МАТЕМАТИЧКИ ВЕСНИК
68, 4 (2016), 298–314
December 2016

оригинални научни рад
research paper

DIFFERENCE SCHEME FOR AN INTERFACE PROBLEM FOR SUBDIFFUSION EQUATION

Aleksandra Delić, Sandra Hodžić and Boško S. Jovanović

Abstract. An implicit finite-difference scheme for numerical approximation of an initial-boundary value problem with an interface for a two-dimensional subdiffusion equation with variable coefficients is proposed. Its stability is investigated and the corresponding convergence rate estimate is obtained. In a special case an efficient factorized scheme is proposed and investigated.

1. Introduction

Fractional partial differential equations have become increasingly popular in recent years. Such equations are used as models for diverse physical and chemical processes, especially those that exhibit memory type effects: anomalous diffusion, turbulent flow, chaotic dynamics, processes in media with fractal geometry, disordered materials, viscoelastic media etc. (see [10, 16, 17]).

Interface problems arise in different situations, for example: in the heat transfer process in composite materials, in transmission and diffraction processes etc. They are characterized by non-zero jump of the flux across the given interface (line or surface). Such jumps can be modelled by various types of conjugation conditions or involving singular distributions in the coefficients of partial differential equation (see [12, 14, 18, 21]).

In this article we consider the first initial-boundary value problem for a two-dimensional fractional in time diffusion equation with variable coefficients. A Dirac distribution concentrated at the line interface is involved in the coefficient of the time fractional derivative. We note that the first space derivatives of its solution may have discontinuities across the given interface. The problem is approximated by an implicit finite difference scheme and its stability and convergence are investigated. In the case when the coefficient of the Dirac distribution is constant an efficient factorized scheme is proposed and investigated.

2010 Mathematics Subject Classification: 35R11, 65M12, 65M15
Keywords and phrases: Fractional derivatives; interface problem; finite differences; factorized scheme; stability; convergence rate.

The paper is organized as follows. In Section 2 we introduce the notion of fractional derivatives. In Section 3 we define the first initial-boundary value problem for a two-dimensional fractional in time diffusion equation with variable coefficients and prove existence and uniqueness of its weak solution. In Section 4 we define the simplest implicit finite difference scheme approximating the considered problem and prove its stability. Section 5 is devoted to the investigation of the convergence of the implicit difference scheme. In Section 6 an efficient factorized difference scheme for the numerical solution of the considered initial-boundary value problem is proposed and investigated.

2. Fractional derivatives

Let u be a function defined on a nonempty bounded interval $[a,b]$ and let $k-1 \le \alpha < k$, $k \in \mathbb{N}$. The left Riemann-Liouville fractional derivative of order α is defined as [17]

$$D^{\alpha}_{a+}u(t) = \frac{1}{\Gamma(k-\alpha)}\frac{d^k}{dt^k}\int_a^t \frac{u(s)}{(t-s)^{\alpha+1-k}}\,ds, \qquad t \ge a, \tag{1}$$

where the $\Gamma(\cdot)$ is the Gamma function. The right Riemann-Liouville fractional derivative is defined analogously

$$D^{\alpha}_{b-}u(t) = \frac{(-1)^k}{\Gamma(k-\alpha)}\frac{d^k}{dt^k}\int_t^b \frac{u(s)}{(s-t)^{\alpha+1-k}}\,ds, \qquad t \le b.$$

For $\alpha = k-1$ from (1) it immediately follows that $D^{k-1}_{a+}u(t) = u^{(k-1)}(t)$. Moreover, under some natural assumptions, $\lim_{\alpha\to k} D^{\alpha}_{a+}u(t) = u^{(k)}(t)$ (see [17]).

The Caputo fractional derivative is obtained by interchanging the derivative and integral operators in (1)

$${}^{C}D^{\alpha}_{a+}u(t) = \frac{1}{\Gamma(k-\alpha)}\int_a^t \frac{u^{(k)}(s)}{(t-s)^{\alpha+1-k}}\,ds.$$

For sufficiently smooth $u(t)$ the following relation holds

$$D^{\alpha}_{a+}u(t) = {}^{C}D^{\alpha}_{a+}u(t) + \sum_{j=0}^{k-1} u^{(j)}(a)\frac{(x-a)^{j-\alpha}}{\Gamma(j-\alpha+1)}.$$

In particular, $D^{\alpha}_{a+}u(t) = {}^{C}D^{\alpha}_{a+}u(t)$ if $u(a) = u'(a) = \cdots = u^{(k-1)}(a) = 0$.

Let us mention some result that will be used in the sequel.

Fractional derivatives satisfy the semigroup property, unlike classical ones, only under certain additional assumptions [17]. For example, for continuous functions:

$$D^{\alpha}_{a+}D^{\beta}_{a+}u(t) = D^{\alpha+\beta}_{a+}u(t) \quad \text{if} \quad 0 < \alpha, \beta < 1, \quad u(a) = 0. \tag{2}$$

Let $0 < \alpha < 1$, and let $u(t)$ and $v(t)$ be continuously differentiable functions. Then:

$$(D^{\alpha}_{a+}u,\, v)_{L^2(a,b)} = (u,\, D^{\alpha}_{b-}v)_{L^2(a,b)}\,. \tag{3}$$

Let $\alpha > 0$ and let u be an infinitely differentiable function in $\mathbb{R}$, with $\operatorname{supp} u \subset (a,b)$. Then (see [5]):

$$(D^{\alpha}_{a+}u,\, D^{\alpha}_{b-}u)_{L^2(a,b)} = \cos\pi\alpha\, \|D^{\alpha}_{a+}u\|^2_{L^2(a,+\infty)}\,. \tag{4}$$

For functions of several variables, partial fractional derivatives are defined in an analogous manner, for example

$$\partial^{\alpha}_{t,a+}u(x,t) = \frac{1}{\Gamma(k-\alpha)}\frac{\partial^k}{\partial t^k}\int_a^t \frac{u(x,s)}{(t-s)^{\alpha+1-k}}\,ds, \quad k-1 \le \alpha < k, \quad k \in \mathbb{N}.$$

3. Problem formulation

Let $0 < \alpha < 1$, $\Omega = (0,1)\times(0,1)$, $\Gamma = \partial\Omega$ and $Q = \Omega \times (0,T)$. We shall consider the time fractional diffusion equation

$$(1 + K\delta_S)\,\partial^{\alpha}_{t,0+}u + \mathcal{L}u = f(x,t), \quad x = (x_1,x_2) \in \bar{\Omega}, \quad t \in (0,T) \tag{5}$$

subject to homogeneous boundary and initial conditions

$$u(x,t) = 0, \quad x \in \Gamma, \quad t \in (0,T), \tag{6}$$

$$u(x,0) = 0, \quad x \in \bar{\Omega}, \tag{7}$$

where $\mathcal{L}$ is an elliptic operator with variable coefficients

$$\mathcal{L}u = -\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(a_{ij}\,\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{2}\left[b_i\,\frac{\partial u}{\partial x_i} + \frac{\partial(b_i u)}{\partial x_i}\right] + cu,$$

and $\delta_S(x) = \delta(x_2 - 1/2)$ is the Dirac distribution concentrated on the straight line S: $x_2 = 1/2$.

Notice that the presence of the Dirac distribution in equation (5) causes a discontinuity of the first space derivatives of the solution across the interface S. An analogous problem for $\alpha = 1$ is considered in [9].

Let us denote $\Omega^- = (0,1)\times(0,1/2)$, $\Omega^+ = (0,1)\times(1/2,1)$ and $Q^{\mp} = \Omega^{\mp}\times(0,T)$. Using the theory of generalized functions equation (5) reduces to

$$\partial^{\alpha}_{t,0+}u + \mathcal{L}u = f(x,t) \quad \text{in } Q^- \text{ and } Q^+, \tag{8}$$

while on S, in the case when $f(x,t)$ does not contain a term with δ_S, we obtain the following conjugation conditions (comp. [9]):

$$[u]_S = u(x_1, 1/2+0, t) - u(x_1, 1/2-0, t) = 0 \tag{9}$$

and

$$K\,\partial^{\alpha}_{t,0+}u\Big|_{x_2=1/2} = \Big[\sum_{j=1}^{2} a_{2j}\,\frac{\partial u}{\partial x_j}\Big]_S. \tag{10}$$

We assume that the coefficients of equation (5) satisfy the standard ellipticity assumptions

$$\begin{gathered} K \in L^{\infty}(S), \quad K \ge K_0 > 0, \quad a_{ij},\, b_i,\, c \in L^{\infty}(\Omega), \quad c \ge 0, \quad a_{ij} = a_{ji}, \\ \sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \ge c_0 \sum_{i=1}^{2}\xi_i^2, \quad x \in \Omega, \quad \xi = (\xi_1,\xi_2) \in \mathbb{R}^2, \quad c_0 > 0. \end{gathered} \tag{11}$$

In the sequel by C we shall denote a positive generic constant which does not depend on the solution of the initial-boundary value problem and the discretization parameters and which may take different values in different formulas.

