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FIXED POINT RESULTS FOR (φ, ψ)-CONTRACTIONS IN METRIC SPACES ENDOWED WITH A GRAPH AND APPLICATIONS

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Abstract. In the present work, we introduce the concepts of (G, φ, ψ)-contraction and (G, φ, ψ)-graphic contraction defined on metric spaces endowed with a graph G and we show that these two types of contractions generalize a large number of contractions. Subsequently, we investigate various results which assure the existence and uniqueness of fixed points for such mappings. According to the applications of our main results, we further obtain a fixed point theorem for cyclic operators and an existence theorem for the solution of a nonlinear integral equation. Moreover, some illustrative examples are provided to demonstrate our obtained results.

1. Introduction

A very interesting approach in the theory of fixed points in some general structures was recently given by Jachymski [7] and Gwóźdź-Lukawska and Jachymski [6] in the context of metric spaces endowed with a graph. Using this simple but very interesting idea, Jachymski [7] investigated the Banach contraction principle in metric spaces with a graph and generalized the same results in metric and partially ordered metric spaces simultaneously. He also presented its applications to the Kelisky-Rivlin theorem on the iterates of the Bernstein operators defined on the Banach space of continuous functions on $[0, 1]$. In the recent years, this idea was followed by many authors for different contraction mappings in metric spaces.

In [1], Bojor proved fixed point theorems for φ-contraction mapping on a metric space endowed with a graph. Recently, Bojor in [2] and [3] investigated the existence and uniqueness of fixed points for Kannan type and Reich type contractions on metric spaces with a graph, respectively. Very recently, Öztürk and Girgin [9] obtained some fixed point results for ψ-contraction and ψ-graphic contraction in a metric space endowed with a graph. Some new fixed point results for φ-graphic contraction and graphic contraction on a complete metric space with a graph have been presented in [10] and [5], respectively.

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In this work, motivated by the work of Jachymski [7] and Petruşel and Chifu [5], we introduce the notions of (G, φ, ψ)-contraction and (G, φ, ψ)-graphic contraction defined on metric spaces involving a graph by using some auxiliary functions and find sufficient conditions which guarantee the existence and uniqueness of a fixed point for these two types of contractions. As an application of our main results, we obtain a fixed point theorem for cyclic contractive mappings. Finally, the existence of solution of an integral equation is proved under appropriate conditions. Some examples are presented in order to verify the effectiveness and applicability of our main results.

2. Preliminaries

We shall begin by presenting some basic notions and notations of graph theory that will be needed throughout the paper. For more details, the reader is referred to [4, 7].

Let (X, d) be a metric space and Δ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Delta$. Assume further that G has no parallel edges. Now we can identify G with the pair $(V(G), E(G))$ and also it is said that the metric space (X, d) is endowed with the graph G. The graph G can be converted to a weighted graph by assigning to each edge a weight equal to the distance between its vertices.

The metric space (X, d) can also be endowed with the graphs G^{-1} and $\widetilde{G}$, where the former is the conversion of G which is obtained from G by reversing the directions of the edges, and the latter is an undirected graph obtained from G by omitting the direction of the edges. Thus, it is clear that $V(G^{-1}) = V(\widetilde{G}) = V(G) = X$ and we have

$$E(G^{-1}) = \big\{(x, y) \in X \times X : (y, x) \in E(G)\big\} \quad \text{and} \quad E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

If x and y are two vertices in a graph G, then a path in G from x to y of length N is a finite sequence $(x_i)_{i=0}^{N}$ consisting of $N+1$ distinct vertices of G such that $x_0 = x$, $x_N = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \dots, N$. The number of edges in G forming the path is called the length of the path. A graph G is connected if there is a path in G between any two vertices of G. If a graph G is not connected, then it is called disconnected and its different paths are called the components of G. Every component of G is a subgraph of it. Furthermore, G is weakly connected if the graph $\widetilde{G}$ is connected.

If G is such that $E(G)$ is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices which are contained in some path in G beginning at x is called the component of G containing x. In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of x in the equivalence relation $\Re$ defined by

$$y \,\Re\, z \iff \text{there exists a path in } G \text{ from } y \text{ to } z \qquad \big(y, z \in V(G)\big).$$

It is clear that the graph G_x is connected for all $x \in X$.

The use of auxiliary functions to generalize the contractive conditions on maps have been a subject of interest in fixed point theory. Now, we define the following class of auxiliary functions which will be used densely in the sequel.

We denote by Ψ the class of all functions $\psi : [0, +\infty) \to [0, +\infty)$ with the following properties:

(ψ_i) ψ is nondecreasing;

(ψ_{ii}) $\psi(t) = 0$ if and only if $t = 0$;

(ψ_{iii}) for every $\{t_n\} \in [0, +\infty)$, $\psi(t_n) \to 0$ if and only if $t_n \to 0$;

(ψ_{iv}) ψ is subadditive, that is, $\psi(t + s) \leq \psi(t) + \psi(s)$ for all $t, s \in [0, +\infty)$.

Also, we denote by Φ the class of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ with the following properties:

(φ_i) φ is nondecreasing;

(φ_{ii}) $\sum_{n=1}^{+\infty} \varphi^n(t) < +\infty$ for all $t > 0$, where φ^n is the n-th iterate of φ.

A function $\varphi \in \Phi$ is called a (c)-comparison function.

Lemma 2.1. *Let $\varphi \in \Phi$. Then*

(i) $\varphi(t) < t$, *for all* $t > 0$;

(ii) $\varphi(0) = 0$;

(iii) φ *is continuous at* $t = 0$.

In what follows, we assume that (X, d) is a metric space that is endowed with a directed graph G with $V(G) = X$ and $E(G) \supseteq \Delta$ and the graph G has no parallel edges unless stated otherwise. We denote by $\mathrm{Fix}(f)$ the set of all fixed points for a self-map f on X.

To present our main results, we make use of the following useful definitions.

Definition 2.2. [7] We say that the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy equivalent in X if both of them are Cauchy and further, $d(x_n, y_n) \to 0$ as $n \to +\infty$.

Definition 2.3. [11] Let f be a mapping from X into itself. We say that

(i) f is a Picard operator if f has a unique fixed point $x^* \in X$ and $f^n x \to x^*$ as $n \to +\infty$ for all $x \in X$.

(ii) f is a weakly Picard operator if the sequence $\{f^n x\}$ is convergent to a fixed point of f for all $x \in X$.

