МАТЕМАТИЧКИ ВЕСНИК
67, 3 (2015), 185–193
September 2015

оригинални научни рад
research paper

FABER POLYNOMIAL COEFFICIENT ESTIMATES FOR A SUBCLASS OF ANALYTIC BI-UNIVALENT FUNCTIONS DEFINED BY SĂLĂGEAN DIFFERENTIAL OPERATOR

Serap Bulut

Abstract. In this work, considering a subclass of analytic bi-univalent functions defined by Sălăgean differential operator, we determine estimates for the general Taylor-Maclaurin coefficients of the functions in this class. For this purpose, we use the Faber polynomial expansions. In certain cases, our estimates improve some of existing coefficient bounds.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

For $f \in \mathcal{A}$, Sălăgean [14] introduced the following operator:

$$\mathcal{D}^0 f(z) = f(z), \tag{1.2}$$

$$\mathcal{D}^1 f(z) = zf'(z) =: \mathcal{D}f(z), \tag{1.3}$$

$$\mathcal{D}^j f(z) = \mathcal{D}(\mathcal{D}^{j-1} f(z)), \quad (j \in \mathbb{N} := \{1, 2, 3, \dots\}). \tag{1.4}$$

If f is given by (1.1), then from (1.3) and (1.4) we see that

$$\mathcal{D}^j f(z) = z + \sum_{n=2}^{\infty} n^j a_n z^n, \quad (j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \tag{1.5}$$

with $\mathcal{D}^j f(0) = 0$.

2010 Mathematics Subject Classification: 30C45, 30C80

Keywords and phrases: Bi-univalent functions; Taylor-Maclaurin series expansion; Coefficient bounds and coefficient estimates; Faber polynomials; Sălăgean differential operator.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by $f^{-1}(f(z)) = z$ $(z \in \mathbb{U})$ and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function $g = f^{-1}$ is given by

$$\begin{aligned} g(w) &= f^{-1}(w) \\ &= w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots \\ &=: w + \sum_{n=2}^{\infty} A_n w^n. \end{aligned} \tag{1.6}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). The class of analytic bi-univalent functions was first introduced and studied by Lewin [11], where it was proved that $|a_2| < 1.51$. Brannan and Clunie [3] improved Lewin's result to $|a_2| \leq \sqrt{2}$ and later Netanyahu [20] proved that $|a_2| \leq 4/3$. Brannan and Taha [4] and Taha [16] also investigated certain subclasses of bi-univalent functions and found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. For a brief history and interesting examples of functions in the class Σ, see [15] (see also [4]). In fact, the aforecited work of Srivastava et al. [15] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Frasin and Aouf [6], Xu et al. [18,19], Hayami and Owa [8], Porwal and Darus [13] and others.

Not much is known about the bounds on the general coefficient $|a_n|$ for $n > 3$. This is because the bi-univalency requirement makes the behavior of the coefficients of the function f and f^{-1} unpredictable. Here, in this paper, we use the Faber polynomial expansions for a subclass of analytic bi-univalent functions to determine estimates for the general coefficient bounds $|a_n|$.

In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions given by (1.1) using Faber polynomial expansions, [7,9,10].

2. The class $\mathcal{H}_{\Sigma}(j, \alpha, \lambda)$

Firstly, we consider the class of analytic bi-univalent functions defined by Porwal and Darus [13].

Definition 1. [13] For $\lambda \geq 1$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{H}_{\Sigma}(j, \alpha, \lambda)$ if the following conditions are satisfied:

$$\operatorname{Re}\left((1-\lambda)\frac{\mathcal{D}^j f(z)}{z} + \lambda\left(\mathcal{D}^j f(z)\right)'\right) > \alpha \tag{2.1}$$

and

$$\operatorname{Re}\left((1-\lambda)\frac{\mathcal{D}^j g(w)}{w} + \lambda\left(\mathcal{D}^j g(w)\right)'\right) > \alpha, \tag{2.2}$$

where $0 \leq \alpha < 1$; $j \in \mathbb{N}_0$; $z, w \in \mathbb{U}$; $g = f^{-1}$ is defined by (1.6); and $\mathcal{D}^j$ is the Sălăgean differential operator.