As usual, with $C^k(\Omega)$, $C^k(\bar{\Omega})$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we denote the spaces of k-fold differentiable functions, with $L^p(\Omega)$, $p \geq 1$, the Lebesgue spaces, while with $H^\alpha(\Omega)$, $\dot{H}^\alpha(\Omega) = H_0^\alpha(\Omega)$, $\alpha \geq 0$, we denote the Sobolev spaces [1]. For $\alpha > 0$ we further set

$$|u|_{C_+^\alpha[a,b]} = \|D_{a+}^\alpha u\|_{C[a,b]}, \qquad |u|_{C_-^\alpha[a,b]} = \|D_{b-}^\alpha u\|_{C[a,b]},$$
$$\|u\|^2_{C_\pm^\alpha[a,b]} = \|u\|^2_{C^{[\alpha]^-}[a,b]} + |u|^2_{C_\pm^\alpha[a,b]},$$
$$|u|_{H_+^\alpha(a,b)} = \|D_{a+}^\alpha u\|_{L^2(a,b)}, \qquad |u|_{H_-^\alpha(a,b)} = \|D_{b-}^\alpha u\|_{L^2(a,b)},$$
$$\|u\|^2_{H_\pm^\alpha(a,b)} = \|u\|^2_{H^{[\alpha]^-}(a,b)} + |u|^2_{H_\pm^\alpha(a,b)},$$

where $[\alpha]^-$ denotes the largest integer $< \alpha$. Then we define $C_\pm^\alpha[a,b]$ as the space of functions $u \in C^{[\alpha]^-}[a,b]$ with finite norm $\|u\|_{C_\pm^\alpha[a,b]}$. The space $H_\pm^\alpha(a,b)$ is defined analogously, while the space $\dot{H}_\pm^\alpha(a,b)$ is defined as the closure of $\dot{C}^\infty(a,b) = C_0^\infty(a,b)$ with respect to the norm $\|\cdot\|_{H_\pm^\alpha(a,b)}$. Since for $\alpha = k \in \mathbb{N}_0$ the fractional derivative reduces to the standard k-th derivative, we have $C_\pm^k[a,b] = C^k[a,b]$ and $H_\pm^k(a,b) = H^k(a,b)$.

The next result holds:

LEMMA 1. (see [11]) *For $\alpha > 0$, $\alpha \neq k + 1/2$, $k \in \mathbb{N}_0$, the spaces $\dot{H}_+^\alpha(a,b)$, $\dot{H}_-^\alpha(a,b)$ and $\dot{H}^\alpha(a,b)$ are equal and their norms are equivalent.*

For vector valued functions mapping a real interval $[0,T]$ or $(0,T)$ into a Banach space X we introduce the spaces $C^k([0,T],X)$, $k \in \mathbb{N}_0$ and $H^\alpha((0,T),X)$, $\alpha \geq 0$, in the usual way [13]. In an analogous manner we define the spaces $C_\pm^\alpha([0,T],X)$ and $H_\pm^\alpha((0,T),X)$.

Let $\tilde{L}^2(\Omega)$ be the space of functions defined on Ω, with the inner product

$$(v,w)_{\tilde{L}^2(\Omega)} = (v,w)_{L^2(\Omega)} + (v,w)_{L^2(S)}.$$

For functions defined on $Q = \Omega \times (0,T)$, we define the space $\tilde{L}^2(Q) = L^2((0,T), \tilde{L}^2(\Omega))$, with inner product

$$(v,w)_{\tilde{L}^2(Q)} = (v,w)_{L^2(Q)} + (v,w)_{L^2(\Sigma)}, \qquad \Sigma = S \times (0,T).$$

Finally, for $\alpha, \beta \geq 0$, we introduce the anisotropic Sobolev type spaces:

$$\tilde{H}^{\alpha,\beta}(Q) = L^2((0,T), H^\alpha(\Omega)) \cap H^\beta((0,T), \tilde{L}^2(\Omega))$$

and

$$\tilde{H}_\pm^{\alpha,\beta}(Q) = L^2((0,T), H^\alpha(\Omega)) \cap H_\pm^\beta((0,T), \tilde{L}^2(\Omega)).$$

Notice that for $0 \leq \beta < 1/2$: $\tilde{H}_+^{\alpha,\beta}(Q) = \tilde{H}_-^{\alpha,\beta}(Q) = \tilde{H}^{\alpha,\beta}(Q)$.

Taking the inner product of equation (5) with a test function v and formally applying partial integration and relations (2)–(4) one obtains the following weak

formulation of the problem (5)–(7): find $u \in \dot{\tilde{H}}^{1,\alpha/2}(Q) = L^2((0,T),\dot{H}^1(\Omega)) \cap \dot{H}^{\alpha/2}((0,T),\tilde{L}^2(\Omega))$ such that

$$a(u,v) = l(v), \qquad \forall v \in \dot{\tilde{H}}^{1,\alpha/2}(Q),$$

where

$$a(u,v) = \left(\partial_{t,0+}^{\alpha/2}u, \partial_{t,T-}^{\alpha/2}v\right)_{L^2(Q)} + \left(K\partial_{t,0+}^{\alpha/2}u, \partial_{t,T-}^{\alpha/2}v\right)_{L^2(\Sigma)} + \sum_{i,j=1}^{2}\left(a_{ij}\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i}\right)_{L^2(Q)} + \sum_{i=1}^{2}\left[\left(\frac{\partial u}{\partial x_i}, b_i v\right)_{L^2(Q)} - \left(\frac{\partial v}{\partial x_i}, b_i u\right)_{L^2(Q)}\right] + (cu,v)_{L^2(Q)}$$

and

$$l(v) = (f,v)_{L^2(Q)}.$$

It is easy to check that the problem (8)–(10), (6), (7) has the same weak formulation. In such a manner, problems (5)–(7) and (8)–(10), (6), (7) are equivalent.

LEMMA 2. *Let $\alpha \in (0,1)$, $f \in L^2(Q)$ and let the assumptions* (11) *hold. Then the problem* (5)–(7) *is well posed in $\dot{\tilde{H}}^{1,\alpha/2}(Q)$ and its weak solution satisfies a priori estimate*

$$\|u\|_{\tilde{H}^{1,\alpha/2}(Q)} \le C\|f\|_{L^2(Q)}. \tag{12}$$

The proof follows immediately using relations (2)–(4), (11) and the Lax-Milgram lemma.

It immediately follows from (12) that the a priori estimate

$$\|u\|_{\tilde{B}^{1,\alpha/2}(Q)} \le C\|f\|_{L^2(Q)}$$

in the weaker norm [13] is

$$\|u\|^2_{\tilde{B}^{1,\alpha/2}(Q)} = \int_0^T \left[(T-t)^{-\alpha}\|u(\cdot,t)\|^2_{\tilde{L}^2(\Omega)} + \|u(\cdot,t)\|^2_{H^1(\Omega)}\right] dt.$$

4. Finite difference approximation – implicit scheme

In the area $\bar{Q} = \bar{\Omega} \times [0,T]$, we define the uniform mesh $\bar{Q}_{h\tau} = \bar{\Omega}_h \times \bar{\Omega}_\tau$, where $\bar{\Omega}_h = \{(x_{1i}, x_{2j}) = (ih, jh) \,|\, i,j = 0,1,\ldots,2n;\ h = 1/2n\}$ and $\bar{\Omega}_\tau = \{t_k = k\tau \,|\, k = 0,1,\ldots,m;\ \tau = T/m\}$. We also define $\Omega_h = \bar{\Omega}_h \cap \Omega$, $\Gamma_h = \bar{\Omega}_h \setminus \Omega_h$, $\Omega_{1h} = \bar{\Omega}_h \cap (0,1] \times (0,1)$, $\Omega_{2h} = \bar{\Omega}_h \cap (0,1) \times (0,1]$, $S_h = \Omega_h \cap S$, $S_h^- = S_h \cup \{(0,1/2)\}$, $\Omega_\tau = \bar{\Omega}_\tau \cap (0,T)$, $\Omega_\tau^- = \bar{\Omega}_\tau \cap [0,T)$, $\Omega_\tau^+ = \bar{\Omega}_\tau \cap (0,T]$ and $\Sigma_{h\tau} = S_h \times \Omega_\tau$. We will use standard notation from the theory of the finite difference schemes (see [19]):

$$v = v(x,t), \quad \hat{v} = v(x,t+\tau), \quad v^k = v(x,t_k), \quad x = (x_1,x_2) \in \bar{\Omega}_h, \quad t \in \bar{\Omega}_\tau,$$

$$v_{x_i} = \frac{v(x+he_i,t) - v(x,t)}{h} = v_{\bar{x}_i}(x - he_i, t), \quad e_i = (2-i, i-1).$$

For a function u defined on $\bar{Q}$ which satisfies a homogeneous initial condition, we approximate the left Riemann-Liouville fractional derivative $\partial^{\alpha}_{t,0+}u(x,t_k)$ by (see [4]):

$$(\partial^{\alpha}_{t,0+,\tau}u)^k = \frac{1}{\Gamma(2-\alpha)}\sum_{l=0}^{k-1}(t_{k-l}^{1-\alpha}-t_{k-l-1}^{1-\alpha})u_t^l.$$