It is clear that each Picard operator is a weakly Picard operator but the converse is not generally true. Moreover, a weakly Picard operator is a Picard operator if and only if its fixed point is unique.

Definition 2.4. [7] Let $f : X \to X$ be an arbitrary mapping. We say that

(i) f is orbitally continuous on X if for all $x, y \in X$ and all sequence $\{a_n\}$ of positive integers, $f^{a_n} x \to y$ as $n \to +\infty$ implies $f\big(f^{a_n} x\big) \to fy$ as $n \to +\infty$.

(ii) f is orbitally G-continuous on X if for all $x, y \in X$ and all sequence $\{a_n\}$ of positive integers with $\big(f^{a_n}x, f^{a_{n+1}}x\big) \in E(G)$ for $n = 1, 2, \ldots$ such that $f^{a_n}x \to y$ as $n \to +\infty$ implies $f\big(f^{a_n}x\big) \to fy$ as $n \to +\infty$.

It is clear that continuity implies orbital continuity, and orbital continuity implies orbital G-continuity. But the converse of these relations is not true in general.

3. (G, φ, ψ)-contractions

In this section, following [7], we study the existence of fixed points in metric spaces with a graph by defining the (G, φ, ψ)-contraction.

Definition 3.1. One says that a mapping $f : X \to X$ is a (G, φ, ψ)-contraction if the following conditions hold:

C1) f preserves the edges of G, i.e., $(x, y) \in E(G)$ implies $(fx, fy) \in E(G)$ for all $x, y \in X$;

C2) f decreased the weight of edges of G, that is, there exist two functions $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\psi\big(d(fx, fy)\big) \leq \varphi\big(\psi\big(d(x, y)\big)\big), \tag{3.1}$$

for all $x, y \in X$ with $(x, y) \in E(G)$.

Remark 3.2. It is interesting to remark at this point that

(1) Any Banach G-contraction (see [7]) is a (G, φ, ψ)-contraction, where $\varphi(t) = \alpha t$ and $\psi(t) = t$ for some $\alpha \in (0, 1)$ and each $t \in [0, +\infty)$.

(2) Any (G, φ)-contraction (see [1]) is a (G, φ, ψ)-contraction, where $\psi(t) = t$ for each $t \in [0, +\infty)$.

(3) Any (G, ψ)-contraction (see [9]) is a (G, φ, ψ)-contraction, where $\varphi(t) = \alpha t$ for some $\alpha \in (0, 1)$ and each $t \in [0, +\infty)$.

We now give some examples of (G, φ, ψ)-contractions in metric spaces endowed with a graph.

Example 3.3. Since $E(G)$ contains all loops, it follows that each constant mapping $f : X \to X$ is a (G, φ, ψ)-contraction for any $\varphi \in \Phi$ and $\psi \in \Psi$.

Example 3.4. Each (G, φ, ψ)-contraction is a (G_0, φ, ψ)-contraction, where G_0 is the complete graph with $V(G_0) = X$, that is, $E(G_0) = X \times X$.

Example 3.5. Suppose that $\preceq$ is a partial order on X and consider the poset graph G_1, that is, $V(G_1) = X$ and

$$E(G_1) = \big\{(x, y) \in X \times X : x \preceq y\big\}.$$

Then $E(G_1)$ contains all loops. Now, (G_1, φ, ψ)-contractions are precisely the nondecreasing order contractive mappings on X. That is, Condition (C1) means that f is nondecreasing with respect to $\preceq$, and Condition (C2) means that f is an order (G_1, φ, ψ)-contraction, i.e., (3.1) holds for all $x, y \in X$ with $x \preceq y$.

Our first proposition is an immediate consequence of symmetry of d and Definition 3.1.

Proposition 3.6. *If a mapping $f : X \to X$ satisfies Condition* (C1) *(respectively, Condition* (C2)*) for a graph G, then it satisfies Condition* (C1) *(respectively, Condition* (C2)*) for the graphs G^{-1} and $\widetilde{G}$. In particular, a (G,φ,ψ)-contraction is both a (G^{-1},φ,ψ)-contraction and a $(\widetilde{G},\varphi,\psi)$-contraction.*

The following result shows that there is a close relationship between convergence of an iteration sequence which can be obtained by using a (G,φ,ψ)-contraction mapping and connectivity of the graph.

Theorem 3.7. *The following statements are equivalent:*

(i) *G is weakly connected;*

(ii) *For each (G,φ,ψ)-contraction $f : X \to X$ and each $x, y \in X$, the sequences $\{f^n x\}$ and $\{f^n y\}$ are Cauchy equivalent;*

(iii) *Each (G,φ,ψ)-contraction has at most one fixed point in X.*

Proof. (i) $\Rightarrow$ (ii): Let $x \in X$ and $y \in [x]_{\widetilde{G}}$. Then there is a path $(x_i)_{i=0}^N$ in $\widetilde{G}$ from x to y, which means $x_0 = x$, $x_N = y$, and $(x_{i-1}, x_i) \in E(\widetilde{G})$ for $i = 1, 2, \ldots, N$. Because f is a (G,φ,ψ)-contraction, it follows by Proposition 3.6 that f is a $(\widetilde{G},\varphi,\psi)$-contraction and so it preserves the edges of $\widetilde{G}$. Hence an easy induction yields

$$\big(f^n x_{i-1}, f^n x_i\big) \in E(\widetilde{G}) \quad (i = 1, 2, \ldots, N,\ n \geq 1).$$

Furthermore, by the contractive condition (3.1) and the fact that φ is nondecreasing, we get

$$\begin{aligned}\psi\big(d\big(f^n x_{i-1}, f^n x_i\big)\big) &\leq \varphi\big(\psi\big(d\big(f^{n-1} x_{i-1}, f^{n-1} x_i\big)\big)\big)\\ &\leq \varphi^2\big(\psi\big(d\big(f^{n-2} x_{i-1}, f^{n-2} x_i\big)\big)\big)\\ &\ \vdots\\ &\leq \varphi^n\big(\psi\big(d(x_{i-1}, x_i)\big)\big),\end{aligned}$$

for all $n \geq 1$ and all $i = 1, 2, \ldots, N$. Thus, by using the property of ψ and the triangle inequality, we get

$$\begin{aligned}\psi\big(d\big(f^n x, f^n y\big)\big) &\leq \sum_{i=1}^N \psi\big(d\big(f^n x_{i-1}, f^n x_i\big)\big)\\ &\leq \sum_{i=1}^N \varphi^n\big(\psi\big(d(x_{i-1}, x_i)\big)\big) < +\infty \quad \text{as} \quad n \to +\infty.\end{aligned} \tag{3.2}$$