Remark 1. In the following special cases of Definition 1, we show how the class of analytic bi-univalent functions $\mathcal{H}_\Sigma(j,\alpha,\lambda)$ for suitable choices of j and λ lead to certain new as well as known classes of analytic bi-univalent functions studied earlier in the literature.

(i) For $j = 0$, we obtain the bi-univalent function class

$$\mathcal{H}_\Sigma(0,\alpha,\lambda) = \mathcal{B}_\Sigma(\alpha,\lambda)$$

introduced by Frasin and Aouf [6]. This class consists of functions $f \in \Sigma$ satisfying

$$\mathrm{Re}\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right) > \alpha$$

and

$$\mathrm{Re}\left((1-\lambda)\frac{g(w)}{w} + \lambda g'(w)\right) > \alpha$$

where $0 \leq \alpha < 1$; $\lambda \geq 1$; $z, w \in \mathbb{U}$ and $g = f^{-1}$ is defined by (1.6).

(ii) For $j = 0$ and $\lambda = 1$, we have the bi-univalent function class

$$\mathcal{H}_\Sigma(0,\alpha,1) = \mathcal{H}_\Sigma(\alpha)$$

introduced by Srivastava et al. [15]. This class consists of functions $f \in \Sigma$ satisfying $\mathrm{Re}(f'(z)) > \alpha$ and $\mathrm{Re}(g'(w)) > \alpha$, where $0 \leq \alpha < 1$; $z, w \in \mathbb{U}$ and $g = f^{-1}$ is defined by (1.6).

3. Coefficient estimates

Using the Faber polynomial expansion of functions $f \in \mathcal{A}$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as [1]:

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \dots) w^n, \tag{3.1}$$

where

$$\begin{aligned} K_{n-1}^{-n} = {} & \frac{(-n)!}{(-2n+1)!\,(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!\,(n-3)!} a_2^{n-3} a_3 \\ & + \frac{(-n)!}{(-2n+3)!\,(n-4)!} a_2^{n-4} a_4 \\ & + \frac{(-n)!}{(2(-n+2))!\,(n-5)!} a_2^{n-5}\left[a_5 + (-n+2)\,a_3^2\right] \\ & + \frac{(-n)!}{(-2n+5)!\,(n-6)!} a_2^{n-6}\left[a_6 + (-2n+5)\,a_3 a_4\right] + \sum_{k\geq 7} a_2^{n-k} V_k, \end{aligned} \tag{3.2}$$

such that V_k with $7 \leq k \leq n$ is a homogeneous polynomial in the variables $a_2, a_3, \ldots, a_n$, [2]. In particular, the first three terms of K_{n-1}^{-n} are

$$\begin{aligned} K_1^{-2} &= -2a_2, \\ K_2^{-3} &= 3\left(2a_2^2 - a_3\right), \\ K_3^{-4} &= -4\left(5a_2^3 - 5a_2a_3 + a_4\right). \end{aligned} \tag{3.3}$$

In general, for any $p \in \mathbb{N}$, an expansion of K_n^p is as, [1],

$$K_n^p = pa_n + \frac{p(p-1)}{2}D_n^2 + \frac{p!}{(p-3)!\,3!}D_n^3 + \cdots + \frac{p!}{(p-n)!\,n!}D_n^n, \tag{3.4}$$

where $D_n^p = D_n^p(a_2, a_3, \ldots)$, and by [17],

$$D_n^m(a_1, a_2, \ldots, a_n) = \sum_{n=1}^{\infty} \frac{m!}{i_1! \ldots i_n!} a_1^{i_1} \ldots a_n^{i_n}$$

while $a_1 = 1$, and the sum is taken over all non-negative integers $i_1, \ldots, i_n$ satisfying

$$i_1 + i_2 + \cdots + i_n = m, \quad i_1 + 2i_2 + \cdots + ni_n = n.$$

It is clear that $D_n^n(a_1, a_2, \ldots, a_n) = a_1^n$.

