TAC-contractive mapping. For any $x \in [0,1]$ and $y \in [1,2]$ we have $\alpha(x)\beta(y) \geq 1$; also $Tx \in [1,2]$ and $Ty \in [0,1]$. Hence $d(Tx,Ty) = \frac{5}{2}$. Now, we choose $\psi(t) = t$, $\phi(t) = \frac{4295}{110000}t$ and $f(a,t) = a - t$. For $x \in [0,1]$ and $y \in [1,2]$ we have

$$\begin{aligned} M_s(x,y) &= \max\left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2s} \right\} \\ &= \max\left\{ \frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5 + \frac{1}{y+2-x} + \frac{7}{2} + x + y - 2}{2(\frac{11}{10})} \right\}, \end{aligned}$$

so that $\frac{3875}{1100} \leq M_s(x,y) \leq 5$. Now, we have

$$\begin{aligned} \psi(s^3 d(Tx,Ty)) &= \psi\left(\left(\frac{11}{10}\right)^3 \left(\frac{5}{2}\right) \right) = \psi\left(\frac{33275}{10000}\right) = \frac{33275}{10000} \\ &= \frac{3875}{1100} - \frac{21475}{110000} = \psi\left(\frac{3875}{1100}\right) - \phi(5) \\ &\leq \psi(M_s(x,y)) - \phi(M_s(x,y)) = f(\psi(M_s(x,y)), \phi(M_s(x,y))). \end{aligned}$$

Hence, T is a generalized TAC-contractive mapping. Clearly condition (ii) of Theorem 3.2 holds. Hence T satisfies all the hypotheses of Theorem 3.2 and $x = 1$ and every element of $(2,\infty)$ are fixed points of T. So T has more than one fixed point in X.

Here we observe that in the usual metric sense, for any $\alpha, \beta : X \to [0,\infty)$ such that T is a cyclic (α,β)-admissible mapping, we can easily verify that

$$\psi(d(Tx,Ty)) \nleq f(\psi(d(x,y)), \phi(d(x,y))),$$

for any $\psi \in \Psi$, $\phi \in \Phi$ and f defined as in Definition 1.1, and for any $x \neq y$ with $\alpha(x)\beta(y) \geq 1$. Hence T is not a TAC-contractive mapping. Therefore Theorem 1.2 is not applicable.

One more example in support of Theorem 3.2 is the following:

EXAMPLE 4.8. Let $X = [0,\infty)$ and let $d : X \times X \to [0,\infty)$ be defined by $d(x,y) = |x-y|^2$ for all $x, y \in X$. Then clearly (X,d) is a b-metric space with coefficient $s = 2$. Let us define $T : X \to X$ by $T(x) = \begin{cases} 1-\frac{x}{4}, & \text{if } x \in [0,1] \\ x, & \text{if } x \in (1,\infty) \end{cases}$ and $\alpha, \beta : X \to [0,\infty)$ by

$$\alpha(x) = \beta(x) = \begin{cases} \frac{2}{x+1}, & \text{if } x \in [0,1] \\ 0, & \text{if } x \in (1,\infty). \end{cases}$$

Since for any $x \in X$, $\alpha(x) \geq 1 \Leftrightarrow x \in [0,1]$, we have $\beta(Tx) = \frac{2}{Tx+1} = \frac{2}{2-\frac{x}{4}} \geq 1$. Since $\alpha(x) = \beta(x)$, clearly T is a cyclic (α,β)-admissible mapping.

Next we show that T is generalized TAC-contractive mapping. We assume that $\alpha(x)\beta(y) \geq 1$. This implies that $x, y \in [0,1]$ and hence $Tx = 1-\frac{x}{4}$ and $Ty = 1-\frac{y}{4}$. We choose

$$\psi(t) = t, \quad f(a,t) = \frac{a}{1+t} \quad \text{and} \quad \phi(x) = \begin{cases} \frac{2}{3}, & \text{if } x \in [0,2] \\ 1, & \text{if } x \in (2,\infty). \end{cases}$$

Then

$$\begin{aligned} M_s(x,y) &= \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\right\} \\ &= \max\left\{|x-y|^2, |x-1-\frac{x}{4}|^2, |y-1-\frac{y}{4}|^2, \frac{|x-1-\frac{y}{4}|^2+|x-1-\frac{x}{4}|^2}{4}\right\}, \end{aligned}$$