So $d\big(f^n x, f^n y\big) \to 0$. Now, the weak connectivity of G gives $fx \in X = [x]_{\widetilde{G}}$, and so, setting $y = fx$ in (3.2) yields

$$\psi\big(d\big(f^n x, f^{n+1} x\big)\big) \leq \sum_{i=1}^N \varphi^n\big(\psi\big(d(x_{i-1}, x_i)\big)\big)$$

for all $n \geq 1$. Hence

$$\sum_{n=0}^{\infty} \psi\big(d(f^n x, f^{n+1}x)\big) = \sum_{i=1}^{N} \sum_{n=0}^{\infty} \varphi^n\big(\psi\big(d(x_{i-1}, x_i)\big)\big) < +\infty,$$

and a standard argument shows that $\{f^n x\}$ is Cauchy. Similarly, $\{f^n y\}$ is Cauchy and hence $\{f^n x\}$ and $\{f^n y\}$ are Cauchy equivalent.

(ii) $\Rightarrow$ (iii): Let f is a (G, φ, ψ)-contraction and $x, y \in \mathrm{Fix}(f)$. Since, by the hypothesis, $\{f^n x\}$ and $\{f^n y\}$ are Cauchy equivalent, it follows that

$$d(x, y) = d\big(f^n x, f^n y\big) \to 0.$$

Therefore, $x = y$.

(iii) $\Rightarrow$ (i): Suppose on the contrary that G is not weakly connected, that is, $\widetilde{G}$ is disconnected. Then there exists an $x_0 \in X$ such that both sets $[x_0]_{\widetilde{G}}$ and $X \setminus [x_0]_{\widetilde{G}}$ are nonempty. Let $y_0 \in X \setminus [x_0]_{\widetilde{G}}$ and define $f : X \to X$ by

$$fx = \begin{cases} x_0, & x \in [x_0]_{\widetilde{G}}, \\ y_0, & x \in X \setminus [x_0]_{\widetilde{G}}. \end{cases}$$

Obviously, $\mathrm{Fix}(f) = \{x_0, y_0\}$. To get a contradiction, we show that f is a (G, φ, ψ)-contraction. If $(x, y) \in E(G)$, then $[x]_{\widetilde{G}} = [y]_{\widetilde{G}}$, and so either $x, y \in [x_0]_{\widetilde{G}}$ or $x, y \in X \setminus [x_0]_{\widetilde{G}}$. Hence, in both cases, $fx = fy$, so $(fx, fy) \in \widetilde{G}$ because $E(G) \supseteq \Delta$. Moreover,

$$\psi\big(d(fx, fy)\big) = 0 \leq \varphi\big(\psi\big(d(x, y)\big)\big),$$

where $\varphi \in \Phi$ and $\psi \in \Psi$. Thereby, f is a (G, φ, ψ)-contraction having two fixed points which violates the assumption. ■

The following result is an easy consequence of Theorem 3.7.

Corollary 3.8. *Let (X, d) be a complete metric space and G be a graph weakly connected. Then for each (G, φ, ψ)-contraction $f : X \to X$, there exists $x^* \in X$ such that $f^n x \to x^*$ for all $x \in X$.*

Proof. Let $f : X \to X$ be a (G, φ, ψ)-contraction and fix any point $x \in X$. Since G is weakly connected, it follows by Theorem 3.6 that the sequence $\{f^n x\}$ is Cauchy in X, and since X is complete, there exists $x^* \in X$ such that $f^n x \to x^*$. Now, let $y \in X$ be given. Then by using the Cauchy equivalence of $\{f^n y\}$ and $\{f^n x\}$, we have

$$d\big(f^n y, x^*\big) \leq d\big(f^n y, f^n x\big) + d\big(f^n x, x^*\big) \to 0.$$

Therefore, $f^n y \to x^*$. ■

The next example shows that we could not add that x^* is a fixed point of f in Corollary 3.7.

Example 3.9. Let $X = [0, 1]$ be endowed with the Euclidean metric and define a graph G by

$$E(G) = \big\{(0, 0)\big\} \cup \big\{(0, 1)\big\} \cup \big\{(x, y) \in (0, 1] \times (0, 1] : x \geq y\big\}.$$

Consider $f : X \to X$ defined by the rule

$$fx = \begin{cases} \frac{x}{3}, & \text{if } 0 < x \leq 1, \\ \frac{1}{2}, & \text{if } x = 0. \end{cases}$$

Take $\varphi(t) = \frac{t}{2}$ and $\psi(t) = \frac{t}{t+1}$ for each $t \in [0, +\infty)$. Then it can be easily seen that G is a weakly connected graph and f is a (G, φ, ψ)-contraction. Clearly, $f^n x \to 0$ for all $x \in X$, but f has no fixed point.

Proposition 3.10. *Let (X, d) be a complete metric space endowed with a graph G and $f : X \to X$ be a (G, φ, ψ)-contraction for which there exists $x_0 \in X$ such that $fx_0 \in [x_0]_{\widetilde{G}}$. Then $[x_0]_{\widetilde{G}}$ is f-invariant and $f|_{[x_0]_{\widetilde{G}}}$ is a $(\widetilde{G}_{x_0}, \varphi, \psi)$-contraction, where $\widetilde{G}_{x_0}$ is the component of $\widetilde{G}$ containing x_0. Moreover, if $x, y \in [x_0]_{\widetilde{G}}$, then $\{f^n x\}$ and $\{f^n y\}$ are Cauchy equivalent.*

Proof. Note that since $fx_0 \in [x_0]_{\widetilde{G}}$, we have $[fx_0]_{\widetilde{G}} = [x_0]_{\widetilde{G}}$. Suppose that $x \in [x_0]_{\widetilde{G}}$. Then there exists a path $(x_i)_{i=0}^N$ in $\widetilde{G}$ from x_0 to x, i.e., $x_N = x$ and $(x_{i-1}, x_i) \in E(\widetilde{G})$ for $i = 1, 2, \dots, N$. Since f is a (G, φ, ψ)-contraction, it follows by Proposition 3.6 that f is a $(\widetilde{G}, \varphi, \psi)$-contraction and so it preserves the edges of $\widetilde{G}$ which yields $(fx_{i-1}, fx_i) \in E(\widetilde{G})$ for $i = 1, 2, \dots, N$, that is, $(fx_i)_{i=0}^N$ is a path in $\widetilde{G}$ from fx_0 to fx. Therefore, $fx \in [fx_0]_{\widetilde{G}} = [x_0]_{\widetilde{G}}$. Consequently, $[x_0]_{\widetilde{G}}$ is f-invariant.

