МАТЕМАТИЧКИ ВЕСНИК
68, 1 (2016), 12–25
March 2016

оригинални научни рад
research paper

EXISTENCE OF A POSITIVE SOLUTION FOR A THIRD-ORDER THREE POINT BOUNDARY VALUE PROBLEM

Ali Rezaiguia and Smail Kelaiaia

Abstract. By applying the Krasnoselskii fixed point theorem in cones and the fixed point index theory, we study the existence of positive solutions of the non linear third-order three point boundary value problem

$$u'''(t) + a(t)f(t, u(t)) = 0, \quad t \in (0,1),$$
$$u'(0) = u'(1) = \alpha u(\eta), \quad u(0) = \beta u(\eta),$$

where α, β and η are constants with $\alpha \in [0, \frac{1}{\eta})$, and $0 < \eta < 1$. The results obtained here generalize the work of Torres [Positive solution for a third-order three point boundary value problem, Electronic J. Diff. Equ. 2013 (2013), 147, 1–11].

1. Introduction

Third order equations arise in a variety of different areas of applied mathematics and physics, as the deflection of a curved beam having a constant or varying cross section, three layer beam, electromagnetic waves or gravity driven flows and so on [9]. Different types of techniques have been used to study such problems: reduce them to first and/or second order equations [5], use Green's functions and comparison principles [2, 3, 15] (for periodic boundary value conditions), [4, 6, 7, 10, 20] (two point ones), and [16, 19] (three point boundary conditions). A large part of the literature on multiple solutions to boundary value problems seems to be traced back to Krasnoseleskii's work on nonlinear operator equations [1], especially the part dealing with the theory of cones in Banach spaces.

In this paper, we are interested in the analysis of qualitative theory of positive solutions of third-order differential equations. Motivated by the papers [12–14,17] and the references therein, we concentrate on the existence of positive solutions for the nonlinear third-order differential equation three point boundary value problem

$$u'''(t) + a(t)f(t, u(t)) = 0, \quad t \in (0,1), \tag{1.1}$$
$$u'(0) = u'(1) = \alpha u(\eta), \quad u(0) = \beta u(\eta), \tag{1.2}$$

2010 Mathematics Subject Classification: 34B10

Keywords and phrases: Third-order differential equations; three point boundary value problem; Krasnoselski fixed point in a cone; fixed point index theory

where α, β and η are constants with $\alpha \in [0, \frac{1}{\eta})$, $0 < \eta < 1$ and $\beta \in [0, 1-\alpha\eta)$. In the case $\beta = 0$, Torres in [18] showed that (1.1) and (1.2) has positive solutions by using Krasnoselskii's fixed point theorem and the fixed point index theory.

This paper is organized as follows. In Section 2, we present some theorems and lemmas that will be used to prove our main results. In Section 3, we discuss the existence of single positive solutions of BVP (1.1) and (1.2). In Section 4, we discuss the existence conditions of multiple positive solutions of BVP (1.1) and (1.2). In Section 5, we give some examples to illustrate our results. The results presented in this paper generalize the main results in [18].

2. Preliminaries

We shall consider the Banach space $X = C[0,1]$ equipped with the norm $\|x\| = \max_{0\le t\le 1} |x(t)|$. Define a cone in X by $C^+[0,1] = \{x \in X : x(t) \geqslant 0 \text{ for } t \in [0,1]\}$, and the ordering $\leqslant$ by $x \leqslant y$ iff $x(t) \leqslant y(t)$ for all $t \in [0,1]$.

Definition 1. A function $u(t)$ is called a positive solution of (1.1) and (1.2) if $u \in C[0,1]$ and $u(t) > 0$ for all $t \in (0,1)$.

The proof of existence of positive solutions is based on applications of the following theorems.

Theorem 1. [8,11] *Let X be a Banach space and K be a cone in X. Assume that Ω_1 and Ω_2 are open subsets of X with $0 \in \Omega_1 \subseteq \overline{\Omega_1} \subseteq \Omega_2$ and let*

$$T\colon K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|Tu\| \le \|u\|$ if $u \in K \cap \partial\Omega_1$ and $\|Tu\| \ge \|u\|$ if $u \in K \cap \partial\Omega_2$, or

(ii) $\|Tu\| \ge \|u\|$if $u \in K \cap \partial\Omega_1$ and $\|Tu\| \le \|u\|$ if $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Theorem 2. [8,11] *Let X be a Banach space and K be a cone in X. For $r > 0$, define $K_r = \{u \in K : \|u\| \le r\}$ and assume $T : K_r \to K$ is a completely continuous operator such that $Tu \ne u$ for $u \in \partial K_r$.*

(1) *If $\|Tu\| \le \|u\|$ for all $u \in \partial K_r$ then $i(T, K_r, K) = 1$.*

(2) *If $\|Tu\| \ge \|u\|$ for all $u \in \partial K_r$ then $i(T, K_r, K) = 0$.*

Lemma 1. *Assume that $\beta \in [0, 1-\alpha\eta)$. Then for $y \in C[0,1]$ the problem*

$$u'''(t) + y(t) = 0, \quad t \in (0,1), \tag{2.1}$$

$$u'(0) = u'(1) = \alpha u(\eta), \quad u(0) = \beta u(\eta), \tag{2.2}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)\,ds + \frac{\alpha t + \beta}{1 - \alpha\eta - \beta}\int_0^1 G(\eta,s)y(s)\,ds,$$

where

$$G(t,s) = \begin{cases} \frac{1}{2}(2t - t^2 - s)s, & 0 \le s \le t \le 1, \\ \frac{1}{2}t^2(1-s), & 0 \le t \le s \le 1. \end{cases} \tag{2.3}$$