Consequently, for functions $f \in \mathcal{H}_\Sigma(j, \alpha, \lambda)$ of the form (1.1), we can write:

$$(1-\lambda)\frac{\mathcal{D}^j f(z)}{z} + \lambda\left(\mathcal{D}^j f(z)\right)' = 1 + \sum_{n=2}^{\infty} F_{n-1}(a_2, a_3, \ldots, a_n) z^{n-1}, \tag{3.5}$$

where

$$F_1 = (1+\lambda)2^j a_2, \quad F_2 = (1+2\lambda)3^j a_3, \quad F_3 = (1+3\lambda)4^j a_4.$$

In general, we have

$$F_{n-1}(a_2, a_3, \ldots, a_n) = [1 + (n-1)\lambda]\, n^j a_n. \tag{3.6}$$

Our first theorem introduces an upper bound for the coefficients $|a_n|$ of analytic bi-univalent functions in the class $\mathcal{H}_\Sigma(j, \alpha, \lambda)$.

Theorem 1. *For $\lambda \geq 1$, $0 \leq \alpha < 1$ and $j \in \mathbb{N}_0$, let the function $f \in \mathcal{H}_\Sigma(j, \alpha, \lambda)$ be given by* (1.1). *If $a_k = 0$ $(2 \leq k \leq n-1)$, then*

$$|a_n| \leq \frac{2(1-\alpha)}{[1 + (n-1)\lambda]\, n^j} \qquad (n \geq 4).$$

Proof. For the function $f \in \mathcal{H}_\Sigma(j, \alpha, \lambda)$ of the form (1.1), we have the expansion (3.5) and for the inverse map $g = f^{-1}$, considering (1.6), we obtain

$$(1-\lambda)\frac{\mathcal{D}^j g(w)}{w} + \lambda\left(\mathcal{D}^j g(w)\right)' = 1 + \sum_{n=2}^{\infty} F_{n-1}(A_2, A_3, \ldots, A_n) w^{n-1}, \tag{3.7}$$

with

$$A_n = \frac{1}{n} K_{n-1}^{-n} \left(a_2, a_3, \dots, a_n\right). \tag{3.8}$$

On the other hand, since $f \in \mathcal{H}_\Sigma (j, \alpha, \lambda)$ and $g = f^{-1} \in \mathcal{H}_\Sigma (j, \alpha, \lambda)$, by definition, there exist two positive real part functions

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{A}$$

and

$$q(w) = 1 + \sum_{n=1}^{\infty} d_n w^n \in \mathcal{A},$$

where $\operatorname{Re}\{p(z)\} > 0$ and $\operatorname{Re}\{q(w)\} > 0$ in $\mathbb{U}$ so that

$$\begin{aligned}(1-\lambda)\frac{\mathcal{D}^j f(z)}{z} + \lambda \left(\mathcal{D}^j f(z)\right)' &= \alpha + (1-\alpha) p(z) \\ &= 1 + (1-\alpha) \sum_{n=1}^{\infty} K_n^1 \left(c_1, c_2, \dots, c_n\right) z^n \end{aligned} \tag{3.9}$$

and

$$\begin{aligned}(1-\lambda)\frac{\mathcal{D}^j g(w)}{w} + \lambda \left(\mathcal{D}^j g(w)\right)' &= \alpha + (1-\alpha) q(w) \\ &= 1 + (1-\alpha) \sum_{n=1}^{\infty} K_n^1 \left(d_1, d_2, \dots, d_n\right) w^n. \end{aligned} \tag{3.10}$$

Note that, by the Caratheodory Lemma (e.g., [5]),

$$|c_n| \leq 2 \qquad \text{and} \qquad |d_n| \leq 2 \quad (n \in \mathbb{N}).$$

Comparing the corresponding coefficients of (3.5) and (3.9), for any $n \geq 2$, yields

$$F_{n-1}\left(a_2, a_3, \dots, a_n\right) = (1-\alpha) K_{n-1}^1 \left(c_1, c_2, \dots, c_{n-1}\right), \tag{3.11}$$

and similarly, from (3.7) and (3.10) we find

$$F_{n-1}\left(A_2, A_3, \dots, A_n\right) = (1-\alpha) K_{n-1}^1 \left(d_1, d_2, \dots, d_{n-1}\right). \tag{3.12}$$