Since f is itself a $(\widetilde{G}, \varphi, \psi)$-contraction and $E(\widetilde{G}_{x_0}) \subseteq E(\widetilde{G})$, to see that it is a $(\widetilde{G}_{x_0}, \varphi, \psi)$-contraction on $[x_0]_{\widetilde{G}}$, it suffices to show that f preserves the edges of $\widetilde{G}_{x_0}$. To this end, suppose that (x, y) is any edge of $\widetilde{G}_{x_0}$, i.e., $(x, y) \in E(\widetilde{G}_{x_0})$. This means that there exists a path $(x_i)_{i=0}^N$ in $\widetilde{G}$ from x_0 to y such that $x_{N-1} = x$. On the other hand, $fx_0 \in [x_0]_{\widetilde{G}}$. So there is another path $(y_j)_{j=0}^M$ in $\widetilde{G}$ from x_0 to fx_0, i.e., $y_0 = x_0$, $y_M = fx_0$, and $(y_{j-1}, y_j) \in E(\widetilde{G})$ for $i = 1, 2, \dots, N$. Repeating the argument above, we realize that

$$(x_0 = y_0, y_1, \cdots, fx_0 = y_M, fx_1, fx_2, \cdots, fx = fx_{N-1}, fy = fx_N)$$

is a path in $\widetilde{G}$ from x_0 to fy. In particular,

$$(fx, fy) = (fx_{N-1}, fx_N) \in E(\widetilde{G}_{x_0}).$$

Furthermore, since the graph $\widetilde{G}_{x_0}$ is connected and $f : [x_0]_{\widetilde{G}} \to [x_0]_{\widetilde{G}}$ is a $(\widetilde{G}_{x_0}, \varphi, \psi)$-contraction, it follows by Theorem 3.6 that the sequences $\{f^n x\}$ and $\{f^n y\}$ are Cauchy equivalent for all $x, y \in [x_0]_{\widetilde{G}}$. ■

For any mapping which satisfies the condition of Corollary 3.8 to have a fixed point we need to add property $(\star)$, which is given in the following theorem.

Theorem 3.11. *Let X be a complete metric space and the triple (X, d, G) satisfies the following property:*

($\star$) *If a sequence $\{x_n\}$ in X converges to some $x \in X$ and it satisfies $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 1$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \geq 1$.*

Suppose that $f : X \to X$ is a (G, φ, ψ)-contraction, and $X_f = \{x \in X : (x, fx) \in E(G)\}$. Then the following assertions hold:

(i) *$f|_{[x]_{\widetilde{G}}}$ is a Picard operator for all $x \in X_f$.*

(ii) *If $X_f \neq \emptyset$ and G is weakly connected, then f is a Picard operator.*

(iii) $\operatorname{card}\big(\operatorname{Fix}(f)\big) = \operatorname{card}\big\{[x]_{\widetilde{G}} : x \in X_f\big\}$.

(iv) $\operatorname{Fix}(f) \neq \emptyset$ *if and only if* $X_f \neq \emptyset$.

(v) *f has a unique fixed point if and only if there exists $x_0 \in X_f$ such that $X_f \subseteq [x_0]_{\widetilde{G}}$.*

(vi) *$f|_{X'}$ is a weakly Picard operator, where $X' = \bigcup\big\{[x]_{\widetilde{G}} : x \in X_f\big\}$.*

(vii) *If $f \subseteq E(G)$, then f is a weakly Picard operator.*

Proof. We prove each part of the theorem separately.

(i) Let $x \in X_f$. Then $(x, fx) \in E(G) \subseteq E(\widetilde{G})$ and hence $fx \in [x]_{\widetilde{G}}$. Now, by Proposition 3.10, the mapping $f|_{[x]_{\widetilde{G}}}$ is a $(\widetilde{G}_x, \varphi, \psi)$-contraction and if $y \in [x]_{\widetilde{G}}$, then the sequences $\{f^n x\}$ and $\{f^n y\}$ are Cauchy equivalent. Since X is complete, an argument similar to that appeared in the proof of Corollary 3.8 establishes that there exists $x^* \in X$ such that $f^n y \to x^*$ for all $y \in [x]_{\widetilde{G}}$. In particular, $f^n x \to x^*$. To prove our claim, note first that since f is a (G, φ, ψ)-contraction and $(x, fx) \in E(G)$, it follows by induction that

$$\big(f^n x, f^{n+1} x\big) \in E(G) \qquad (n \geq 1). \tag{3.3}$$

By applying the property ($\star$), there exists a subsequence $\{f^{n_k} x\}$ of $\{f^n x\}$ such that $(f^{n_k} x, x^*) \in E(G)$ for all $k \geq 1$.

Now, by using (3.3), we conclude that $(x, fx, f^2 x, \ldots, f^{n_1} x, x^*)$ is a path in G and hence in $\widetilde{G}$ from x to x^*, and this means that $x^* \in [x]_{\widetilde{G}}$. Moreover, because f is a (G, φ, ψ)-contraction, it follows that

$$\psi\big(d\big(f^{n_k+1} x, fx^*\big)\big) \leq \varphi\big(\psi\big(d\big(f^{n_k} x, x^*\big)\big)\big) \qquad (k \geq 1).$$

By taking the limit as $k \to +\infty$, we deduce $x^* = fx^*$. Finally, if $y^* \in [x]_{\widetilde{G}}$ is a fixed point for f, then $y^* = f^n y^* \to x^*$ and so by the uniqueness of the limits of convergent sequences, we have $x^* = y^*$. Consequently, $f|_{[x]_{\widetilde{G}}}$ is a Picard operator.

(ii) If $x \in X_f$, since G is weakly connected, it then follows that $[x]_{\widetilde{G}} = X$. So by using (i), f is a Picard operator.

(iii) Put $\mathcal{C} = \big\{[x]_{\widetilde{G}} : x \in X_f\big\}$ and define a mapping $\Gamma : \operatorname{Fix}(f) \to \mathcal{C}$ by

$$\Gamma(x) = [x]_{\widetilde{G}}.$$

We are going to show that Γ is a bijection. Suppose first that $x \in \operatorname{Fix}(f)$. Then $(x, fx) = (x, x) \in E(G)$ since $E(G) \supseteq \Delta$. So $x \in X_f$ and $\Gamma(x) \in \mathcal{C}$. Moreover,

$x_1 = x_2$ implies $\Gamma(x_1) = \Gamma(x_2)$ for all $x_1, x_2 \in \mathrm{Fix}(f)$. Hence, the mapping Γ is well-defined.