Now, we have

$$\begin{aligned} \psi(s^3 d(Tx,Ty)) &= \psi(8|\frac{x}{4}-\frac{y}{4}|^2) = \psi(\frac{1}{2}|x-y|^2) = |x-y|^2 \\ &\leq M_s(x,y) = \frac{2M_s(x,y))}{1+1} \leq \frac{2M_s(x,y))}{1+\frac{2}{3}} \\ &= \frac{\psi(M_s(x,y))}{1+\phi(M_s(x,y))} = f(\psi(M_s(x,y)), \phi(M_s(x,y))). \end{aligned}$$

Hence T is generalized TAC-contractive mapping. For a sequence $\{x_n\}$ in X such that $x_n \to x$ and $\alpha(x_n) \geq 1$ for all n, this implies that $\{x_n\} \subseteq [0,1]$. Since $[0,1]$ is a closed subset of X then $x \in [0,1]$, therefore $\beta(x) \geq 1$. Hence T satisfies all the hypotheses of Theorem 3.2 and $x = \frac{4}{5}$ and also every element of the interval $(1,\infty)$ is a fixed point of T.

Here we observe that with the usual metric on $[0,\infty)$, the inequality (2.1) fails to hold: for any $x, y \in (1,\infty)$ with $x \neq y$, we have $d(x,y) = M(x,y)$, and hence

$$\psi(d(Tx,Ty)) = \psi(d(x,y)) \nleq \psi(d(x,y)) - \varphi(d(x,y)) = \psi(M(x,y)) - \varphi(M(x,y)),$$

for any $\psi \in \Psi$ and $\varphi \in \Phi$. Hence, Theorem 2.11 is not applicable.

EXAMPLE 4.9. Let $X = \mathbb{R}$ and let $d : X \times X \to [0,\infty)$ be defined by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ \frac{5}{2}+|x|+|y|, & \text{if } x, y \in (-\frac{3}{2},\frac{3}{2}),\ x \neq y \\ 5+\frac{1}{|x|+|y|}, & \text{if } x, y \in (-\infty,-\frac{3}{2}] \cup (\frac{3}{2},\infty),\ x \neq y \\ \frac{5}{2}, & \text{otherwise.} \end{cases}$$

Clearly, d is a b-metric with coefficient $s=\frac{11}{10}$. We define $T:X\to X$ by $Tx=3-x$ and $\alpha,\beta:X\to[0,\infty)$ by

$$\alpha(x)=\begin{cases}1, & \text{if } x\in[0,\frac{3}{2}]\\ 0, & \text{otherwise,}\end{cases} \quad \text{and} \quad \beta(x)=\begin{cases}e, & \text{if } x\in[\frac{3}{2},3]\\ 0, & \text{otherwise.}\end{cases}$$

Since for any $x\in X$, $\alpha(x)\geq 1 \Leftrightarrow x\in[0,\frac{3}{2}]$, where $Tx=3-x\in[\frac{3}{2},3]$, hence $\beta(Tx)\geq 1$. Also for $x\in X$, $\beta(x)\geq 1\Leftrightarrow x\in[\frac{3}{2},3]$, where $Tx=3-x\in[0,\frac{3}{2}]$, hence $\alpha(Tx)\geq 1$. Therefore T is a cyclic (α,β)-admissible mapping.

We now show that T is a generalized TAC-contractive mapping. For any $x\in[0,\frac{3}{2}]$ and $y\in[\frac{3}{2},3]$ we have $\alpha(x)\beta(y)\geq 1$; also $Tx\in[\frac{3}{2},3]$ and $Ty\in[0,\frac{3}{2}]$. Hence $d(Tx,Ty)=\frac{5}{2}$. Now, for $t,s\geq 0$ we choose

$$\psi(t)=t,\quad f(a,t)=\frac{a}{1+t}\quad \text{and}\quad \phi(t)=\begin{cases}t, & \text{if } t\in[0,\frac{3}{2}]\\ \frac{2077}{43923}, & \text{if } t\in\mathbb{R}\setminus[0,\frac{3}{2}].\end{cases}$$

Then, for $x\in[0,\frac{3}{2}]$ and $y\in[\frac{3}{2},3]$, we have

$$\begin{aligned}M_s(x,y)&=\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2s}\right\}\\&=\max\left\{\frac{5}{2},\frac{5}{2},\frac{5}{2},\frac{5+\frac{1}{|y|+|3-x|}+\frac{5}{2}+|x|+|3-y|}{2(\frac{11}{10})}\right\},\end{aligned}$$

hence $\frac{230}{66}\leq M_s(x,y)\leq\frac{325}{66}$.