Proof. Rewriting the differential equation as $u'''(t) = -y(t)$ and integrating three times, we obtain

$$u(t) = -\frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds + A_1 t^2 + A_2 t + A_3 \text{ where } A_1, A_2, A_3 \in \mathbf{R}. \qquad (2.4)$$

Since $u'(0) = u'(1)$,

$$\begin{aligned} A_2 &= -\int_0^1 (1-s)y(s)\,ds + 2A_1 + A_2, \\ A_1 &= \frac{1}{2}\int_0^1 (1-s)y(s)\,ds. \end{aligned}$$

Since $u'(0) = \alpha u(\eta)$, we obtain

$$\begin{aligned} A_2 &= \alpha\left(-\frac{1}{2}\int_0^\eta (\eta-s)^2 y(s)\,ds + \frac{\eta^2}{2}\int_0^1 (1-s)y(s)\,ds + A_2\eta + A_3\right), \\ \alpha A_3 &= (1-\alpha\eta)A_2 + \frac{\alpha}{2}\int_0^\eta (\eta-s)^2 y(s)\,ds - \frac{\alpha\eta^2}{2}\int_0^1 (1-s)y(s)\,ds. \end{aligned}$$

From $u(0) = \beta u(\eta)$, we have

$$\begin{aligned} A_3 &= \beta\left(-\frac{1}{2}\int_0^\eta (\eta-s)^2 y(s)\,ds + A_1\eta^2 + A_2\eta + A_3\right), \\ A_3 &= \frac{\beta\eta^2}{2(1-\alpha\eta-\beta)}\int_0^1 (1-s)y(s)\,ds - \frac{\beta}{2(1-\alpha\eta-\beta)}\int_0^\eta (\eta-s)^2 y(s)\,ds, \\ \alpha A_3 &= \frac{\alpha\beta\eta^2}{2(1-\alpha\eta-\beta)}\int_0^1 (1-s)y(s)\,ds - \frac{\alpha\beta}{2(1-\alpha\eta-\beta)}\int_0^\eta (\eta-s)^2 y(s)\,ds. \end{aligned}$$

Then

$$A_2 = \frac{\alpha\eta^2}{(1-\alpha\eta-\beta)}\int_0^1 \frac{1}{2}(1-s)y(s)\,ds - \frac{\alpha}{(1-\alpha\eta-\beta)}\int_0^\eta \frac{1}{2}(\eta-s)^2 y(s)\,ds$$

Replacing these expressions in (2.4),

$$\begin{aligned} u(t) &= -\frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds + \frac{t^2}{2}\int_0^1 (1-s)y(s)\,ds \\ &\quad + \frac{1}{2}\frac{\alpha\eta^2 t}{(1-\alpha\eta-\beta)}\int_0^1 (1-s)y(s)\,ds - \frac{1}{2}\frac{\alpha t}{(1-\alpha\eta-\beta)}\int_0^\eta (\eta-s)^2 y(s)\,ds \\ &\quad + \frac{\beta\eta^2}{2(1-\alpha\eta-\beta)}\int_0^1 (1-s)y(s)\,ds - \frac{\beta}{2(1-\alpha\eta-\beta)}\int_0^\eta (\eta-s)^2 y(s)\,ds \\ &= -\frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds + \frac{t^2}{2}\int_0^1 (1-s)y(s)\,ds \\ &\quad - \frac{1}{2}\frac{\alpha t+\beta}{(1-\alpha\eta-\beta)}\int_0^\eta (\eta-s)^2 y(s)\,ds + \frac{\alpha t+\beta}{(1-\alpha\eta-\beta)}\frac{\eta^2}{2}\int_0^1 (1-s)y(s)\,ds \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds + \frac{t^2}{2}\int_0^1 (1-s)y(s)\,ds \\
&\quad + \frac{\alpha t+\beta}{(1-\alpha\eta-\beta)}\Big(-\frac{1}{2}\int_0^\eta (\eta-s)^2 y(s)\,ds + \frac{\eta^2}{2}\int_0^1 (1-s)y(s)\,ds\Big) \\
&= -\frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds + \frac{t^2}{2}\int_0^t (1-s)y(s)\,ds + \frac{t^2}{2}\int_t^1 (1-s)y(s)\,ds \\
&\quad + \frac{\alpha t+\beta}{(1-\alpha\eta-\beta)}\Big(-\frac{1}{2}\int_0^\eta (\eta-s)^2 y(s)\,ds + \frac{\eta^2}{2}\int_0^1 (1-s)y(s)\,ds\Big) \\
&= \frac{1}{2}\Big[\int_0^t (2t-t^2-s)sy(s)\,ds + \int_t^1 t^2(1-s)y(s)\,ds\Big] \\
&\quad + \frac{\alpha t+\beta}{(1-\alpha\eta-\beta)}\Big(-\frac{1}{2}\int_0^\eta (\eta-s)^2 y(s)\,ds + \frac{\eta^2}{2}\int_0^1 (1-s)y(s)\,ds\Big) \\
&= \int_0^1 G(t,s)y(s)\,ds + \frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)y(s)\,ds \quad \blacksquare
\end{aligned}
$$

LEMMA 2. *For all t and s such that $0 \le s \le 1$ and $0 < \tau \le t \le 1$, we have*

$$\theta G(1,s) \le G(t,s) \le G(1,s) = \frac{1}{2}(1-s)s \quad \text{where} \quad \theta = \tau^2. \tag{2.5}$$