To see that Γ is surjective, let x be any point of X_f. Since by using (i), $f|_{[x]_{\widetilde{G}}}$ is a Picard operator, it has a unique fixed point in $[x]_{\widetilde{G}}$, say x^*. Thus, we have

$$\Gamma(x^*) = [x^*]_{\widetilde{G}} = [x]_{\widetilde{G}}.$$

Finally, if x_1 and x_2 are two fixed points for f such that

$$[x_1]_{\widetilde{G}} = \Gamma(x_1) = \Gamma(x_2) = [x_2]_{\widetilde{G}},$$

then $x_1 \in X_f$, and by (i), $f|_{[x_1]_{\widetilde{G}}}$ is a Picard operator. Therefore, x_1 and x_2 are two fixed points for f in $[x_1]_{\widetilde{G}}$ and because f must have only one fixed point in $[x_1]_{\widetilde{G}}$, it follows that $x_1 = x_2$. Hence, Γ is injective and consequently, it is a bijection.

(iv) It is an immediate consequence of (iii).

(v) Suppose first that x_0 is the unique fixed point for f. Then $x_0 \in X_f$ and by (iii), for any $y \in X_f$, we have $[y]_{\widetilde{G}} = [x]_{\widetilde{G}}$. So $y \in [x]_{\widetilde{G}}$.

For the converse, note that since X_f is nonempty, it follows by (iv) that f has at least one fixed point in X. Now, if $x^*, x^* \in \mathrm{Fix}(f)$, then $x^*, y^* \in X_f \subseteq [x]_{\widetilde{G}}$ and so $[x^*]_{\widetilde{G}} = [y^*]_{\widetilde{G}} = [x]_{\widetilde{G}}$. Consequently, the one-to-one correspondence in (iii) implies that $x^* = y^*$.

(vi) If $X_f = \emptyset$, then so is X' then so is X and vice versa, and there is nothing to prove. So let $x \in X'$. Then there exists $x_0 \in X_f$ such that $x \in [x_0]_{\widetilde{G}}$. Since, by (i), $T|_{[x_0]_{\widetilde{G}}}$ is a Picard operator, it follows that the sequence $\{f^n x\}$ converges to a fixed point of f. Therefore, $f|_{X'}$ is a weakly Picard operator.

(vii) If $f \subseteq E(G)$, then $X_f = X$ which implies that the set X' in (vi) coincides with X. Hence it is concluded from (vi) that f is a weakly Picard operator. ∎

COROLLARY 3.12. *Let (X, d) be complete metric space and the triple (X, d, G) satisfy the property $(\star)$. Then the following statements are equivalent:*

(i) *G is weakly connected.*

(ii) *Every (G, φ, ψ)-contraction $f : X \to X$ such that $(x_0, fx_0) \in E(G)$ for some $x_0 \in X$, is a Picard operator.*

(iii) *Every (G, φ, ψ)-contraction $f : X \to X$ has at most one fixed point in X.*

Proof. (i) $\Rightarrow$ (ii): It is an immediate consequence of Assertion (ii) of Theorem 3.11.

(ii) $\Rightarrow$ (iii): Let f be a (G, φ, ψ)-contraction. If $X_f = \emptyset$, then f is fixed point free since $\mathrm{Fix}(f) \subseteq X_f$ and then there is nothing to prove. Otherwise, if $X_f \neq \emptyset$, it then follows by the hypotheses that f is a Picard operator and so it has a unique fixed point. Therefore, f has at most one fixed point in X.

(iii) $\Rightarrow$ (i): This implication follows immediately from Theorem 3.6. ∎

Corollary 3.13. *Let X be a complete and ε-chainable metric space for some $\varepsilon > 0$, i.e., for each $x, y \in X$ there exist $N \in \mathbb{N}$ and a sequence $(x_i)_{i=0}^N$ such that*

$$x_0 = x, \quad x_N = y \quad \text{and} \quad d(x_{i-1}, x_i) < \varepsilon \quad \text{for } i = 1, 2, \cdots, N.$$

Suppose that $f : X \to X$ be such that

$$d(x, y) < \varepsilon \Longrightarrow \psi\big(d(fx, fy)\big) \leq \varphi\big(\psi\big(d(x, y)\big)\big) \tag{3.4}$$

for all $x, y \in X$. Then f is a Picard operator.

Proof. Consider a graph G consisting of $V(G) := X$ and

$$E(G) := \big\{(x, y) \in X \times X : d(x, y) < \varepsilon\big\}.$$

Since X is ε-chainable, it follows that G is weakly connected. Let $(x, y) \in E(G)$. Thus, by using (3.4), we have

$$\psi\big(d(fx, fy)\big) \leq \varphi\big(\psi\big(d(x, y)\big)\big) < \psi\big(d(x, y)\big).$$

As ψ is nondecreasing, we deduce that $d(fx, fy) < d(x, y) < \varepsilon$. Hence, $d(fx, fy) \in E(G)$. Therefore, in view of (3.4), f is a (G, φ, ψ)-contraction. Further, (3.4) implies that f is continuous. Now,the conclusion follows by using Assertion (ii) of Theorem 3.11. ∎.

Remark 3.14. In all theorems and corollaries above, setting $G = G_0$ ($G = G_1$), we get the usual (ordered) version of fixed point theorems in (partially ordered) metric spaces.

4. (G, φ, ψ)-graphic contractions

In this section, we establish some fixed point theorems in metric spaces with a graph by defining the (G, φ, ψ)-graphic contraction.

Definition 4.1. One says that a mapping $f : X \to X$ is a (G, φ, ψ)-graphic contraction if the following conditions hold;

GC1) the edges of G are preserved by f, i.e., $(x, y) \in E(G)$ implies $(fx, fy) \in E(G)$ for all $x, y \in X$;

GC2) there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\psi\big(d\big(fx, f^2x\big)\big) \leq \varphi\big(\psi\big(d(x, fx)\big)\big), \tag{4.1}$$

for all $x \in X^f$, where $X^f = \big\{x \in X : (x, fx) \in E(G) \text{ or } (fx, x) \in E(G)\big\}$.