Note that for $a_k = 0$ $(2 \leq k \leq n-1)$, we have $A_n = -a_n$ and so

$$\begin{aligned} [1 + (n-1)\lambda] n^j a_n &= (1-\alpha) c_{n-1}, \\ -[1 + (n-1)\lambda] n^j a_n &= (1-\alpha) d_{n-1}. \end{aligned}$$

Taking the absolute values of the above equalities, we obtain

$$|a_n| = \frac{(1-\alpha)|c_{n-1}|}{[1+(n-1)\lambda] n^j} = \frac{(1-\alpha)|d_{n-1}|}{[1+(n-1)\lambda] n^j} \leq \frac{2(1-\alpha)}{[1+(n-1)\lambda] n^j},$$

which completes the proof of Theorem 1. ■

The following corollary is an immediate consequence of the above theorem.

Corollary 1. [9] *For* $\lambda \geq 1$ *and* $0 \leq \alpha < 1$, *let the function* $f \in \mathcal{B}_{\Sigma}(\alpha, \lambda)$ *be given by* (1.1). *If* $a_k = 0$ $(2 \leq k \leq n-1)$, *then*

$$|a_n| \leq \frac{2(1-\alpha)}{1+(n-1)\lambda} \qquad (n \geq 4).$$

Relaxing the coefficient restrictions imposed in Theorem 1, we see the unpredictable behavior of the initial Taylor-Maclaurin coefficients of functions $f \in \mathcal{H}_{\Sigma}(j, \alpha, \lambda)$ illustrated in the following theorem.

Theorem 2. *For* $\lambda \geq 1$, $0 \leq \alpha < 1$ *and* $j \in \mathbb{N}_0$, *let the function* $f \in \mathcal{H}_{\Sigma}(j, \alpha, \lambda)$ *be given by* (1.1). *Then one has the following*

$$|a_2| \leq \min\left\{ \sqrt{\frac{2(1-\alpha)}{(1+2\lambda)3^j}}, \frac{2(1-\alpha)}{(1+\lambda)2^j} \right\}, \tag{3.13}$$

$$|a_3| \leq \frac{2(1-\alpha)}{(1+2\lambda)3^j} \tag{3.14}$$

$$\left|a_3 - 2a_2^2\right| \leq \frac{2(1-\alpha)}{(1+2\lambda)3^j}.$$

Proof. If we set $n = 2$ and $n = 3$ in (3.11) and (3.12), respectively, we get

$$(1+\lambda)2^j a_2 = (1-\alpha)c_1, \tag{3.15}$$

$$(1+2\lambda)3^j a_3 = (1-\alpha)c_2, \tag{3.16}$$

$$-(1+\lambda)2^j a_2 = (1-\alpha)d_1, \tag{3.17}$$

$$(1+2\lambda)3^j\left(2a_2^2 - a_3\right) = (1-\alpha)d_2. \tag{3.18}$$

From (3.15) and (3.17), we find (by the Caratheodory Lemma)

$$|a_2| = \frac{(1-\alpha)|c_1|}{(1+\lambda)2^j} = \frac{(1-\alpha)|d_1|}{(1+\lambda)2^j} \leq \frac{2(1-\alpha)}{(1+\lambda)2^j}. \tag{3.19}$$

Also from (3.16) and (3.18), we obtain

$$2(1+2\lambda)3^j a_2^2 = (1-\alpha)(c_2+d_2). \tag{3.20}$$

Using the Caratheodory Lemma, we get

$$|a_2| \leq \sqrt{\frac{2(1-\alpha)}{(1+2\lambda)3^j}},$$

and combining this with the inequality (3.19), we obtain the desired estimate on the coefficient $|a_2|$ as asserted in (3.13).