Proof. For all $t, s \in [0,1]$, if $s \le t$,

$$
\begin{aligned}
G(t,s) &= \frac{1}{2}(2t-t^2-s)s = \frac{1}{2}[(1-s)-(1-t)^2]s \\
&\le \frac{1}{2}(1-s)s = G(1,s),
\end{aligned}
$$

and

$$
\begin{aligned}
G(t,s) &= \frac{1}{2}(2t-t^2-s)s = \frac{1}{2}st^2(1-s) + \frac{1}{2}(1-t)[(t-s)+(1-s)t]s \\
&\ge \theta G(1,s);
\end{aligned}
$$

if $t \le s$,

$$\frac{t^2}{2}(1-s)s \le G(t,s) = \frac{1}{2}t^2(1-s) \le G(1,s).$$

Therefore

$$\theta G(1,s) \le G(t,s) \le G(1,s), \ \forall (t,s) \in [\tau,1]\times[0,1]. \quad \blacksquare \tag{2.6}$$

LEMMA 3. *For all $y \in C^+[0,1]$, the unique solution $u(t)$ of (2.1) and (2.2) is nonnegative and satisfies*

$$\min_{\tau\le t\le 1} u(t) \ge \theta\|u\|.$$

Proof. From Lemma 1 and Lemma 2, $u(t)$ is nonnegative,

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)y(s)\,ds + \frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)y(s)\,ds \\ &\le \frac{1}{2}\int_0^1 s(1-s)y(s)\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)y(s)\,ds; \end{aligned}$$

then

$$\|u\| \le \frac{1}{2}\int_0^1 s(1-s)y(s)\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)y(s)\,ds. \tag{2.7}$$

For $t \in [\tau,1]$,

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)y(s)\,ds + \frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)y(s)\,ds \\ &\ge \tau^2\int_0^1 \frac{1}{2}s(1-s)y(s)\,ds + \frac{\alpha t^2+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)y(s)\,ds \\ &\ge \tau^2\int_0^1 \frac{1}{2}s(1-s)y(s)\,ds + \frac{\alpha \tau^2+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)y(s)\,ds \\ &\ge \tau^2\int_\tau^1 \frac{1}{2}s(1-s)y(s)\,ds + \tau^2\frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)y(s)\,ds \\ &= \theta\left[\frac{1}{2}\int_0^1 s(1-s)y(s)\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)y(s)\,ds\right] \\ &\ge \theta\|u\|. \end{aligned}$$

Therefore $\min_{\tau\le t\le 1}(t) \ge \theta\|u\|$. ■

Now, we assume the following

$(J1)$ $f \in C([0,1]\times[0,\infty),[0,\infty))$,

$(J2)$ $a \in L^1[0,1]$ is nonnegative and $a(t) \not\equiv 0$ on any subinterval $[0,1]$.

Define the cone

$$K = \left\{u \in C[0,1] : u(t) \ge 0,\ \min_{\tau\le t\le 1} u(t) \ge \theta\|u\|\right\},$$

and the operator $T\colon K \to X$ by

$$Tu(t) = \int_0^1 G(t,s)a(s)f(s,u(s))\,ds + \frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)f(s,u(s))\,ds. \tag{2.8}$$

By Lemma1 and Lemma 3, it is easy to see that the BVP (1.1) and (1.2) has a positive solution $u(t)$ if and only if u is a fixed point of T.

LEMMA 4. *The operator defined in* (2.8) *is completely continuous and satisfies* $T(K) \subseteq K$.

Proof. By Lemma 3, $T(K) \subseteq K$, and by Ascoli-Arzela theorem we prove that T is a completely continuous operator. ■

3. Existence results

Throughout this paper, we shall use the following notations

$$
\begin{aligned}
f^0 &= \limsup_{u\to 0} \max_{0\le t\le 1} \frac{f(t,u)}{u}, \quad f_0 = \liminf_{u\to 0} \min_{0\le t\le 1} \frac{f(t,u)}{u},\\
f^\infty &= \limsup_{u\to \infty} \max_{0\le t\le 1} \frac{f(t,u)}{u}, \quad f_\infty = \liminf_{u\to \infty} \min_{0\le t\le 1} \frac{f(t,u)}{u},\\
A &= \int_0^1 G(1,s)a(s)\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)\,ds, \quad R = A^{-1},\\
B &= \tau^2\Big(\int_\tau^1 G(1,s)a(s)\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_\tau^1 G(\eta,s)a(s)\,ds\Big), \quad r = B^{-1}.
\end{aligned}
$$

Theorem 3. *Assume that $f^0 = 0$ and $f_\infty = \infty$ are valid. Then the problem* (1.1) *and* (1.2) *has at least one positive solution.*

Proof. Since $f^0 = 0$, there exists $H_1 > 0$ such that $f(t,u) \le \epsilon u$, for all $t \in [0,1]$ where $0 < u \le H_1$ and $\epsilon > 0$. Then for $u \in K\cap\partial\Omega_1$ with $\Omega_1 = \{u \in X : \|u\| < H_1\}$, we have

$$
\begin{aligned}
Tu(t) &= \int_0^1 G(t,s)a(s)f(s,u(s))\,ds + \frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)f(s,u(s))\,ds\\
&\le \int_0^1 G(1,s)a(s)f(s,u(s))\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)f(s,u(s))\,ds\\
&\le \int_0^1 G(1,s)a(s)\epsilon u(s)\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)\epsilon u(s)\,ds\\
&\le \epsilon\Big[\int_0^1 G(1,s)a(s)\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)\,ds\Big]\|u\|.
\end{aligned}
$$