The next result holds:

LEMMA 3. (see [20]) *Suppose that* $u \in C^2([0,T],C(\bar{\Omega}))$, $t \in \Omega_\tau^+$ *and* $u(x,0) = 0$. *Then*

$$|\partial^{\alpha}_{t,0+}u - \partial^{\alpha}_{t,0+,\tau}u| \le \tau^{2-\alpha}\frac{1}{1-\alpha}\left[\frac{1-\alpha}{12}+\frac{2^{2-\alpha}}{2-\alpha}-(1+2^{-\alpha})\right]\max_{0\le s\le t}\left|\frac{\partial^2 u}{\partial t^2}(x,s)\right|.$$

We approximate the initial-boundary value problem (5)–(7) with the following implicit finite difference scheme:

$$(1+K\delta_{S_h})\left(\partial^{\alpha}_{t,0+,\tau}v\right)^k + \mathcal{L}_h v^k = \bar{f}^k, \quad x\in\Omega_h, \quad k=1,2,\dots,m, \tag{13}$$

subject to homogeneous boundary and initial conditions:

$$\begin{gathered} v(x,t)=0, \quad (x,t)\in\Gamma_h\times\Omega_\tau^+, \\ v(x,0)=0, \quad x\in\bar{\Omega}_h, \end{gathered} \tag{14}$$

where it is denoted

$$\mathcal{L}_h v = -\frac{1}{2}\sum_{i,j=1}^{2}\left[(a_{ij}v_{\bar{x}_j})_{x_i}+(a_{ij}v_{x_j})_{\bar{x}_i}\right]+\frac{1}{2}\sum_{i=1}^{2}\left[b_i v_{\bar{x}_i}+b_i v_{x_i}+(b_i v)_{\bar{x}_i}+(b_i v)_{x_i}\right]+\bar{c}v$$

and

$$\delta_{S_h}(x)=\delta_h(x_2-1/2)=\begin{cases}0, & x\in\Omega_h\setminus S_h,\\ 1/h, & x\in S_h.\end{cases}$$

When the right-hand side f is a continuous function, we set $\bar{f}=f$, otherwise we must use some averaged value, for example $\bar{f}=T_1^2T_2^2f$, where T_1 and T_2 are Steklov averaging operators:

$$T_i f(x,t)=T_i^- f(x+0.5he_i,t)=T_i^+ f(x-0.5he_i,t)=\int_{-1/2}^{1/2} f(x+hse_i,t)\,ds, \quad i=1,2.$$

Analogously, we set $\bar{c}=c$ or $\bar{c}=T_1^2T_2^2c$.

We define the following discrete inner products and norms:

$$(v,w)_h=(v,w)_{L^2(\Omega_h)}=h^2\sum_{x\in\Omega_h}vw, \quad \|v\|_h^2=\|v\|^2_{L^2(\Omega_h)}=(v,v)_h,$$

$$(v,w)_{ih}=(v,w)_{L^2(\Omega_{ih})}=h^2\sum_{x\in\Omega_{ih}}vw, \quad \|v\|_{ih}^2=\|v\|^2_{L^2(\Omega_{ih})}=(v,v)_{ih}, \quad i=1,2,$$

$$(v,w)_{L^2(S_h)}=h\sum_{x\in S_h}vw, \quad \|v\|^2_{L^2(S_h)}=(v,v)_{L^2(S_h)},$$

$$|v|^2_{H^{1/2}(S_h)}=h^2\sum_{x\in S_h^-}\sum_{x'\in S_h^-,\ x'\ne x}\frac{|v(x)-v(x')|^2}{|x_1-x_1'|^2},$$

$$(v,w)_{\tilde{L}^2(\Omega_h)} = (v,w)_{L^2(\Omega_h)} + (v,w)_{L^2(S_h)}, \quad \|v\|^2_{\tilde{L}^2(\Omega_h)} = (v,v)_{\tilde{L}^2(\Omega_h)},$$
$$|v|^2_{H^1(\Omega_h)} = \|v_{\bar{x}_1}\|^2_{1h} + \|v_{\bar{x}_2}\|^2_{2h}, \quad \|v\|^2_{H^1(\Omega_h)} = |v|^2_{H^1(\Omega_h)} + \|v\|^2_h,$$
$$\|v\|^2_{L^2(Q_{h\tau})} = \tau \sum_{k=1}^{m} \|v^k\|^2_h, \quad \|v\|^2_{L^2(Q_{ih\tau})} = \tau \sum_{k=1}^{m} \|v^k\|^2_{ih},$$
$$\|v\|^2_{\tilde{L}^2(Q_{h\tau})} = \tau \sum_{k=1}^{m} \|v^k\|^2_{\tilde{L}^2(\Omega_h)},$$
$$\|v\|^2_{L^2(\Sigma_{h\tau})} = \tau \sum_{k=1}^{m} \|v^k\|^2_{L^2(S_h)}, \quad |v|^2_{L^2(\Omega_\tau;H^{1/2}(S_h))} = \tau \sum_{k=1}^{m} |v^k|^2_{H^{1/2}(S_h)},$$
$$\|v\|^2_{\tilde{B}^{1,\alpha/2}(Q_{h\tau})} = \tau \sum_{k=1}^{m} \|v^k\|^2_{H^1(\Omega_h)} + \tau \sum_{k=1}^{m} \left(\partial^\alpha_{t,0+,\tau}\left(\|v\|^2_{\tilde{L}^2(\Omega_h)}\right)\right)^k.$$

For every function $v(\cdot,t)$ defined on the mesh $\bar{\Omega}_\tau$, which satisfies the initial condition $v(\cdot,0)=0$, the following equality is valid (see [4])

$$\tau \sum_{k=1}^{m} \left(\partial^\alpha_{t,0+,\tau}(v^2)\right)^k = \frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^{m} \left(t^{1-\alpha}_{m-k+1} - t^{1-\alpha}_{m-k}\right)(v^k)^2.$$

In particular, from here it follows that the norm $\|v\|_{\tilde{B}^{1,\alpha/2}(Q_{h\tau})}$ is well defined. It can be treated as the discrete analogue of $\|\cdot\|_{\tilde{B}^{1,\alpha/2}(Q)}$, since

$$(1-\alpha)\tau(T-t_{k-1})^{-\alpha} \le t^{1-\alpha}_{m-k+1} - t^{1-\alpha}_{m-k} \le (1-\alpha)\tau(T-t_k)^{-\alpha}.$$

LEMMA 4. (see [2, 7]) *For $0<\alpha<1$ and any function $v(\cdot,t)$ defined for $t\in\bar{\Omega}_\tau$ the following inequality is valid*

$$v^k\left(\partial^\alpha_{t,0+,\tau}v\right)^k \ge \frac{1}{2}\left(\partial^\alpha_{t,0+,\tau}(v^2)\right)^k + \frac{\tau^{2-\alpha}(1-2^{-\alpha})}{\Gamma(2-\alpha)}(v^{k-1}_t)^2, \quad k=1,2,\dots,m. \tag{15}$$

THEOREM 1. *Let $\alpha\in(0,1)$, $f\in L^2(Q)$ and let the assumptions* (11) *hold. Let also a_{ij}, b_i and K be continuous functions. Then the finite difference scheme* (13)-(14) *is absolutely stable and its solution satisfies the following a priori estimate:*

$$\|v\|_{\tilde{B}^{1,\alpha/2}(Q_{h\tau})} \le C\|\bar{f}\|_{L^2(Q_{h\tau})}. \tag{16}$$

Proof. Taking the inner product of (13) with v^k, we obtain

$$\left(v^k,(1+K\delta_{S_h})\partial^\alpha_{t,0+,\tau}v^k\right)_h + \left(v^k,\mathcal{L}v^k\right)_h = \left(v^k,\bar{f}^k\right)_h.$$

From inequality (15) it follows that

$$\begin{aligned}\left(v^k,(1+K\delta_{S_h})\partial^\alpha_{t,0+,\tau}v^k\right)_h &= \left(v^k,\partial^\alpha_{t,0+,\tau}v^k\right)_h + \left(Kv^k,\partial^\alpha_{t,0+,\tau}v^k\right)_{L^2(S_h)}\\ &\ge \frac{1}{2}\left(\partial^\alpha_{t,0+,\tau}(\|v\|^2_h)\right)^k + \frac{K_0}{2}\left(\partial^\alpha_{t,0+,\tau}(\|v\|^2_{L^2(S_h)})\right)^k.\end{aligned}$$

Using partial summation and assumptions (11) we obtain

$$\left(v^k,\mathcal{L}v^k\right)_h \ge c_0|v^k|^2_{H^1(\Omega_h)}$$

while the right-hand side we estimate by applying the Cauchy-Schwarz and the ε-inequality

$$\left(v^k,\bar{f}^k\right)_h \le \varepsilon\|v^k\|^2_h + \frac{1}{4\varepsilon}\|\bar{f}^k\|^2_h.$$

In such a way, the result follows by taking a sufficiently small ε, applying the discrete Poincaré inequality (see [19])

$$\|v\|_h \le \frac{1}{4}\,|v|_{H^1(\Omega_h)} \tag{17}$$

and summing obtained inequalities for $k = 1, 2, \dots, m$. ■

We also need the following assertion.