Remark 4.2. It is interesting to remark at this point that

(1) Any Banach G-graphic contraction (see [5]) is a (G, φ, ψ)-graphic contraction, where $\varphi(t) = \alpha t$ and $\psi(t) = t$ for some $\alpha \in [0, 1)$ and each $t \in [0, +\infty)$.

(2) Any (G, φ)-graphic contraction (see [10]) is a (G, φ, ψ)-graphic contraction, where $\psi(t) = t$ for each $t \in [0, +\infty)$.

(3) Any (G,ψ)-graphic contraction (see [9]) is a (G,φ,ψ)-graphic contraction, where $\varphi(t)=\alpha t$ for some $\alpha\in[0,1)$ and each $t\in[0,+\infty)$.

The following example demonstrates that the (G,φ,ψ)-graphic contraction is more general than the (G,φ,ψ)-contraction.

EXAMPLE 4.3. Let $X=[0,1]$ be endowed with the Euclidean metric. Consider

$$E(G)=\big\{(0,0)\big\}\cup\big\{(0,1)\big\}\cup\big\{(x,y)\in(0,1]:x\geq y\big\}$$

and define $f:X\to X$ by

$$fx=\begin{cases}\frac{x}{4}, & \text{if } 0<x\leq 1,\\ \frac{3}{8}, & \text{if } x=0.\end{cases}$$

Take $\varphi(t)=\frac{t}{2}$ and $\psi(t)=\sqrt{t}$ for each $t\in[0,+\infty)$. Then it is clear that G is a weakly connected graph and $X^f\neq\emptyset$, and with simple calculations it can be easily seen that f is a (G,φ,ψ)-graphic contraction. Take

$$\psi\Big(d\Big(f0,f\frac{1}{2}\Big)\Big)\leq\varphi\Big(\psi\Big(d\Big(0,\frac{1}{2}\Big)\Big)\Big)\quad\Longrightarrow\quad\frac{2}{8}\leq\frac{1}{8},$$

which is a contradiction. Thereby, f is not a (G,φ,ψ)-contraction.

The next proposition is an immediate consequence of symmetry of the metric d and Definition 4.1.

PROPOSITION 4.4. *If a mapping f from X into itself satisfies Condition* (GC1) *(respectively, Condition* (GC2)*) for a graph G, then it satisfies Condition* (GC1) *(respectively, Condition* (GC2)*) for the graphs G^{-1} and $\widetilde{G}$. In particular, a (G,φ,ψ)-graphic contraction is both a (G^{-1},φ,ψ)-graphic contraction and a $(\widetilde{G},\varphi,\psi)$-graphic contraction.*

To investigate the existence of fixed points for (G,φ,ψ)-graphic contractions, we need following lemmas.

LEMMA 4.5. *Let (X,d) be a metric space endowed with a graph G and $f:X\to X$ be a (G,φ,ψ)-graphic contraction. If $x\in X^f$, then there exists $r(x)\geq 0$ such that*

$$\psi\big(d\big(f^nx,f^{n+1}x\big)\big)\leq\varphi^n\big(r(x)\big),$$

for all $n\geq 1$, where $r(x):=\psi\big(d(x,fx)\big)$.

Proof. Take $x\in X^f$, i.e., $(x,fx)\in E(G)$ or $(fx,x)\in E(G)$. Now, if $(x,fx)\in E(G)$, then by induction, we have $\big(f^nx,f^{n+1}x\big)\in E(G)$ for each $n\geq 1$ since f is a (G,φ,ψ)-graphic contraction. Moreover, by using (4.1) and the fact that ψ is nondecreasing, we get

$$\psi\big(d\big(f^nx,f^{n+1}x\big)\big)\leq\varphi\big(\psi\big(d\big(f^{n-1}x,f^nx\big)\big)\big)\leq\cdots\leq\varphi^n\big(\psi\big(d(x,fx)\big)\big)=\varphi^n\big(r(x)\big).$$

If $(fx,x)\in E(G)$, then again by induction, we get $\big(f^{n+1}x,f^nx\big)\in E(G)$ for each $n\geq 1$. Thus, we have

$$\psi\big(d\big(f^nx,f^{n+1}x\big)\big)\leq\varphi\big(\psi\big(d\big(f^{n-1}x,f^nx\big)\big)\big)\leq\cdots\leq\varphi^n\big(\psi\big(d(x,fx)\big)\big)=\varphi^n\big(r(x)\big).\ \blacksquare$$

Lemma 4.6. *Let (X,d) be a metric space endowed with a graph G and $f : X \to X$ be a (G,φ,ψ)-graphic contraction. Then there exists $x^* \in X$ such that $f^n x \to x^*$ for all $x \in X^f$.*

Proof. Suppose that $x \in X^f$. By Lemma 4.5, we obtain

$$\sum_{n=0}^{\infty} \psi\big(d\big(f^n x, f^{n+1}x\big)\big) \le \sum_{n=0}^{\infty} \varphi^n\big(r(x)\big) < \infty,$$

which implies that $\psi\big(d(f^n x, f^{n+1}x)\big) \to 0$. Hence, by using the property of ψ, we have $d(f^n x, f^{n+1}x) \to 0$. Now, let $m > n \ge 1$. By using the property of ψ and the triangle inequality, we obtain

$$\begin{aligned}
\psi\big(d\big(f^n x, f^{n+m}x\big)\big) &\le \psi\big(d\big(f^n x, f^{n+1}x\big)\big) + \psi\big(d\big(f^{n+1}x, f^{n+2}x\big)\big) + \cdots \\
&\quad + \psi\big(d\big(f^{n+m-1}x, f^{n+m}x\big)\big) \\
&\le \varphi^n\big(r(x)\big) + \varphi^{n+1}\big(r(x)\big) + \cdots + \varphi^{n+m-1}\big(r(x)\big) \\
&= \sum_{j=1}^{m} \varphi^{n+j-1}\big(r(x)\big) < \infty \quad \text{as} \quad n, m \to +\infty,
\end{aligned}$$

from which it follows that $\{f^n x\}$ is Cauchy sequence. Because (X,d) is a complete metric space, we get that there exists $x^* \in X$ such that the sequence $\{f^n x\}$ converges to x^* as $n \to +\infty$. ■