If $\epsilon A \leqslant 1$, then $Tu(t) \le \|u\|$. Therefore

$$\|Tu\| \le \|u\| \text{ for } u \in K\cap\partial\Omega_1.$$

On the other hand, since $f_\infty = \infty$, there exists $\widehat{H}_2 > 0$ such that $f(t,u) \ge \delta u$, for all $t \in [\tau,1]$ where $u \ge \widehat{H}_2$ and $\delta > 0$. Then for $u \in K\cap\partial\Omega_2$ where $\Omega_2 = \{u \in X, \|u\| < H_2\}$ with $H_2 = \max\{2H_1, \frac{\widehat{H}_2}{\theta}\}$. Then $u \in K\cap\partial\Omega_2$ implies that $\min_{\tau\le t\le 1} u(t) \ge \theta\|u\| = \theta H_2 > \widehat{H}_2$, and

$$
\begin{aligned}
Tu(1) &= \int_0^1 G(1,s)a(s)f(s,u(s))\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)f(s,u(s))\,ds\\
&\ge \int_\tau^1 \frac{1}{2}s(1-s)a(s)f(s,u(s))\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_\tau^1 G(\eta,s)a(s)f(s,u(s))\,ds\\
&\ge \int_\tau^1 \frac{1}{2}s(1-s)a(s)\delta u(s)\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_\tau^1 G(\eta,s)a(s)\delta u(s)\,ds
\end{aligned}
$$

$$\geq \delta\left[\int_\tau^1 \frac{\tau^2}{2}s(1-s)a(s)\,ds+\tau^2\frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_\tau^1 G(\eta,s)a(s)\,ds\right]\|u\|.$$

If $\delta B\geq 1$ then $Tu(1)\geq\|u\|$. We conclude that

$$\|Tu\|\geq\|u\| \text{ for } u\in K\cap\partial\Omega_2.$$

Then T has a fixed point in $K\cap(\overline{\Omega_2}\setminus\Omega_1)$ such that $H_1\leq\|u\|\leq H_2$. ■

Theorem 4. *Assume that $f_0=\infty$ and $f^\infty=0$ are valid. Then the problem* (1.1) *and* (1.2) *has at least one positive solution.*

Proof. Since $f_0=\infty$, there exists $H_1>0$ such that $f(t,u)\geq\delta u$, for all $t\in[\tau,1]$ where $0<u\leq H_1$ and $\delta>0$. Then for $u\in K\cap\partial\Omega_1$ with $\Omega_1=\{u\in X:\|u\|<H_1\}$, we have

$$\begin{aligned}Tu(1)&=\int_0^1 G(1,s)a(s)f(s,u(s))\,ds+\frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)f(s,u(s))\,ds\\
&\geq\int_\tau^1\frac{1}{2}s(1-s)a(s)f(s,u(s))\,ds+\frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_\tau^1 G(\eta,s)a(s)f(s,u(s))\,ds\\
&\geq\int_\tau^1\frac{1}{2}s(1-s)a(s)\delta u(s)\,ds+\frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_\tau^1 G(\eta,s)a(s)\delta u(s)\,ds\\
&\geq\delta\left[\int_\tau^1\frac{\tau^2}{2}s(1-s)a(s)\,ds+\tau^2\frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_\tau^1 G(\eta,s)a(s)\,ds\right]\|u\|.\end{aligned}$$

If $\delta B\geq 1$, then $Tu(1)\geq\|u\|$. Therefore

$$\|Tu\|\geq\|u\| \text{ for } u\in K\cap\partial\Omega_1.$$

On the other hand, Since $f^\infty=0$, there exists $\widehat{H}_2>0$ such that $f(t,u)\leq\epsilon u$, for all $t\in[0,1]$ where $u\geq\widehat{H}_2$ and $\epsilon>0$.

We consider two cases:

Case 1. Suppose f is bounded. Let L be such that $f(t,u)\leq L$ and $\Omega_2=\{u\in X:\|u\|<H_2\}$ with $H_2=\max\{2H_1,LA\}$. Then for $u\in K\cap\partial\Omega_2$, we have

$$\begin{aligned}Tu(t)&=\int_0^1 G(t,s)a(s)f(s,u(s))\,ds+\frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)f(s,u(s))\,ds\\
&\leq\int_0^1\frac{1}{2}s(1-s)a(s)f(s,u(s))\,ds+\frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)f(s,u(s))\,ds\\
&\leq\int_0^1\frac{1}{2}s(1-s)a(s)L\,ds+\frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)L\,ds\\
&=L\left[\int_0^1 G(1,s)a(s)\,ds+\frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)\,ds\right]\\
&\leq H_2=\|u\|.\end{aligned}$$

Then $\|Tu\|\leq\|u\|$.

CASE 2. Suppose f is unbounded. Then from $(J1)$, there exists $H_2 > \max\{2H_1, \widehat{H}_2\}$ such that $f(t,u) \le f(t,H_2)$ with $0 < u \le H_2$ and let $\Omega_2 = \{u \in X, \|u\| < H_2\}$. Then for $u \in K \cap \partial\Omega_2$, we have

$$\begin{aligned}
Tu(t) &= \int_0^1 G(t,s)a(s)f(s,u(s))\,ds + \frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)f(s,u(s))\,ds \\
&\le \int_0^1 \frac{1}{2}s(1-s)a(s)f(s,H_2)\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)f(s,H_2)\,ds \\
&\le \int_0^1 \frac{1}{2}s(1-s)a(s)\epsilon H_2\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)\epsilon H_2\,ds \\
&= \epsilon H_2\left[\int_0^1 G(1,s)a(s)\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)\,ds\right].
\end{aligned}$$

If $\epsilon A \le 1$, then $Tu(t) \le H_2 = \|u\|$. Therefore $\|Tu\| \le \|u\|$. ■