LEMMA 5. (see [8]) *Let the function v be defined on the mesh $\bar\Omega_h$, $v = 0$ on Γ_h and let the function w be defined on the mesh S_h^-. Then*

$$\big|(v, w_{\bar x_1})_{L^2(S_h)}\big| \le C\,\|v\|_{H^1(\Omega_h)} |w|_{H^{1/2}(S_h)}.$$

5. Convergence of implicit scheme

Let u be the solution of the initial-boundary value problem (5)–(7) and v the solution of the finite difference scheme (13)–(14), where $\bar f = T_1^2 T_2^2 f$ and $\bar c = T_1^2 T_2^2 c$. The error $z = u - v$ satisfies the finite difference scheme

$$\begin{aligned} (1 + K\delta_{S_h})\,\partial^\alpha_{t,0+,\tau} z + \mathcal{L}_h\, z &= \varphi \quad \text{in } \Omega_h \times \Omega_\tau^+, \\ z &= 0 \quad \text{on } \Gamma_h \times \Omega_\tau^+, \\ z(x,0) &= 0 \quad \text{on } \Omega_h, \end{aligned} \tag{18}$$

where

$$\begin{aligned} \varphi &= \sum_{i,j=1}^{2} \eta_{ij,\bar x_i} + \sum_{i=1}^{2} \eta_{i,\bar x_i} + \sum_{i=1}^{2} \zeta_i + \eta + \chi + \delta_{S_h}\mu\,, \\ \eta_{ij} &= T_i^+ T_{3-i}^2 \Big(a_{ij}\,\frac{\partial u}{\partial x_j}\Big) - \frac{1}{2}\Big[a_{ij} u_{x_j} + \big(a_{ij} u_{\bar x_j}\big)\,\Big|_{(x+he_i,t)}\Big], \\ \eta_i &= \frac{1}{2}\Big[\,(b_i u)\,\Big|_{(x+he_i,t)} + b_i u\Big] - T_i^+ T_{3-i}^2 (b_i u), \\ \zeta_i &= \frac{1}{2}\big(b_i u_{\bar x_i} + b_i u_{x_i}\big) - T_1^2 T_2^2 \Big(b_i\,\frac{\partial u}{\partial x_i}\Big), \\ \eta &= (T_1^2 T_2^2 c) u - T_1^2 T_2^2 (cu), \\ \chi &= \partial^\alpha_{t,0+,\tau} u - T_1^2 T_2^2 (\partial^\alpha_{t,0+} u), \\ \mu &= K \partial^\alpha_{t,0+,\tau} u - T_1^2 (K \partial^\alpha_{t,0+} u). \end{aligned}$$

Let us set

$$\eta_{1j} = \tilde\eta_{1j} + \delta_{S_h}\,\hat\eta_{1j}, \quad \eta_1 = \tilde\eta_1 + \delta_{S_h}\,\hat\eta_1, \quad \zeta_j = \tilde\zeta_j + \delta_{S_h}\,\hat\zeta_j,$$
$$\eta = \tilde\eta + \delta_{S_h}\,\hat\eta, \quad \chi = \tilde\chi + \delta_{S_h}\,\hat\chi,$$

where

$$\hat\eta_{11} = \frac{h^2}{6}\,T_1^+\bigg(\Big[a_{11}\,\frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial a_{11}}{\partial x_2}\,\frac{\partial u}{\partial x_1}\Big]_S\bigg),$$

$$\hat{\eta}_{12} = \frac{h^2}{6}\,T_1^+\Big(\Big[a_{12}\,\frac{\partial^2 u}{\partial x_2^2} + \frac{\partial a_{12}}{\partial x_2}\,\frac{\partial u}{\partial x_2}\Big]_S\Big) - \frac{h^2}{4}\,T_1^+\Big(\Big[\frac{\partial}{\partial x_1}\Big(a_{12}\,\frac{\partial u}{\partial x_2}\Big)\Big]_S\Big),$$
$$\hat{\eta}_1 = -\frac{h^2}{6}\,T_1^+\Big(\Big[\frac{\partial(b_1 u)}{\partial x_2}\Big]_S\Big),$$
$$\hat{\zeta}_1 = -\frac{h^2}{6}\,T_1^2\Big(\Big[\frac{\partial}{\partial x_2}\Big(b_1\,\frac{\partial u}{\partial x_1}\Big)\Big]_S\Big),$$
$$\hat{\zeta}_2 = \frac{h^2}{4}\,\Big[b_2\,T_1^2\Big(\frac{\partial^2 u}{\partial x_2^2}\Big)\Big]_S - \frac{h^2}{6}\,T_1^2\Big(\Big[\frac{\partial}{\partial x_2}\Big(b_2\,\frac{\partial u}{\partial x_2}\Big)\Big]_S\Big),$$
$$\hat{\eta} = -\frac{h^2}{3}\,\Big[(T_1^2 c)\Big(T_1^2\,\frac{\partial u}{\partial x_2}\Big)\Big]_S,$$
$$\hat{\chi} = \frac{h^2}{6}\,T_1^2\Big(\Big[\frac{\partial(\partial^\alpha_{t,0+}u)}{\partial x_2}\Big]_S\Big).$$

THEOREM 2. *Under the assumptions of Theorem 1 the finite difference scheme* (18) *is absolutely stable and its solution satisfies the following a priori estimate:*

$$\begin{aligned}\|z\|_{\tilde{B}^{1,1/2}(Q_{h\tau})} \le C\Big\{ \sum_{j=1}^{2} \Big(&\|\eta_{2j}\|_{L^2(Q_{2h\tau})} + \|\tilde{\eta}_{1j}\|_{L^2(Q_{1h\tau})} \\ &+ |\hat{\eta}_{1j}|_{L^2(\Omega_\tau;H^{1/2}(S_h))} + \|\tilde{\zeta}_j\|_{L^2(Q_{h\tau})} + \|\hat{\zeta}_j\|_{L^2(\Sigma_{h\tau})}\Big) \\ &+ \|\eta_2\|_{L^2(Q_{2h\tau})} + \|\tilde{\eta}_1\|_{L^2(Q_{1h\tau})} + |\hat{\eta}_1|_{L^2(\Omega_\tau;H^{1/2}(S_h))} + \|\tilde{\eta}\|_{L^2(Q_{h\tau})} \\ &+ \|\hat{\eta}\|_{L^2(\Sigma_{h\tau})} + \|\tilde{\chi}\|_{L^2(Q_{h\tau})} + \|\hat{\chi}\|_{L^2(\Sigma_{h\tau})} + \|\mu\|_{L^2(\Sigma_{h\tau})}\Big\}. \quad (19)\end{aligned}$$

The proof is analogous to the proof of Theorem 1, while the right-hand side terms are estimated using summation by part, Lemma 5 and the discrete trace theorem [19]

$$\|v\|_{L^2(S_h)} \le 2\,\|v_{\bar{x}_2}\|_{2h}.$$

In such a way, in order to estimate the rate of convergence of the finite difference scheme (13)-(14) it is sufficient to estimate the right-hand side terms of (19).

THEOREM 3. *Let the assumptions of Theorem 1 hold,* $a_{ij}, b_i \in H^2(\Omega^\pm)$, $c \in H^1(\Omega^\pm)$, $K \in H^2(S)$, *and let the solution* u *of initial-boundary value problem* (5)–(7) *belong to the space* $C^2([0,T], C(\bar{\Omega})) \cap C([0,T], H^3(\Omega^\pm)) \cap C^\alpha_+([0,T], H^2(\Omega^\pm)) \cap C^\alpha_+([0,T], H^2(S))$. *Then the solution* v *of finite difference scheme* (13)–(14) *with* $\bar{c} = T_1^2 T_2^2 c$ *and* $\bar{f} = T_1^2 T_2^2 f$ *converges to* u *and the following convergence rate estimate holds:*

$$\|u - v\|_{\tilde{B}^{1,\alpha/2}(Q_{h\tau})} = O(h^2 + \tau^{2-\alpha}).$$

Proof. The terms η_{2j}, $\tilde{\eta}_{1j}$, $\hat{\eta}_{1j}$, η_2, $\tilde{\eta}_1$, $\hat{\eta}_1$, $\tilde{\zeta}_j$, $\hat{\zeta}_j$, $\tilde{\eta}$ and $\hat{\eta}$ are estimated in [9] for fixed $t = t_k$:

$$\|\eta^k_{2j}\|_{L^2(\Omega_{2h})} \le Ch^2\big(\|a_{2j}\|_{H^2(\Omega^-)}\|u(\cdot,t_k)\|_{H^3(\Omega^-)} + \|a_{2j}\|_{H^2(\Omega^+)}\|u(\cdot,t_k)\|_{H^3(\Omega^+)}\big),$$
$$\|\tilde{\eta}^k_{1j}\|_{L^2(\Omega_{1h})} \le Ch^2\big(\|a_{1j}\|_{H^2(\Omega^-)}\|u(\cdot,t_k)\|_{H^3(\Omega^-)} + \|a_{1j}\|_{H^2(\Omega^+)}\|u(\cdot,t_k)\|_{H^3(\Omega^+)}\big),$$

$$
\begin{aligned}
|\hat\eta_{1j}^k|_{H^{1/2}(S_h)} &\le Ch^2\big(\|a_{1j}\|_{H^2(\Omega^-)}\|u(\cdot,t_k)\|_{H^3(\Omega^-)} + \|a_{1j}\|_{H^2(\Omega^+)}\|u(\cdot,t_k)\|_{H^3(\Omega^+)}\big),\\
\|\eta_2^k\|_{L^2(\Omega_{2h})} &\le Ch^2\big(\|b_2\|_{H^2(\Omega^-)}\|u(\cdot,t_k)\|_{H^3(\Omega^-)} + \|b_2\|_{H^2(\Omega^+)}\|u(\cdot,t_k)\|_{H^3(\Omega^+)}\big),\\
\|\tilde\eta_1^k\|_{L^2(\Omega_{1h})} &\le Ch^2\big(\|b_1\|_{H^2(\Omega^-)}\|u(\cdot,t_k)\|_{H^3(\Omega^-)} + \|b_1\|_{H^2(\Omega^+)}\|u(\cdot,t_k)\|_{H^3(\Omega^+)}\big),\\
|\hat\eta_1^k|_{H^{1/2}(S_h)} &\le Ch^2\big(\|b_1\|_{H^2(\Omega^-)}\|u(\cdot,t_k)\|_{H^3(\Omega^-)} + \|b_1\|_{H^2(\Omega^+)}\|u(\cdot,t_k)\|_{H^3(\Omega^+)}\big),\\
\|\tilde\zeta_j^k\|_{L^2(\Omega_h)} &\le Ch^2\big(\|b_j\|_{H^2(\Omega^-)}\|u(\cdot,t_k)\|_{H^3(\Omega^-)} + \|b_j\|_{H^2(\Omega^+)}\|u(\cdot,t_k)\|_{H^3(\Omega^+)}\big),\\
\|\hat\zeta_j^k\|_{L^2(S_h)} &\le Ch^2\big(\|b_j\|_{H^2(\Omega^-)}\|u(\cdot,t_k)\|_{H^3(\Omega^-)} + \|b_j\|_{H^2(\Omega^+)}\|u(\cdot,t_k)\|_{H^3(\Omega^+)}\big),\\
\|\tilde\eta^k\|_{L^2(\Omega_h)} &\le Ch^2\big(\|c\|_{H^1(\Omega^-)}\|u(\cdot,t_k)\|_{H^3(\Omega^-)} + \|c\|_{H^1(\Omega^+)}\|u(\cdot,t_k)\|_{H^3(\Omega^+)}\big),\\
\|\hat\eta^k\|_{L^2(S_h)} &\le Ch^2\big(\|c\|_{H^1(\Omega^-)}\|u(\cdot,t_k)\|_{H^3(\Omega^-)} + \|c\|_{H^1(\Omega^+)}\|u(\cdot,t_k)\|_{H^3(\Omega^+)}\big).
\end{aligned}
$$

From these inequalities we immediately obtain the following bounds:

$$
\begin{aligned}
\|\eta_{2j}\|_{L^2(Q_{2h\tau})} &\le Ch^2\big(\|a_{2j}\|_{H^2(\Omega^-)}\|u\|_{C([0,T],H^3(\Omega^-))}\\
&\quad +\|a_{2j}\|_{H^2(\Omega^+)}\|u\|_{C([0,T],H^3(\Omega^+))}\big), \qquad (20)\\
\|\tilde\eta_{1j}\|_{L^2(Q_{1h\tau})},\ |\hat\eta_{1j}|_{L^2(\Omega_\tau;H^{1/2}(S_h))} &\le Ch^2\big(\|a_{1j}\|_{H^2(\Omega^-)}\|u\|_{C([0,T],H^3(\Omega^-))}\\
&\quad +\|a_{1j}\|_{H^2(\Omega^+)}\|u\|_{C([0,T],H^3(\Omega^+))}\big), \qquad (21)\\
\|\eta_2\|_{L^2(Q_{2h\tau})} &\le Ch^2\big(\|b_2\|_{H^2(\Omega^-)}\|u\|_{C([0,T],H^3(\Omega^-))}\\
&\quad +\|b_2\|_{H^2(\Omega^+)}\|u\|_{C([0,T],H^3(\Omega^+))}\big), \qquad (22)\\
\|\tilde\eta_1\|_{L^2(Q_{1h\tau})},\ |\hat\eta_1|_{L^2(\Omega_\tau;H^{1/2}(S_h))} &\le Ch^2\big(\|b_1\|_{H^2(\Omega^-)}\|u\|_{C([0,T],H^3(\Omega^-))}\\
&\quad +\|b_1\|_{H^2(\Omega^+)}\|u\|_{C([0,T],H^3(\Omega^+))}\big), \qquad (23)\\
\|\tilde\zeta_j\|_{L^2(Q_{h\tau})},\ \|\hat\zeta_j\|_{L^2(\Sigma_{h\tau})} &\le Ch^2\big(\|b_j\|_{H^2(\Omega^-)}\|u\|_{C([0,T],H^3(\Omega^-))}\\
&\quad +\|b_j\|_{H^2(\Omega^+)}\|u\|_{C([0,T],H^3(\Omega^+))}\big) \qquad (24)
\end{aligned}
$$

and

$$
\begin{aligned}
\|\tilde\eta\|_{L^2(Q_{h\tau})},\ \|\hat\eta\|_{L^2(\Sigma_{h\tau})} &\le Ch^2\big(\|c\|_{H^1(\Omega^-)}\|u\|_{C([0,T],H^3(\Omega^-))}\\
&\quad +\|c\|_{H^1(\Omega^+)}\|u\|_{C([0,T],H^3(\Omega^+))}\big), \qquad (25)
\end{aligned}
$$

Let us set $\tilde\chi = \tilde\chi_1 + \tilde\chi_2$, where

$$
\begin{aligned}
\tilde\chi_1 &= \partial^\alpha_{t,0+,t}u - \partial^\alpha_{t,0+}u,\\
\tilde\chi_2 &= \partial^\alpha_{t,0+}u - T_1^2T_2^2\partial^\alpha_{t,0+}u, \quad x\in\Omega_h\setminus S_2,\\
\tilde\chi_2 &= \tilde\chi_2^- + \tilde\chi_2^+, \quad x\in S_2,\\
\tilde\chi_2^\pm &= \frac{1}{2}\left(\partial^\alpha_{t,0+}u - T_1^2T_2^{2\pm}\partial^\alpha_{t,0+}u \mp \frac{h}{3}T_1^2\left(\frac{\partial(\partial^\alpha_{t,0+}u)}{\partial x_2}\right)\right)\Bigg|_{x_2=1/2\pm 0},\\
T_2^{2\pm}u &= \int_0^1 (1-s)u(x_1,x_2\pm hs,t)\,ds.
\end{aligned}
$$

Using Lemma 3 we immediately obtain the bound:

$$
\|\tilde\chi_1\|_{L^2(Q_{h\tau})} \le C\tau^{2-\alpha}\|u\|_{C^2([0,T],C(\bar\Omega))}. \qquad (26)
$$

For $x \in \Omega_h \setminus S_2$, $\tilde{\chi}_2$ is a linear bounded functional of $\partial^{\alpha}_{t,0+}u \in H^2(e)$, $e = e(x) = (x_1 - h, x_1 + h) \times (x_2 - h, x_2 + h)$, which vanishes on first order polynomials in x_1 and x_2. Using the Bramble-Hilbert lemma [3], we obtain

$$|\tilde{\chi}_2(x,t)| \le Ch\,|\partial^{\alpha}_{t,0+}u(\cdot,t)|_{H^2(e)}.$$

Analogous result holds for $\tilde{\chi}_2^{\pm}$:

$$|\tilde{\chi}_2^{\pm}(x,t)| \le Ch\,|\partial^{\alpha}_{t,0+}u(\cdot,t)|_{H^2(e^{\pm})}, \quad x = (x_1, 1/2) \in S_h,$$

where we denoted $e^+ = (x_1 - h, x_1 + h) \times (1/2, 1/2 + h)$ and $e^- = (x_1 - h, x_1 + h) \times (1/2 - h, 1/2)$. Summing these inequalities over the mesh $Q_{h\tau}$ we obtain the bound:

$$\|\tilde{\chi}_2\|_{L^2(Q_{h\tau})} \le Ch^2\big(\|u\|_{C^{\alpha}_{+}([0,T],H^2(\Omega^-))} + \|u\|_{C^{\alpha}_{+}([0,T],H^2(\Omega^+))}\big). \tag{27}$$