Proposition 4.7. *Let (X,d) be a metric space endowed with a graph G and $f : X \to X$ be a (G,φ,ψ)-graphic contraction for which there exists $x_0 \in X$ such that $fx_0 \in [x_0]_{\widetilde{G}}$. Then $[x_0]_{\widetilde{G}}$ is f-invariant and $f|_{[x_0]_{\widetilde{G}}}$ is a $(\widetilde{G}_{x_0},\varphi,\psi)$-graphic contraction, where $\widetilde{G}_{x_0}$ is the component of $\widetilde{G}$ containing x_0.*

Proof. The proof of this proposition can obtained by using similar arguments as given in the proof of Proposition 3.10. ■

Theorem 4.8. *Let X be a complete metric space and the triple (X,d,G) satisfies the following property:*

($\star\star$) *If a sequence $\{x_n\}$ in X converges to some $x \in X$ and it satisfies $(x_n, x_{n+1}) \in E(G)$ (or respectively, $(x_{n+1}, x_n) \in E(G)$) for all $n \ge 1$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(\widetilde{G})$ (or respectively, $(x, x_{n_k}) \in E(\widetilde{G})$) for all $k \ge 1$.*

Suppose that $f : X \to X$ is a (G,φ,ψ)-graphic contraction which is orbitally G-continuous. Then the following statements hold:

(i) *$f|_{[x]_{\widetilde{G}}}$ is a weakly Picard operator for each $x \in X^f$.*

(ii) *$\mathrm{Fix}(f) \ne \emptyset$ if and only if $X^f \ne \emptyset$.*

(iii) *If $X^f \ne \emptyset$ and G is weakly connected, then f is a weakly Picard operator.*

Proof. (i) Suppose that $x \in X^f$. By applying Lemma 4.5, there exists $r(x) \ge 0$ such that

$$\psi\big(d\big(f^n x, f^{n+1}x\big)\big) \le \varphi^n\big(r(x)\big),$$

for all $n \geq 1$. This yields, as in the proof of Lemma 4.6, that there exists an $x^* \in X$ such that $f^n x \to x^*$. Because $x \in X^f$, it follows that $f^n x \in X^f$ for every $n \geq 1$. Now, assume that $(x, fx) \in E(G)$. (A similar deduction can be made if $(fx, x) \in E(G)$. By using the property $(\star\star)$, there exists a subsequence $\{f^{n_k}x\}$ of $\{f^n x\}$ such that $(f^{n_k}x, x^*) \in E(G)$ for all $k \geq 1$. A path in G can be formed by using the points $(x, fx, f^2x, \dots, f^{n_1}x, x^*)$ and hence $x^* \in [x]_{\widetilde{G}}$. Since f is orbitally G-continuous, it follows that $f(f^n x) \to fx^*$. On the other hand, since

$$f(f^n x) = f^{n+1}x \to x^*,$$

we deduce $x^* = fx^*$. Therefore, $f|_{[x]_{\widetilde{G}}}$ is a weakly Picard operator.

(ii) If $X^f \neq \emptyset$, then, by using (i), $\mathrm{Fix}(f) \neq \emptyset$. Conversely, suppose that $\mathrm{Fix}(f) \neq \emptyset$. By using the assumption that $E(G) \supseteq \Delta$, we immediately get that $X^f \neq \emptyset$.

(iii) Choose any $x \in X^f$. Since G is weakly connected, it follows that $[x]_{\widetilde{G}} = X$, and thus by using (i), f is a weakly Picard operator. ■

The next example illustrates that f must be orbitally G-continuous in order to obtain statements which are given in Theorem 4.8.

Example 4.9. Let $X = [0, 1]$ be endowed with the Euclidean metric. Consider

$$E(G) = \{(0,0)\} \cup \{(0,x) : x \geq 1/2\} \cup \{(x,y) : x, y \in (0,1]\}$$

and define $f : X \to X$ by

$$fx = \begin{cases} \frac{x}{2}, & \text{if } 0 < x \leq 1, \\ \frac{1}{2}, & \text{if } x = 0. \end{cases}$$

Then it can be easily checked that G is a weakly connected graph, X^f is nonempty and f is a (G, φ, ψ)-graphic contraction where $\varphi(t) = \frac{t}{2}$ and $\psi(t) = \frac{t}{3}$ for each $t \in [0, +\infty)$, but it is not orbitally G-continuous. Thus, f does not have a fixed point.

Example 4.10. Let $X = [0, 1]$ be endowed with the Euclidean metric. Consider

$$E(G) = \{(0,0)\} \cup \{(0,x) : x \geq 1/2\} \cup \{(x,y) : x, y \in (0,1]\}$$

and $f : X \to X$,

$$fx = \begin{cases} \frac{x}{2}, & \text{if } 0 < x < 1, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x = 1. \end{cases}$$

Then G is a weakly connected graph, X^f is nonempty and f is a (G, φ, ψ)-graphic contraction with $\varphi(t) = \frac{t}{2}$ and $\psi(t) = t$ for each $t \in [0, +\infty)$. An easy argument shows that f is G-orbitally continuous. Moreover, $\mathrm{Fix}(f) = \{0, 1\}$.

Remark 4.11. In the above result, we can remove the condition that the triple (X, d, G) satisfies the property $(\star\star)$ and f is G-orbitally continuous if we assume that the mapping f is orbitally continuous.

5. Applications

Recently, Kirk et al. [8] introduced the idea of cyclic contractions and established fixed point results for such mappings.

Let (X, d) be a metric space, $m \geq 2$ be a positive integer, $\{X_i\}_{i=1}^m$ be nonempty closed subsets of X and let $f : \bigcup_{i=1}^m X_i \to \bigcup_{i=1}^m X_i$ be an operator. Then $Y := \bigcup_{i=1}^m X_i$ is known as a cyclic representation of X with respect to f if

$$f(X_1) \subseteq X_2, f(X_2) \subseteq X_3, \dots, f(X_{m-1}) \subseteq X_m, f(X_m) \subseteq X_1$$

and the operator f is called a cyclic operator on X [8].

THEOREM 5.1. *Let (X, d) be a complete metric space, $m \geq 2$ be a positive integer, $\{X_i\}_{i=1}^m$ be nonempty closed subsets of X, $Y := \bigcup_{i=1}^m X_i$ and $f : X \to X$ be an operator. Suppose that*

(i) *$\bigcup_{i=1}^m X_i$ is cyclic representation of X with respect to f;*

(ii) *there exist two functions $\varphi \in \Phi$ and $\psi \in \Psi$ such that $\psi\big(d(fx, fy)\big) \leq \varphi\big(\psi\big(d(x, y)\big)\big)$ whenever $x \in X_i$ and $y \in X_{i+1}$ where $X_{m+1} = X_1$.*

Then f has a unique fixed point $x^* \in \bigcap_{i=1}^m X_i$ and $f^n y \to x^*$ for any $y \in \bigcup_{i=1}^m X_i$.