THEOREM 5. *Assume that* $0 \le f^0 < R$ *and* $r < f_\infty \le \infty$ *hold. Then the problem* (1.1) *and* $(1,2)$ *has at least one positive solution.*

Proof. Since $0 \le f^0 < R$, there exist $H_1 > 0$ and $0 < \epsilon_1 < R$ such that $f(t,u) \le (R-\epsilon_1)u$, $0 \le t \le 1$, $0 < u \le H_1$. Let $\Omega_1 = \{u \in X : \|u\| < H_1\}$. So for any $u \in K \cap \partial\Omega_1$ and $RA = 1$ we have

$$\begin{aligned}
Tu(t) &= \int_0^1 G(t,s)a(s)f(s,u(s))\,ds + \frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)f(s,u(s))\,ds \\
&\le \int_0^1 \frac{1}{2}s(1-s)a(s)(R-\epsilon_1)u\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)(R-\epsilon_1)u\,ds \\
&\le (R-\epsilon_1)\left[\int_0^1 G(1,s)a(s)\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)\,ds\right]\|u\| \\
&= (R-\epsilon_1)A\|u\| < \|u\|.
\end{aligned}$$

Thus $\|Tu\| < \|u\|$.

Since $r \le f_\infty \le \infty$ there exist $\overline{H}_2 > 0$ and $0 < \epsilon_2$ such that $f(t,u) \ge (r+\epsilon_2)u$, $u \ge \overline{H}_2$ and $\tau \le t \le 1$. Let $H_2 > \max\{2H_1, \frac{\overline{H}_2}{\theta}\}$ and $\Omega_2 = \{u \in X : \|u\| < H_2\}$. If $u \in K \cap \partial\Omega_2$, then $\min_{\tau\le t\le 1} u(t) \ge \theta\|u\| = \theta H_2 > \overline{H}_2$. So by a property of $G(t,s)$ we have

$$\begin{aligned}
Tu(1) &= \int_0^1 G(1,s)a(s)f(s,u(s))\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)f(s,u(s))\,ds \\
&\ge \int_\tau^1 G(1,s)a(s)(r+\epsilon_2)u\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_\tau^1 G(\eta,s)a(s)(r+\epsilon_2)u\,ds \\
&\ge (r+\epsilon_2)\left[\int_\tau^1 \tau^2 G(1,s)a(s)\,ds + \tau^2\frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_\tau^1 G(\eta,s)a(s)\,ds\right]\|u\| \\
&= (r+\epsilon_2)B\|u\| > \|u\|.
\end{aligned}$$

Thus $\|Tu\| > \|u\|$.

Therefore, by Theorem 1, the operator T has at least one fixed point, which is a positive solution of (1.1) and (1.2). ■

THEOREM 6. *Suppose that* $r < f_0 \leq \infty$ *and* $0 \leq f^\infty < R$ *hold. Then the problem* (1.1) *and* (1.2) *has at least one positive solution.*

Proof. Since $r < f_0 \leq \infty$ there exist $H_1 > 0$ and $0 < \epsilon_1$ such that $f(t,u) > (r+\epsilon_1)u$, $0 < u \leq H_1$ and $\tau \leq t \leq 1$. Let $\Omega_1 = \{u \in X : \|u\| < H_1\}$. If $u \in K \cap \partial\Omega_1$, then we have

$$\begin{aligned}
Tu(1) &= \int_0^1 G(1,s)a(s)f(s,u(s))\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)f(s,u(s))\,ds \\
&\geq \int_\tau^1 G(1,s)a(s)f(s,u(s))\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_\tau^1 G(\eta,s)a(s)f(s,u(s))\,ds \\
&\geq \int_\tau^1 G(1,s)a(s)(r+\epsilon_1)u\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_\tau^1 G(\eta,s)a(s)(r+\epsilon_1)u\,ds \\
&\geq (r+\epsilon_1)\left[\int_\tau^1 \tau^2 G(1,s)a(s)\,ds + \tau^2\frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_\tau^1 G(\eta,s)a(s)\,ds\right]\|u\| \\
&= (r+\epsilon_1)B\|u\| > \|u\|.
\end{aligned}$$

Thus, $Tu(1) > \|u\|$. Then $\|Tu\| > \|u\|$.

Since $0 \leq f^\infty < R$, there exist $\overline{H}_2 > 0$ and $0 < \epsilon_2 < R$ such that $f(t,u) < (R-\epsilon_2)u$, $\overline{H}_2 \leq u$ and $0 \leq t \leq 1$. Let $\Omega_2 = \{u \in X : \|u\| < H_2\}$ where $H_2 > \max\{2H_1, \frac{\overline{H}_2}{\theta}\}$. If $u \in K \cap \partial\Omega_1$ then $\min_{\tau \leq t \leq 1} u(t) \geq \theta\|u\| = \theta H_2 > \overline{H}_2$. So by a property of $G(t,s)$ we have

$$\begin{aligned}
Tu(t) &= \int_0^1 G(t,s)a(s)f(s,u(s))\,ds + \frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)f(s,u(s))\,ds \\
&\leq \int_0^1 G(t,s)a(s)(R-\epsilon_2)u\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)(R-\epsilon_2)u\,ds \\
&\leq (R-\epsilon_2)\left[\int_0^1 G(1,s)a(s)\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)\,ds\right]\|u\| \\
&= (R-\epsilon_1)A\|u\| < \|u\|.
\end{aligned}$$

Then $Tu(t) < \|u\|$. Therefore, by Theorem 1, the operator T has at least one fixed point, which is a positive solution of (1.1) and (1.2). ■

4. Multiplicity results

Now, we will study the problem (1.1) and (1.2) in the following cases:

(a) $\exists \rho > 0 : f(t,u) < R\rho,\ 0 < u \leq \rho, t \in [0,1]$.