The term $\hat{\chi}$ can be estimated directly, using the trace theorem for Sobolev spaces [1]:

$$\|\hat{\chi}\|_{L^2(\Sigma_{h\tau})} \le Ch^2\big(\|u\|_{C^{\alpha}_{+}([0,T],H^2(\Omega^-))} + \|u\|_{C^{\alpha}_{+}([0,T],H^2(\Omega^+))}\big). \tag{28}$$

Let us set $\mu = \mu_1 + \mu_2$, where

$$\mu_1 = K(\partial^{\alpha}_{t,0+,t}u - \partial^{\alpha}_{t,0+}u),$$
$$\mu_2 = K\partial^{\alpha}_{t,0+}u - T_1^2T_2^2\left(K\partial^{\alpha}_{t,0+}u\right).$$

Using Lemma 3 we immediately obtain the bound:

$$\|\mu_1\|_{L^2(\Sigma_{h\tau})} \le C\tau^{2-\alpha}\|K\|_{C(\bar{S})}\|u\|_{C^2([0,T],C(\bar{\Omega}))}. \tag{29}$$

Using the Bramble-Hilbert lemma and properties of multipliers in Sobolev spaces [15] we obtain:

$$\|\mu_2\|_{L^2(\Sigma_{h\tau})} \le Ch^2 \max_{t\in[0,T]} \|K\partial^{\alpha}_{t,0+}u\|_{H^2(S)} \le Ch^2\|K\|_{H^2(S)}\|u\|_{C^{\alpha}_{+}([0,T],H^2(S))}. \tag{30}$$

The result follows from (19)–(30). ■

6. Factorized scheme

The finite difference scheme (13)–(14) is not efficient, because a discrete elliptic problem has to be solved at every time level $t = t_k$. In this Section, assuming for the sake of simplicity that $K = \text{const} > 0$, we propose the following factorized difference scheme approximating the initial-boundary value problem (5)–(7):

$$(I+\theta\tau^{\alpha}\Lambda_1)(B+\theta\tau^{\alpha}\Lambda_2)(\partial_{t,0+,\tau}v)^k+\mathcal{L}_hv^{k-1}=\bar{f}^k, \quad x \in \Omega_h, \quad k = 1, 2, \ldots, m, \tag{31}$$

subject to homogeneous boundary and initial conditions (14). Here I is the identity operator, $\Lambda_i v = -v_{x_i\bar{x}_i}$, $Bv = (1 + K\delta_{S_h})v$ and θ is a real positive parameter. Obviously, when the values $v = v^{k-1}$ are known, for the determination of values $v = v^k$ on the next time level $t = t_k$ we need to invert the operators $I+\theta\tau\Lambda_1$ and $B+\theta\tau\Lambda_2$. Both operators can be represented by tridiagonal matrices and consequently the required values of the solution may be obtained by two applications of the Thomas algorithm. In this sense the finite difference scheme (31), (14) is efficient.

Notice that the scheme (31), (14) can be regarded as a kind of alternating-direction-implicit (ADI) scheme (see [19]). In [6, 22], analogous schemes are constructed for the subdiffusion equation with constant coefficients, without an interface.

THEOREM 4. *Let $K = const > 0$ and let the assumptions of Theorem 1 hold. Then, for sufficiently large θ, the finite difference scheme* (31), (14) *is absolutely stable and its solution satisfies the a priori estimate* (16).

Proof. Let us denote $\tilde{B} = (I+\theta\tau^\alpha\Lambda_1)(B+\theta\tau^\alpha\Lambda_2)$. The operators B and $\tilde{B}$ are positive and selfadjoint, so the corresponding energy norms (see [19]) $\|v\|_B = (Bv,v)_h^{1/2}$ and $\|v\|_{\tilde{B}} = (\tilde{B}v,v)_h^{1/2}$ are well defined. Further:

$$\tilde{B} = B + \theta\tau^\alpha(\Lambda_1+\Lambda_2) + \theta K\tau^\alpha\delta_{S_h}\Lambda_1 + \theta^2\tau^{2\alpha}\Lambda_1\Lambda_2,$$

whereby it follows that

$$\|v\|_{\tilde{B}}^2 \geq \|v\|_B^2 + \theta\tau^\alpha\|v\|_{\Lambda_1+\Lambda_2}^2 = \|v\|_{\tilde{L}^2(\Omega_h)}^2 + \theta\tau^\alpha|v|_{H^1(\Omega_h)}^2. \tag{32}$$

Taking the inner product of equation (31) with v^k we obtain

$$\left(\tilde{B}(\partial_{t,0+,\tau}v)^k, v^k\right)_h + (\mathcal{L}_h v^k, v^k)_h = \tau(\mathcal{L}_h v_t^{k-1}, v^k)_h + (\bar{f}^k, v^k)_h.$$

From Lemma 4 and inequality (32) it follows that

$$\left(\tilde{B}(\partial_{t,0+,\tau}v)^k, v^k\right)_h \geq \frac{1}{2}\left(\partial_{t,0+,\tau}\big(\|v\|_{\tilde{L}^2(\Omega_h)}^2\big)^k\right) + \theta\tau^\alpha\tau^{2-\alpha}\frac{1-2^{-\alpha}}{\Gamma(2-\alpha)}\,|v_t^{k-1}|_{H^1(\Omega_h)}^2.$$

Using (11) and (17) we obtain

$$(\mathcal{L}_h v^k, v^k)_h \geq c_0|v^k|_{H^1(\Omega_h)}^2,$$

$$\tau(\mathcal{L}_h v_t^{k-1}, v^k)_h \leq c_1\tau\|v_t^{k-1}\|_{H^1(\Omega_h)}\|v^k\|_{H^1(\Omega_h)} \leq c_2\tau^2\|v_t^{k-1}\|_{H^1(\Omega_h)}^2 + \frac{c_0}{4}\,\|v^k\|_{H^1(\Omega_h)}^2,$$

where c_1 and c_2 are computable constants, and

$$(\bar{f}^k, v^k)_h \leq \frac{c_0}{4}\,\|v^k\|_{H^1(\Omega_h)}^2 + \frac{1}{16c_0}\,\|\bar{f}^k\|_h^2.$$

Setting $\theta \geq c_2\Gamma(2-\alpha)/(1-2^{-\alpha})$ we obtain the desired result after summation through $k = 1, 2, \ldots, m$. ■

Let u be the solution of the initial-boundary value problem (5)–(7) and v the solution of the finite difference scheme (31), (14). The error $z = u - v$ satisfies the finite difference scheme

$$(I+\theta\tau^\alpha\Lambda_1)(B+\theta\tau^\alpha\Lambda_2)(\partial_{t,0+,\tau}z)^k + \mathcal{L}_h\, z^{k-1} = \bar{\varphi}^k, \quad x \in \Omega_h, \quad k = 1, 2, \ldots, m, \tag{33}$$

subject to homogeneous boundary and initial conditions

$$\begin{gathered} z(x,t) = 0, \quad (x,t) \in \Gamma_h \times \Omega_\tau^+, \\ z(x,0) = 0 \quad x \in \Omega_h, \end{gathered} \tag{34}$$

where

$$\bar{\varphi} = \varphi + \sum_{i,j=1}^{2} \bar{\eta}_{ij,\bar{x}_i} + \sum_{i=1}^{2} (\bar{\eta}_{i,\bar{x}_i} + \xi_{i,\bar{x}_i}) + \xi_{\bar{x}_2} + \sum_{i=1}^{2} \bar{\zeta}_i + \bar{\eta} + \delta_{S_h}\nu\,,$$

φ is the same as before and

$$\begin{aligned}
\bar{\eta}_{ij} &= \frac{\tau}{2}\big[a_{ij}u_{\bar{t}x_j} + \big(a_{ij}u_{\bar{t}\bar{x}_j}\big)\big|_{(x+he_i,t)}\big],\\
\bar{\eta}_i &= -\frac{\tau}{2}\big[\,(b_iu_{\bar{t}})\big|_{(x+he_i,t)} + b_iu_{\bar{t}}\big],\\
\bar{\zeta}_i &= -\frac{\tau}{2}\big(b_iu_{\bar{t}\bar{x}_i} + b_iu_{\bar{t}x_i}\big),\\
\bar{\eta} &= -\tau(T_1^2T_2^2c)u_{\bar{t}},\\
\xi_i &= -\theta\tau^{\alpha}\partial^{\alpha}_{t,0+,\tau}u_{x_i},\\
\xi &= \theta^2\tau^{2\alpha}\partial^{\alpha}_{t,0+,\tau}u_{x_1\bar{x}_1x_2},\\
\nu &= -\theta\tau^{\alpha}K\partial^{\alpha}_{t,0+,\tau}u_{x_1\bar{x}_1}.
\end{aligned}$$