Proof. Since $Y = \bigcup_{i=1}^m X_i$ is closed so (Y, d) is a complete metric space. Let us consider a graph G consisting of $V(G) := Y$ and

$$E(G) := \Omega \cup \big\{(x, y) \in Y \times Y : x \in X_i \text{ and } y \in X_{i+1} \text{ for some } i = 1, 2, \dots, m\big\}.$$

By using (i), it follows that f preserves edges. In view of (ii), the mapping f is a (G, φ, ψ)-contraction on Y. Now, let $x_n \to x$ in Y such that $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 1$. If the sequence $\{x_n\}$ is eventually constant, then clearly we have a subsequence $\{x, x, x, \dots\}$ that satisfies the property $(\star)$ in Theorem 3.11. Otherwise, by the construction of G, there exists at least one pair of closed sets $\{X_j, X_{j+1}\}$ for some $j = 1, 2, \dots, m$ such that both sets contain infinitely many terms of sequence $\{x_n\}$. Since X_i is closed for every i, it follows that $x \in X_j \cap X_{j+1}$ and thus one can easily extract a subsequence $\{x_{n_k}\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \geq 1$. So the property $(\star)$ in Theorem 3.11 is satisfied in this case too. The way we constructed G makes it weakly connected. Also for any $x \in Y$, $(x, fx) \in E(G)$ for $f(X_i) \subseteq X_{i+1}$ for all i. Therefore, by applying Theorem 3.11, f has a unique fixed point $x^* \in Y$ and $f^n y \to x^*$ for any $y \in Y$. Since $fx^* = x^*$ and $f(X_i) \subseteq X_{i+1}$ for all i, it follows that $x^* \in \bigcap_{i=1}^m X_i$. ■

Now, we wish to study the existence of a unique solution to an integral equation, as an application of our results.

Consider the nonlinear integral equation

$$x(t) = g(t) + \int_a^b f\big(t, s, x(s)\big)\, ds \qquad \text{for all } t \in [a, b], \tag{5.1}$$

where $b > a \geq 0$. Assume that the functions f and g in (5.1) satisfy the following conditions.

(i) $f : [a, b] \times [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ ($\mathbb{R}$ stands for real numbers) is a continuous function.

(ii) $f(t, s, .) : \mathbb{R}^n \to \mathbb{R}^n$ is nondecreasing for all $t, s \in [a, b]$.

(iii) $g : [a, b] \to \mathbb{R}^n$ is a continuous function.

(iv) There exist a (c)-comparison function $\varphi : [0, +\infty) \to [0, +\infty)$ and a continuous function $p : [a, b] \times [a, b] \to [0, +\infty)$ such that for all $t, s \in [a, b]$ and $x \leq y$ (i.e., $x(t) \leq y(t)$ for all $t \in [a, b]$),

$$\bigl|f\bigl(t, s, x(s)\bigr) - f\bigl(t, s, y(s)\bigr)\bigr| \leq p(t, s)\varphi\bigl(|x(s) - y(s)|\bigr).$$

(v) $\max_{t \in [a,b]} \int_a^b p(t, s)\, ds \leq 1$.

(vi) There exists $x_0 \in C([a, b], \mathbb{R})$ such that $x_0(t) \leq g(t) + \int_a^b f(t, s, x_0(s))\, ds$ for all $t \in [a, b]$.

Denote by $X = C([a, b], \mathbb{R}^n)$ the set of all continuous functions defined on $[a, b]$ with values in $\mathbb{R}^n$ and pointwise partial order. Obviously, X with the metric given by

$$d(x, y) = \|x - y\|_\infty := \max_{t \in [a,b]} \bigl|x(t) - y(t)\bigr| \quad \text{for all } x, y \in X,$$

is a complete metric space.

THEOREM 5.2. *Under the assumptions* (i)–(vi)*, the integral equation* (5.1) *has a unique solution and it belongs to* $\mathcal{C} = \bigl\{x \in X : x \leq x_0 \text{ or } x \geq x_0\bigr\}$.

Proof. Define a mapping $T : X \to X$ by

$$Tx(t) = g(t) + \int_a^b f(t, s, x(s))\, ds \qquad \text{for all } t \in [a, b]. \tag{5.2}$$

Note that, if $x^* \in X$ is a solution of (5.1) if and only if x^* is a fixed point of f.

By virtue of our assumptions, T is well defined (this means that if $x \in X$, then $Tx \in X$). Consider a graph G with $V(G) := X$ and $E(G) = \{(x, y) \in X \times X : x \leq y\}$. From (ii), it is easy to check that the mapping T is nondecreasing and thus T preserves edges. Furthermore, G satisfies the property $(\star)$ in Theorem 3.11 because $\mathbb{R}^n$ has this property for a sequence described there. By applying (iv), for every $x, y \in X$ with $(x, y) \in E(G)$, we have

$$\begin{aligned}
\bigl|Tx(t) - Ty(t)\bigr| &\leq \int_a^b \bigl|f\bigl(t, s, x(s)\bigr) - f\bigl(t, s, y(s)\bigr)\bigr|\, ds \\
&\leq \int_a^b p(t, s)\varphi\bigl(|x(s) - y(s)|\bigr)\, ds \\
&\leq \Bigl(\int_a^b p(t, s)\, ds\Bigr)\varphi\bigl(\|x - y\|_\infty\bigr) \\
&\leq \varphi\bigl(\|x - y\|_\infty\bigr).
\end{aligned}$$

Hence, we have

$$\|Tx - Ty\|_\infty \leq \varphi\bigl(\|x - y\|_\infty\bigr),$$

which implies that $d(Tx, Ty) \leq \varphi\big(d(x, y)\big)$. From (vi), we have $(x_0, Tx_0) \in E(G)$, so that $[x_0]_{\widetilde{G}} = \{x \in X : x \leq x_0 \text{ or } x \geq x_0\}$. This proves that the mapping T satisfies the contractive condition (3.1) by considering $\psi(t) = t$. Thus, T satisfies all the hypotheses of Theorem 3.11 and so T has a unique fixed point, that is, the integral equation (5.1) has a unique solution in the set $\mathcal{C}$. ∎

Note that Theorem 5.2 specifies region of solution which invokes the novelty of our result.

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