(b) $\exists \rho > 0 : f(t,u) > r\rho,\ 0 < u \leq \frac{\rho}{\theta}, t \in [\tau,1]$.

THEOREM 7. *Assume that* (a) *holds,* $f_0 = \infty$ *and* $f_\infty = \infty$. *Then the problem* (1.1) *and* (1.2) *has at least two positive solutions.*

Proof. Since $f_0 = \infty$, there exists $H_1 > 0$ where $0 < H_1 < \rho$ such that $f(t,u) > ru$, for all $t \in [\tau, 1]$ where $0 < u \le H_1$. Then for $u \in K \cap \partial\Omega_1$ with $\Omega_1 = \{u \in X : \|u\| < H_1\}$, we have

$$\begin{aligned} Tu(1) &= \int_0^1 G(1,s)a(s)f(s,u(s))\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)f(s,u(s))\,ds \\ &> \int_\tau^1 G(1,s)a(s)ru\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_\tau^1 G(\eta,s)a(s)ru\,ds \\ &> r\Big[\int_\tau^1 \tau^2 G(1,s)a(s)\,ds + \tau^2\frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_\tau^1 G(\eta,s)a(s)\,ds\Big]\|u\| \\ &= rB\|u\| = \|u\|. \end{aligned}$$

Then $Tu(1) > \|u\|$. Therefore

$$\|Tu\| > \|u\| \text{ for } u \in K \cap \partial\Omega_1.$$

By Theorem 2, we have $i(T, K_{H_1}, K) = 0$.

Since $f_\infty = \infty$, there exists $\widehat{H}_2 > \rho$ such that $f(t,u) > ru$, for all $t \in [\tau, 1]$, where $u > \widehat{H}_2$. Then $u \in K \cap \partial\Omega_2$ where $\Omega_2 = \{u \in X : \|u\| < H_2\}$ with $H_2 > \frac{\widehat{H}_2}{\theta}$. This implies that $\min_{\tau \le t \le 1} u(t) \ge \theta\|u\| = \theta H_2 > \widehat{H}_2$ and

$$\begin{aligned} Tu(1) &= \int_0^1 G(1,s)a(s)f(s,u(s))\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)f(s,u(s))\,ds \\ &> \int_\tau^1 G(1,s)a(s)ru\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_\tau^1 G(\eta,s)a(s)ru\,ds \\ &> r\Big[\int_\tau^1 \tau^2 G(1,s)a(s)\,ds + \tau^2\frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_\tau^1 G(\eta,s)a(s)\,ds\Big]\|u\| \\ &= rB\|u\|. \end{aligned}$$

Then $Tu(1) > \|u\|$. Therefore

$$\|Tu\| > \|u\| \text{ for } u \in K \cap \partial\Omega_2.$$

By Theorem 2, we have $i(T, K_{H_2}, K) = 0$.

On the other hand, let $\Omega_3 = \{u \in X : \|u\| < \rho\}$. Then for $u \in K \cap \partial\Omega_3$ such that $f(t,u) < R\rho$ for all $t \in [0,1]$ we get

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)a(s)f(s,u(s))\,ds + \frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)f(s,u(s))\,ds \\ &< \int_0^1 G_1(t,s)a(s)R\rho\ \,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)R\rho\ \,ds \\ &< R\Big[\int_0^1 G(1,s)a(s)\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)\,ds\Big]\rho \\ &= RA\rho \leqslant \|u\|. \end{aligned}$$

Then $Tu(t) \leqslant \|u\|$. Therefore

$$\|Tu\| \leqslant \|u\| \text{ for } u \in K \cap \partial\Omega_3.$$

By Theorem 2, we have $i(T, K_\rho, K) = 1$. Hence

$$i(T, K_{H_2} \setminus \overline{K}_\rho, K) = i(T, K_{H_2}, K) - i(T, K_\rho, K) = -1.$$
$$i(T, K_\rho \setminus \overline{K}_{H_1}, K) = i(T, K_\rho, K) - i(T, K_{H_1}, K) = +1.$$

Therefore, there exist at least two positive solutions $u_1 \in K \cap (\overline{\Omega_3} \setminus \Omega_1)$ and $u_2 \in K \cap (\overline{\Omega_2} \setminus \Omega_3)$ of (1.1) and (1.2) in K, such that

$$0 < \|u_1\| < \rho < \|u_2\|. \quad \blacksquare$$

THEOREM 8. *Assume that* (b) *holds,* $f^0 = 0$ *and* $f^\infty = 0$. *Then the problem* (1.1) *and* $(1, 2)$ *has at least two positive solutions.*

Proof. Since $f^0 = 0$ there exists $H_1 > 0$ such that $f(t, u) \leq \epsilon u$, for all $t \in [0, 1]$ where $0 < u \leq H_1$ and $\epsilon > 0$. Then for $u \in K \cap \partial\Omega_1$ with $\Omega_1 = \{u \in X : \|u\| < H_1\}$, we have

$$\begin{aligned}
Tu(t) &= \int_0^1 G(t,s)a(s)f(s,u(s))\,ds + \frac{\alpha t + \beta}{1 - \alpha\eta - \beta}\int_0^1 G(\eta,s)a(s)f(s,u(s))\,ds \\
&\leq \int_0^1 G(1,s)a(s)f(s,u(s))\,ds + \frac{\alpha + \beta}{1 - \alpha\eta - \beta}\int_0^1 G(\eta,s)a(s)f(s,u(s))\,ds \\
&\leq \int_0^1 \frac{1}{2}s(1-s)a(s)\epsilon u(s)\,ds + \frac{\alpha + \beta}{1 - \alpha\eta - \beta}\int_0^1 G(\eta,s)a(s)\epsilon u(s)\,ds \\
&\leq \epsilon\left[\int_0^1 G(1,s)a(s)\,ds + \frac{\alpha + \beta}{1 - \alpha\eta - \beta}\int_0^1 G(\eta,s)a(s)\,ds\right]\|u\|.
\end{aligned}$$