THEOREM 5. *Under the assumptions of Theorems 1 and 4 the finite difference scheme* (33)–(34) *is absolutely stable and its solution satisfies the following a priori estimate:*

$$\begin{aligned}
\|z\|_{\tilde{B}^{1,1/2}(Q_{h\tau})} \le C\Big\{ &\sum_{i,j=1}^{2}\|\bar{\eta}_{ij}\|_{L^2(Q_{ih\tau})} + \sum_{j=1}^{2}\big(\|\eta_{2j}\|_{L^2(Q_{2h\tau})}\\
&+ \|\tilde{\eta}_{1j}\|_{L^2(Q_{1h\tau})} + |\hat{\eta}_{1j}|_{L^2(\Omega_\tau;H^{1/2}(S_h))} + \|\xi_j\|_{L^2(Q_{jh\tau})} + \|\bar{\eta}_j\|_{L^2(Q_{jh\tau})}\\
&+ \|\bar{\zeta}_j\|_{L^2(Q_{h\tau})} + \|\tilde{\zeta}_j\|_{L^2(Q_{h\tau})} + \|\hat{\zeta}_j\|_{L^2(\Sigma_{h\tau})}\big) + \|\xi\|_{L^2(Q_{2h\tau})} + \|\eta_2\|_{L^2(Q_{2h\tau})}\\
&+ \|\tilde{\eta}_1\|_{L^2(Q_{1h\tau})} + |\hat{\eta}_1|_{L^2(\Omega_\tau;H^{1/2}(S_h))} + \|\tilde{\eta}\|_{L^2(Q_{h\tau})} + \|\bar{\eta}\|_{L^2(Q_{h\tau})}\\
&+ \|\hat{\eta}\|_{L^2(\Sigma_{h\tau})} + \|\tilde{\chi}\|_{L^2(Q_{h\tau})} + \|\hat{\chi}\|_{L^2(\Sigma_{h\tau})} + \|\mu\|_{L^2(\Sigma_{h\tau})} + \|\nu\|_{L^2(\Sigma_{h\tau})}\Big\}. \quad (35)
\end{aligned}$$

The proof is similar to the proof of Theorem 4.

THEOREM 6. *Let the assumptions of Theorem 1 hold, $a_{ij}, b_i \in H^2(\Omega^{\pm})$, $c \in H^1(\Omega^{\pm})$, $K = const > 0$, and let the solution u of the initial-boundary value problem* (5)–(7) *belong to the space $C^2([0,T], C(\bar{\Omega}))\cap C^2([0,T], H^3(\Omega^{\pm}))$. Then the solution v of finite difference scheme* (33)–(34) *with $\bar{c} = T_1^2T_2^2c$, $\bar{f} = T_1^2T_2^2f$ and sufficiently large θ converges to u and the following convergence rate estimate holds:*

$$\|u - v\|_{\tilde{B}^{1,\alpha/2}(Q_{h\tau})} = O(h^2 + \tau^{\alpha}).$$

Proof. The terms containing in φ were estimated in (20)-(30) while $\bar{\eta}_{ij}$, $\bar{\eta}_i$, $\bar{\eta}$, $\bar{\zeta}_i$, ξ_i, ξ and ν can be estimated directly. For example, from the definition of $\bar{\eta}_{11}$ it follows that

$$|\bar{\eta}_{11}(x,t)| \le \frac{\tau}{\sqrt{h\tau}}\max_{x_1\le x_1'\le x_1+h}|a_{11}(x_1',x_2)|\left\|\frac{\partial^2u}{\partial x_1\partial t}(\cdot,x_2,\cdot)\right\|_{L^2((x_1,x_1+h)\times(t-\tau,t))},$$

whereby, after summation over the mesh $\Omega_{1h}\times\Omega_\tau^+$ and using the trace theorem for Sobolev spaces [1], one obtains

$$\begin{aligned}
\|\bar{\eta}_{11}\|_{L^2(Q_{1h\tau})} &\le \tau\max_{x_2\in\{h,\dots,(2n-1)h\}}\|a_{11}(\cdot,x_2)\|_{C[0,1]}\left\|\frac{\partial^2u}{\partial x_1\partial t}(\cdot,x_2,\cdot)\right\|_{L^2((0,1)\times(0,T))}\\
&\le C\tau\left(\|a_{11}\|_{C(\Omega^-)}\|u\|_{H^1((0,T),H^2(\Omega^-))} + \|a_{11}\|_{C(\Omega^+)}\|u\|_{H^1((0,T),H^2(\Omega^+))}\right).
\end{aligned}$$

Analogous bounds hold for the other $\bar{\eta}_{ij}$, $\bar{\eta}_i$, $\bar{\eta}$ and $\bar{\bar{\zeta}}_i$:

$$\begin{aligned}\|\bar{\eta}_{ij}\|_{L^2(Q_{ih\tau})} &\le C\tau\big(\|a_{ij}\|_{C(\Omega^-)}\|u\|_{H^1((0,T),H^2(\Omega^-))}\\ &\quad+\|a_{ij}\|_{C(\Omega^+)}\|u\|_{H^1((0,T),H^2(\Omega^+))}\big),\end{aligned}\tag{36}$$

$$\begin{aligned}\|\bar{\eta}_i\|_{L^2(Q_{ih\tau})} &\le C\tau\big(\|b_i\|_{C(\Omega^-)}\|u\|_{H^1((0,T),H^2(\Omega^-))}\\ &\quad+\|b_i\|_{C(\Omega^+)}\|u\|_{H^1((0,T),H^2(\Omega^+))}\big),\end{aligned}\tag{37}$$

$$\begin{aligned}\|\bar{\bar{\zeta}}_{ij}\|_{L^2(Q_{h\tau})} &\le C\tau\big(\|b_i\|_{C(\Omega^-)}\|u\|_{H^1((0,T),H^2(\Omega^-))}\\ &\quad+\|b_i\|_{C(\Omega^+)}\|u\|_{H^1((0,T),H^2(\Omega^+))}\big),\end{aligned}\tag{38}$$

$$\begin{aligned}\|\bar{\eta}\|_{L^2(Q_{h\tau})} &\le C\tau\big(\|c\|_{L^2(\Omega^-)}\|u\|_{H^1((0,T),H^2(\Omega^-))}\\ &\quad+\|c\|_{L^2(\Omega^+)}\|u\|_{H^1((0,T),H^2(\Omega^+))}\big).\end{aligned}\tag{39}$$

Let us set $\xi_i = \xi_{i1} + \xi_{i2} = -\theta\tau^\alpha \partial^\alpha_{t,0+} u_{x_i} + \theta\tau^\alpha \left(\partial^\alpha_{t,0+} u_{x_i} - \partial^\alpha_{t,0+,\tau} u_{x_i}\right)$. Then, similarly as in the previous cases:

$$\begin{aligned}\|\xi_{i1}\|_{L^2(Q_{ih\tau})} &\le C\tau^\alpha \left(\|\partial^\alpha_{t,0+}u\|_{C([0,T],H^2(\Omega^-))} + \|\partial^\alpha_{t,0+}u\|_{C([0,T],H^2(\Omega^+))}\right)\\ &\le C\tau^\alpha\big(\|u\|_{C^\alpha_+([0,T],H^2(\Omega^-))} + \|u\|_{C^\alpha_+([0,T],H^2(\Omega^+))}\big).\end{aligned}$$

The second term we estimate using Lemma 3:

$$\|\xi_{i2}\|_{L^2(Q_{ih\tau})} \le C\tau^\alpha\tau^{2-\alpha}\left(\|u\|_{C^2([0,T],H^2(\Omega^-))} + \|u\|_{C^2([0,T],H^2(\Omega^+))}\right).$$

Hence, after obvious majorization, we obtain:

$$\|\xi_i\|_{L^2(Q_{ih\tau})} \le C\tau^\alpha\left(\|u\|_{C^2([0,T],H^2(\Omega^-))} + \|u\|_{C^2([0,T],H^2(\Omega^+))}\right).\tag{40}$$

The terms ξ and ν can be estimated in analogous manner:

$$\|\xi\|_{L^2(Q_{2h\tau})} \le C\tau^{2\alpha}\left(\|u\|_{C^2([0,T],H^3(\Omega^-))} + \|u\|_{C^2([0,T],H^3(\Omega^+))}\right),\tag{41}$$

$$\|\nu\|_{L^2(\Sigma_{h\tau})} \le C\tau^\alpha\left(\|u\|_{C^2([0,T],H^3(\Omega^-))} + \|u\|_{C^2([0,T],H^3(\Omega^+))}\right).\tag{42}$$

Finally, the result follows from (35), (20)–(30) and (36)–(42). ■

7. Numerical experiment

We consider the problem (5)–(7) in the domain $\Omega \times (0,T)$ with forcing term

$$\begin{aligned}f(x_1,x_2,t) = \sin(\pi x_1)\Bigg[\sin(\pi x_2)\left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2\pi^2 t^2\right)\\ +2\left|\sin(2\pi x_2)\right|\left(\frac{t^{2-2\alpha}}{\Gamma(3-2\alpha)} + \frac{5\pi^2 t^{2-\alpha}}{\Gamma(3-\alpha)}\right)\Bigg],\end{aligned}$$

$a_{ii} = 1$, $a_{ij}, b_i, c = 0$ for $i,j = 1,2$, $K = 4\pi$ and $T = 1$. The exact solution is

$$u(x_1,x_2,t) = \sin(\pi x_1)\left(\sin(\pi x_2)\, t^2 + \left|\sin(2\pi x_2)\right|\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\right).$$

We compute the problem using the factorized scheme (31), (14). We test the temporal errors and convergence orders by letting τ vary and fixing $h = 2^{-8}$.