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (3.18) from (3.16). We thus get

$$(1+2\lambda)\,3^j\left(-2a_2^2+2a_3\right)=(1-\alpha)\,(c_2-d_2)$$

or

$$a_3=a_2^2+\frac{(1-\alpha)\,(c_2-d_2)}{2\,(1+2\lambda)\,3^j}. \tag{3.21}$$

Upon substituting the value of a_2^2 from (3.15) into (3.21), it follows that

$$a_3=\frac{(1-\alpha)^2\,c_1^2}{(1+\lambda)^2\,2^{2j}}+\frac{(1-\alpha)\,(c_2-d_2)}{2\,(1+2\lambda)\,3^j}.$$

We thus find (by the Caratheodory Lemma) that

$$|a_3|\leq\frac{4\,(1-\alpha)^2}{(1+\lambda)^2\,2^{2j}}+\frac{2\,(1-\alpha)}{(1+2\lambda)\,3^j}. \tag{3.22}$$

On the other hand, upon substituting the value of a_2^2 from (3.20) into (3.21), it follows that

$$a_3=\frac{(1-\alpha)\,c_2}{(1+2\lambda)\,3^j}.$$

Consequently, (by the Caratheodory Lemma) we have

$$|a_3|\leq\frac{2\,(1-\alpha)}{(1+2\lambda)\,3^j}. \tag{3.23}$$

Combining (3.22) and (3.23), we get the desired estimate on the coefficient $|a_3|$ as asserted in (3.14).

Finally, from (3.18), we deduce (by the Caratheodory Lemma) that

$$\left|a_3-2a_2^2\right|=\frac{(1-\alpha)\,|d_2|}{(1+2\lambda)\,3^j}\leq\frac{2\,(1-\alpha)}{(1+2\lambda)\,3^j}.$$

This evidently completes the proof of Theorem 2. ■

Remark 2. The above estimates for $|a_2|$ and $|a_3|$ show that Theorem 2 is an improvement of the estimates obtained by Porwal and Darus [13].

Corollary 2. [13] *For $\lambda\geq 1$, $0\leq\alpha<1$ and $j\in\mathbb{N}_0$, let the function $f\in\mathcal{H}_\Sigma(j,\alpha,\lambda)$ be given by* (1.1). *Then one has the following*

$$|a_2|\leq\sqrt{\frac{2\,(1-\alpha)}{(1+2\lambda)\,3^j}}$$

and

$$|a_3|\leq\frac{4\,(1-\alpha)^2}{(1+\lambda)^2\,2^{2j}}+\frac{2\,(1-\alpha)}{(1+2\lambda)\,3^j}.$$

By setting $j = 0$ in Theorem 2, we obtain the following consequence which is an improvement of the estimates obtained by Frasin and Aouf [12].

Corollary 3. [9] *For* $\lambda \geq 1$ *and* $0 \leq \alpha < 1$*, let the function* $f \in \mathcal{B}_\Sigma(\alpha, \lambda)$ *be given by* (1.1)*. Then one has the following*

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\alpha)}{1+2\lambda}}, & 0 \leq \alpha < \frac{1+2\lambda-\lambda^2}{2(1+2\lambda)} \\ \frac{2(1-\alpha)}{1+\lambda}, & \frac{1+2\lambda-\lambda^2}{2(1+2\lambda)} \leq \alpha < 1, \end{cases}$$

$$|a_3| \leq \frac{2(1-\alpha)}{1+2\lambda} \quad \textit{and} \quad \left|a_3 - 2a_2^2\right| \leq \frac{2(1-\alpha)}{1+2\lambda}.$$

By setting $j = 0$ and $\lambda = 1$ in Theorem 2, we obtain the following consequence.

Corollary 4. *For* $0 \leq \alpha < 1$*, let the function* $f \in \mathcal{H}_\Sigma(\alpha)$ *be given by* (1.1)*. Then one has the following*

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\alpha)}{3}}, & 0 \leq \alpha < \frac{1}{3} \\ 1-\alpha, & \frac{1}{3} \leq \alpha < 1 \end{cases}$$

and $|a_3| \leq \dfrac{2(1-\alpha)}{3}$.

Remark 3. The above estimates for $|a_2|$ and $|a_3|$ show that Corollary 4 is an improvement of the estimates obtained by Srivastava et al. [15].