If $\epsilon A \leq 1$, then $Tu(t) \leq \|u\|$. Therefore

$$\|Tu\| \leq \|u\| \text{ for } u \in K \cap \partial\Omega_1.$$

By Theorem 1, we have

$$i(T, K_{H_1}, K) = 1.$$

Since $f^\infty = 0$, there exists $\widehat{H}_2 > 0$ such that $f(t, u) \leq \epsilon u$, for all $t \in [0, 1]$ where $u \geq \widehat{H}_2$ and $\epsilon > 0$.

We consider two cases:

CASE 1. Suppose f is bounded. Let L be such that $f(t, u) \leq L$ and $\Omega_2 = \{u \in X : \|u\| < H_2\}$ with $H_2 = \max\{2H_1, LA\}$. Then for $u \in K \cap \partial\Omega_2$, we have

$$\begin{aligned}
Tu(t) &= \int_0^1 G(t,s)a(s)f(s,u(s))\,ds + \frac{\alpha t + \beta}{1 - \alpha\eta - \beta}\int_0^1 G(\eta,s)a(s)f(s,u(s))\,ds \\
&\leq \int_0^1 \frac{1}{2}s(1-s)a(s)f(s,u(s))\,ds + \frac{\alpha + \beta}{1 - \alpha\eta - \beta}\int_0^1 G(\eta,s)a(s)f(s,u(s))\,ds
\end{aligned}$$

$$\begin{aligned}
&\le \int_0^1 \frac{1}{2}s(1-s)a(s)L\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)L\,ds\\
&= L\left[\int_0^1 G(1,s)a(s)\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)\,ds\right]\\
&\le H_2 = \|u\|.
\end{aligned}$$

Then $\|Tu\| \le \|u\|$.

Case 2. Suppose f is unbounded. Then from $(J1)$, there exists $H_2 > \max\{2H_1, \widehat{H}_2\}$ such that $f(t,u) \le f(t,H_2)$ with $0 < u \le H_2$. Let $\Omega_2 = \{u \in X : \|u\| < H_2\}$. Then for $u \in K \cap \partial\Omega_2$, we have

$$\begin{aligned}
Tu(t) &= \int_0^1 G(t,s)a(s)f(s,u(s))\,ds + \frac{\alpha t+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)f(s,u(s))\,ds\\
&\le \int_0^1 \frac{1}{2}s(1-s)a(s)f(s,H_2)\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)f(s,H_2)\,ds\\
&\le \int_0^1 \frac{1}{2}s(1-s)a(s)\epsilon H_2\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)\epsilon H_2\,ds\\
&= \epsilon H_2\left[\int_0^1 G(1,s)a(s)\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)\,ds\right]
\end{aligned}$$

If $\epsilon A \le 1$ then $Tu(t) \le H_2 = \|u\|$, and $\|Tu\| \le \|u\|$. Therefore

$$\|Tu\| \le \|u\| \text{ for } u \in K \cap \partial\Omega_2.$$

By Theorem 2, we have $i(T, K_{H_2}, K) = 1$.

On the other hand, let $\Omega_3 = \{u \in X : \|u\| < \rho\}$. Then for $u \in K \cap \partial\Omega_3$ such that $f(t,u) > r\rho$ for all $t \in [\tau,1]$ we get

$$\begin{aligned}
Tu(1) &= \int_0^1 G(1,s)a(s)f(s,u(s))\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_0^1 G(\eta,s)a(s)f(s,u(s))\,ds\\
&> \int_\tau^1 G(1,s)a(s)r\rho\,ds + \frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_\tau^1 G(\eta,s)a(s)r\rho\,ds\\
&> r\rho\left[\int_\tau^1 \tau^2 G(1,s)a(s)\,ds + \tau^2\frac{\alpha+\beta}{1-\alpha\eta-\beta}\int_\tau^1 G(\eta,s)a(s)\,ds\right]\|u\|\\
&= r\rho B\|u\|.
\end{aligned}$$

If $\rho = 1$ then $Tu(1) \ge \|u\|$. Therefore

$$\|Tu\| > \|u\| \ for\ u \in K \cap \partial\Omega_3.$$

By Theorem 2, we have $i(T, K_\rho, K) = 0$. Hence

$$\begin{aligned}
i(T, K_{H_2} \setminus \overline{K_\rho}, K) &= i(T, K_{H_2}, K) - i(T, K_\rho, K) = 1 - 0 = 1.\\
i(T, K_\rho \setminus \overline{K_{H_1}}, K) &= i(T, K_\rho, K) - i(T, K_{H_1}, K) = 0 - 1 = -1.
\end{aligned}$$

Therefore, there exist at least two positive solutions $u_1 \in K \cap (\overline{\Omega_3} \setminus \Omega_1)$ and $u_2 \in K \cap (\overline{\Omega_2} \setminus \Omega_3)$ of (1.1) and (1.2) in K, such that

$$0 < \|u_1\| < \rho < \|u_2\|. \quad \blacksquare$$

5. Examples

EXAMPLE 1. If $f(t,u) = (u^2+t)e^{-u}$, then the condition of Theorem 3 hold (superlinear case). If $f(t,u) = \alpha t + ch(u)$, $\alpha > 0$, then the conditions of Theorem 4 hold (sublinear case).