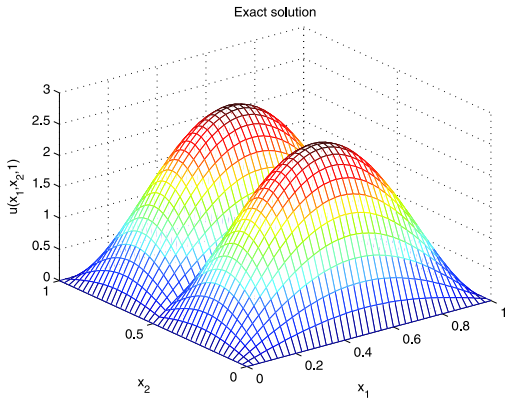

Fig. 1. The exact solution for $\alpha = 0.9$ and $t = 1$

Table 1 presents the computational results. It shows that the proposed factorized scheme generates temporal convergence rate α.

Table 1. The experimental error results and temporal convergence orders for the factorized scheme when $\theta = \frac{\Gamma(2-\alpha)}{4(1-2^{-\alpha})}$ and $h = 2^{-8}$ is fixed

α	τ	$\|z\|_{\tilde{B}^{1,\alpha/2}(Q_{h,\tau})}$	$\log_2 \frac{\|z\|_{\tilde{B}^{1,\alpha/2}(Q_{h,\tau})}}{\|z\|_{\tilde{B}^{1,\alpha/2}(Q_{h,\tau/2})}}$
0.9	2^{-5}	$1.62311806 \cdot 10^{-1}$	0.7931
	2^{-6}	$9.36718006 \cdot 10^{-2}$	0.8391
	2^{-7}	$5.23618878 \cdot 10^{-2}$	0.8614
	2^{-8}	$2.88210355 \cdot 10^{-2}$	0.8709
	2^{-9}	$1.57594461 \cdot 10^{-2}$	/
0.4	2^{-5}	1.54556906	0.3577
	2^{-6}	1.20613588	0.3783
	2^{-7}	$9.27935048 \cdot 10^{-1}$	0.3887
	2^{-8}	$7.08795916 \cdot 10^{-1}$	0.3901
	2^{-9}	$5.40849987 \cdot 10^{-1}$	/

The computational results for the spatial errors and convergence orders are given in Table 2. We fixed τ sufficiently small to make sure that the dominant error is from the space discretization. It can be seen that the factorized scheme achieves second order spatial accuracy.

In Figure 2 we have displayed the exact and numerical solutions on the last time level for comparison.

Table 2. The experimental error results and spatial convergence orders for the factorized scheme when $\theta = \frac{\Gamma(2-\alpha)}{4(1-2^{-\alpha})}$ and $\tau = 2^{-14}$ is fixed

α	h	$\|z\|_{\tilde{B}^{1,\alpha/2}(Q_{h,\tau})}$	$\log_2 \frac{\|z\|_{\tilde{B}^{1,\alpha/2}(Q_{h,\tau})}}{\|z\|_{\tilde{B}^{1,\alpha/2}(Q_{h/2,\tau})}}$
0.9	2^{-2}	$7.20603969 \cdot 10^{-1}$	2.0100
	2^{-3}	$1.78901098 \cdot 10^{-1}$	2.0009
	2^{-4}	$4.46971707 \cdot 10^{-2}$	1.9017
	2^{-5}	$1.12387233 \cdot 10^{-2}$	1.9317
	2^{-6}	$2.94593731 \cdot 10^{-3}$	/

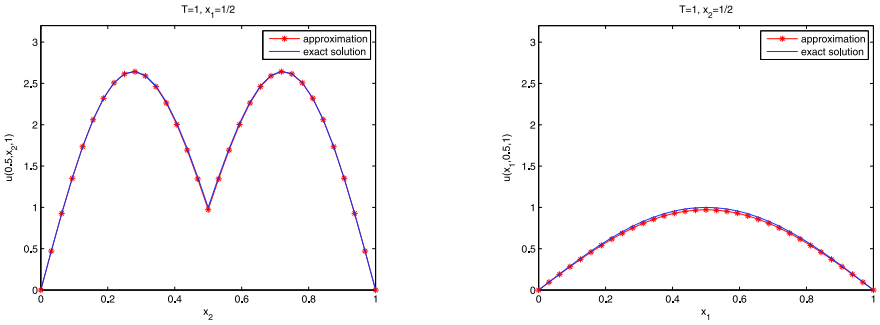

Fig. 2. Solution behavior for $\alpha = 0.9$, $T = 1$, $h = 2^{-5}$ and $\tau = 2^{-8}$.

Acknowledgement. This research was supported by Ministry of Education, Science and Technological Development of Republic of Serbia under project 174015.

REFERENCES

[1] R. A. Adams, J. J. F. Fournier, *Sobolev Spaces*, 2nd ed., Pure and Applied Mathematics Series, **140**, Elsevier/Academic Press, Amsterdam, 2003.

[2] A. A. Alikhanov, *Boundary value problems for the diffusion equation of the variable order in differential and difference settings*, Appl. Math. Comput. **219** (2012), 3938–3946.

[3] J. H. Bramble, S. R. Hilbert, *Estimation of linear functionals on Sobolev spaces with application to Fourier transform and spline interpolation*, SIAM J. Numer. Anal., **7** (1970), 112–124.

[4] A. Delić, B. S. Jovanović, *Numerical approximation of an interface problem for fractional in time diffusion equation*, Appl. Math. Comput., **229** (2014), 467–479.

[5] V. J. Ervin, J. P. Roop, *Variational formulation for the stationary fractional advection dispersion equation*, Numer. Methods Partial Differ. Equ. **23** (2006), 558–576.

[6] S. Hodžić, *Factorized difference scheme for 2D fractional in time diffusion equation*, Appl. Anal. Discrete Math. **9**, 2 (2015), 199–208.

[7] S. Hodžić, B. S. Jovanović, *Additive difference scheme for two-dimensional fractional in time diffusion equation*, Filomat (to appear).

[8] B. S. Jovanović, E. Süli, *Analysis of Finite Difference Schemes*, Springer Series in Computational Mathematics, **46**, Springer, 2013.

[**9**] B. S. Jovanović, L. G. Vulkov, *Finite difference approximations for some interface problems with variable coefficients*, Appl. Numer. Math. **59**, 2 (2009), 349–372.

[**10**] A. Kilbas, H. Srivastava, J. Trujillo, *Theory and Applications of Fractional Differential Equations*, N-H Mathematics Studies, **204**, North-Holland, 2006.

[**11**] X. Li, C. Xu, *A space-time spectral method for the time fractional diffusion equation*, SIAM J. Numer. Anal., **47** (2009), 2108–2131.

[**12**] Z. Li, K. Ito, *The Immersed Interface Method*, Frontiers in Applied Mathematics, SIAM, Philadelphia, 2006.

[**13**] J. L. Lions, E. Magenes, *Non Homogeneous Boundary Value Problems and Applications*, Grundlehren Math. Wiss., **181**, Springer-Verlag, New York 1972.

[**14**] A. V. Lykov, *Heat–Mass Transfer*, Energiya, Moscow, 1978 [in Russian].

[**15**] V. G. Maz′ya, T. O. Shaposhnikova, *Theory of Multipliers in Spaces of Differentiable Functions*, Monographs and Studies in Mathematics **23**, Pitman, Boston, Mass., 1985.

[**16**] K. B. Oldham, J. Spanier, *The Fractional Calculus*, Academic Press, New York, 1974.

[**17**] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[**18**] R. K. Sinha, B. Deka, *On the convergence of finite element method for second order elliptic interface problems*, Numer. Funct. Anal. Optim. **27**, No 1 (2006) 99–115.

[**19**] A. A. Samarskiĭ, *Theory of Difference Schemes*, Pure and Appl. Math., **240**, Marcel Dekker Inc., 2001.

[**20**] Z. Z. Sun, X. N. Wu, *A fully discrete difference scheme for a diffusion-wave system*, Appl. Numer. Math., **56** (2006), 193–209.

[**21**] A. N. Tikhonov, A. A. Samarskiĭ, *Equations of Mathematical Physics*, Nauka, Moscow, 1966 [in Russian].

[**22**] W. Yao, J. Sun, B. Wu, S. Shi, *Numerical simulation of a class of fractional subdiffusion equations via the alternating direction implicit method*, Numer. Methods Part. Differ. Equat. **32** (2016), 531–547.

(received 18.04.2016; available online 27.06.2016)

University of Belgrade, Faculty of Mathematics, Studentski trg 16, 11000 Belgrade, Serbia

E-mail: adelic@matf.bg.ac.rs, sandra@matf.bg.ac.rs, bosko@matf.bg.ac.rs