Corollary 5. [15] *For* $0 \leq \alpha < 1$*, let the function* $f \in \mathcal{H}_\Sigma(\alpha)$ *be given by* (1.1)*. Then one has the following*

$$|a_2| \leq \sqrt{\frac{2(1-\alpha)}{3}} \qquad \textit{and} \qquad |a_3| \leq \frac{(1-\alpha)(5-3\alpha)}{3}.$$

REFERENCES

[**1**] H. Airault and A. Bouali, *Differential calculus on the Faber polynomials*, Bull. Sci. Math. **130** (3) (2006), 179–222.

[**2**] H. Airault and J. Ren, *An algebra of differential operators and generating functions on the set of univalent functions*, Bull. Sci. Math. **126** (5) (2002), 343–367.

[**3**] D.A. Brannan, J.G. Clunie (Eds.), *Aspects of Contemporary Complex Analysis* (Proceedings of the NATO Advanced Study Institute held at the University of Durham, Durham; July 1 20, 1979), Academic Press, New York and London, 1980.

[**4**] D.A. Brannan and T.S. Taha, *On some classes of bi-univalent functions*, in Mathematical Analysis and Its Applications (S. M. Mazhar, A. Hamoui and N. S. Faour, Editors) (Kuwait; February 18–21, 1985), KFAS Proceedings Series, Vol. 3, Pergamon Press (Elsevier Science Limited), Oxford, 1988, pp. 53–60; see also Studia Univ. Babeş-Bolyai Math. **31** (2) (1986), 70–77.

[**5**] P.L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, vol. 259, Springer, New York, 1983.

[**6**] B.A. Frasin and M.K. Aouf, *New subclasses of bi-univalent functions*, Appl. Math. Lett. **24** (2011) 1569–1573.

[**7**] S.G. Hamidi and J.M. Jahangiri, *Faber polynomial coefficient estimates for analytic bi-close-to-convex functions*, C. R. Acad. Sci. Paris, Ser. I **352**, 1 (2014), 17–20.

[**8**] T. Hayami and S. Owa, *Coefficient bounds for bi-univalent functions*, Pan Amer. Math. J. **22** (4) (2012), 15–26.

[**9**] J.M. Jahangiri and S.G. Hamidi, *Coefficient estimates for certain classes of bi-univalent functions*, Int. J. Math. Math. Sci. **2013**, Art. ID 190560, 4 pp.

[**10**] J.M. Jahangiri, S.G. Hamidi and S.A. Halim, *Coefficients of bi-univalent functions with positive real part derivatives*, Bull. Malaysian Math. Soc. (2) **37**, 3 (2014), 633–640.

[**11**] M. Lewin, *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc. **18** (1967), 63–68.

[**12**] E. Netanyahu, *The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in* $|z| < 1$, Arch. Rational Mech. Anal. **32** (1969), 100–112.

[**13**] S. Porwal and M. Darus, *On a new subclass of bi-univalent functions*, J. Egyptian Math. Soc. **21** (3) (2013), 190–193.

[**14**] G.Ş. Sălăgean, *Subclasses of univalent functions*, Complex Analysis-Fifth Romanian-Finnish seminar, Part 1 (Bucharest, 1981), Lecture Notes in Math., vol. 1013, Springer, Berlin, 1983, pp. 362–372.

[**15**] H.M. Srivastava, A.K. Mishra and P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett. **23** (2010) 1188–1192.

[**16**] T.S. Taha, *Topics in Univalent Function Theory*, Ph.D. Thesis, University of London, 1981.

[**17**] P.G. Todorov, *On the Faber polynomials of the univalent functions of class* Σ, J. Math. Anal. Appl. **162** (1) (1991), 268–276.

[**18**] Q.-H. Xu, Y.-C. Gui and H.M. Srivastava, *Coefficient estimates for a certain subclass of analytic and bi-univalent functions*, Appl. Math. Lett. **25** (2012) 990–994.

[**19**] Q.-H. Xu, H.-G. Xiao and H.M. Srivastava, *A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems*, Appl. Math. Comput. **218** (2012) 11461–11465.

(received 25.03.2014; in revised form 13.11.2014; available online 23.12.2014)

Kocaeli University, Civil Aviation College, Arslanbey Campus, 41285 Izmit-Kocaeli, Turkey

E-mail: `serap.bulut@kocaeli.edu.tr`