EXAMPLE 2. Consider the boundary value problem

$$u'''(t) + \mathrm{ch}(u) = 0, \quad t \in (0,1), \tag{5.1}$$

$$u'(0) = u'(1) = \frac{1}{2}u\Big(\frac{1}{2}\Big), \quad u(0) = \frac{1}{2}u\Big(\frac{1}{2}\Big), \tag{5.2}$$

where $f(t,u) = f(u) = \mathrm{ch}(u)$, $a(t) = 1$. Then $f_0 = \infty$ and $f_\infty = \infty$. By simple calculation, $A = \frac{1}{4}$ and $R = 4$.

On the other hand, we could chose $\rho = 1$. Then $f(t,u) \le \frac{1}{2}(e + e^{-1}) \le 4 = \rho R$, for $(t,u) \in [0,1] \times [0,\rho]$. By Theorem 7, (5.1) and (5.2) have at least two positive solutions.

EXAMPLE 3. Consider the boundary value problem

$$u'''(t) + ue^u \ln(t+u) = 0, \quad t \in (0,1), \tag{5.3}$$

$$u'(0) = u'(1) = \frac{1}{2}u\Big(\frac{1}{2}\Big), \quad u(0) = \frac{1}{4}u\Big(\frac{1}{2}\Big), \tag{5.4}$$

where $f(t,u) = f(u) = ue^u \ln(t+u)$, $a(t) = 1$. Then $f^0 = 0$ and $f_\infty = \infty$. By simple calculation $A = \frac{7}{48}$, $R = \frac{48}{7}$, $B = \frac{40\sigma^5+3\sigma^4+14\sigma^2}{96}$ and $r = \frac{96}{40\sigma^5+3\sigma^4+14\sigma^2}$. Therefore $0 \le f^0 = 0 < \frac{48}{7} = R$ and $r < f_\infty \le \infty$. By Theorem 3, (5.3) and (5.4) have at least one positive solution.

REFERENCES

[1] R.P. Agarwal, M. Bohner, P.J.Y. Wong, *Positive solutions and eigenvalues of conjugate boundary value problems*, Proc. Edinburgh Math. Soc. **42** (1999), 349–374.

[2] A. Cabada, *The method of lower and upper solutions for second, third, fourth, and higher order boundary value problems*, J. Math. Anal. Appl. **185** (1994), 302–320.

[3] A. Cabada, *The method of lower and upper solutions for third-order periodic boundary value problems*, J. Math. Anal. Appl. **195** (1995), 568–589.

[4] A. Cabada, M.R. Grossinbo, F. Minhos, *On the solvability of some discontinuous third order nonlinear differential equations with two point boundary conditions*, J. Math. Anal. Appl. **285** (2003), 174–190.

[5] A. Cabada, S. Heikkilä, *Extremality and comparison results for discontinuous third order functional initial-boundary value problems*, J. Math. Anal. Appl. **255** (2001), 195–212.

[6] A. Cabada, S. Heikkilä, *Uniqueness, comparison and existence results for third order initial-boundary value problems*, Comput. Math. Appl. **41** (2001), 607–618.

[7] A. Cabada, S. Lois, *Existence of solution for discontinuous third order boundary value problems*, J. Comput. Appl.Math. **110** (1999), 105–114.

[8] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.

[9] M. Gregus, *Third Order Linear Differential Equations*, Math. Appl., Reidel, Dordrecht, 1987.

[10] M.R. Grossinho, F. Minhös, *Existence result for some third order separated boundary value problems*, Nonlinear Anal. **47** (2001), 2407–2418.

[**11**] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.

[**12**] S. Li, *Positive solutions of nonlinear singular third-order two-point boundary value problems*, J. Math. Anal. Appl. **323** (2006), 413–425.

[**13**] Z. Liu, L. Debnath, S. Kang, *Existence of Positive solutions to a third-order two-point generalized right focal boundary value problems*, Comput. Math. Appl. **55** (2008), 356–367.

[**14**] S. Luan, H Su, Q Sun, *Positive solutions to third-order two-point semipositive boundary value problems*, Int. J. Math. Anal. **3** (2009), 99–108.

[**15**] P. Omari, M. Trombetta, *Remarks on the lower and upper solutions method for second- and third-order periodic boundary value problems*, Appl. Math. Comput. **50** (1992), 1–21.

[**16**] I. Rachunkova, *On some three-point problems for third-order differential equations*, Math. Bohem. **117** (1992), 98–110.

[**17**] Y. Sun, *Positive solutions for third-order three-point nonhomogeneous boundary value problem*, Appl. Math. Lett., **22** (2009), 45–51.

[**18**] F.J. Torres, *Positive solution for a third-order three point boundary value problem*, Electronic J. Diff. Equa. **147** (2013), 1–11.

[**19**] Q. Yao, *The existence and multiplicity of positive solutions for a third-order three-point boundary value problem*, Acta Math. Appl. Sinica **19** (1) (2003), 117–122.

[**20**] W. Zhao, *Existence and uniqueness of solutions for third order nonlinear boundary value problems*, Tohoku Math. J. **44** (2) (1992), 545–555.

(received 15.01.2015; in revised form 28.08.2015; available online 10.10.2015)

A.R.: Univ. Souk Ahras Fact sci Dep MI 41000 Souk Ahras Algeria

E-mail: ali_rezaig@yahoo.fr

S.K.: Department of Mathematics, Faculty of Sciences, University of Annaba, P.O. Box12 Annaba, Algerie

E-mail: kelaiaiasmail@yahoo.fr