Now, we have

$$\begin{aligned}\psi(s^3d(Tx,Ty))&=\psi\left(\left(\frac{11}{10}\right)^3\left(\frac{5}{2}\right)\right)=\psi\left(\frac{33275}{10000}\right)=\frac{33275}{10000}\\&=\frac{\frac{115}{33}}{1+\frac{2077}{43923}}\leq\frac{M_s(x,y)}{1+\frac{2077}{43923}}=\frac{\psi(M_s(x,y))}{1+\phi(M_s(x,y))}\\&=f(\psi(M_s(x,y)),\phi(M_s(x,y))).\end{aligned}$$

Hence, T is a generalized TAC-contractive mapping. Thus, T satisfies all the hypotheses of Theorem 3.3 and $x=\frac{3}{2}$ is the (unique) fixed point of T.

Here we observe that in the usual metric sense, for any $\alpha,\beta:X\to[0,\infty)$ such that T is a cyclic (α,β)-admissible mapping, we can easily verify that

$$\psi(d(Tx,Ty))\nleq f(\psi(d(x,y)),\phi(d(x,y))),$$

for any $\psi\in\Psi$, $\phi\in\Phi$ and f defined as in Definition 1.1, and for any $x\neq y$ with $\alpha(x)\beta(y)\geq 1$, so that T is not a TAC-contractive mapping. Hence Theorem 1.2 is not applicable.

Example 4.10. Let $X=[0,1]$ and let $d:X\times X\to[0,\infty)$ be defined by $d(x,y)=|x-y|^2$. Then (X,d) is a b-metric space with $s=2$. Let $A=[0,\frac{7}{24}]$ and $B=[\frac{1}{8},1]$, and define $T:A\cup B\to A\cup B$ by $T(x)=\frac{1}{3}-\frac{x}{3}$. Hence, we have $TA=[\frac{17}{72},\frac{1}{3}]\subset B$ and $TB=[0,\frac{7}{24}]=A$ which implies that T is cyclic.

We now show that T is a generalized TAC-cyclic contractive mapping. We choose $\psi(t) = t$, $\phi(t) = \frac{1}{8}$, $t \geq 0$ and $f(a,t) = \frac{a}{1+t}$. For $x \in A$ and $y \in B$ we have

$$M_s(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s}\right\}$$
$$= \max\left\{|x-y|^2, |\frac{4x}{3}-\frac{1}{3}|^2, |\frac{4y}{3}-\frac{1}{3}|^2, \frac{|x-\frac{y}{3}+\frac{1}{3}|^2+|y-\frac{x}{3}+\frac{1}{3}|^2}{4}\right\},$$

Now, we obtain

$$\psi(s^3 d(Tx,Ty)) = \psi\big(2^3 d(\frac{x}{3},\frac{y}{3})\big) = \psi\big((8|\frac{x}{3}-\frac{y}{3}|^2\big) \leq \psi\big((\frac{8}{9}|x-y|^2)\big)$$
$$= \frac{8}{9}|x-y|^2 \leq \frac{8}{9}M_s(x,y) = \frac{M_s(x,y)}{1+\frac{1}{8}}$$
$$= \frac{\psi(M_s(x,y))}{1+\phi(M_s(x,y))} = f(\psi(M_s(x,y)), \phi(M_s(x,y)).$$

Therefore, T is a generalized TAC-cyclic contractive mapping. Hence T satisfies all the hypotheses of Theorem 3.6 and $x = \frac{1}{4}$ is the fixed point of T.

Acknowledgement. The authors express their gratitude to the referee(s) for their useful comments and suggestions.

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(received 09.04.2016; in revised form 10.01.2017; available online 26.01.2017)

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