МАТЕМАТИЧКИ ВЕСНИК
67, 1 (2015), 1–16
March 2015

оригинални научни рад
research paper

# CONSTRUCTIONS OF $(m, n)$-HYPERRINGS

**S. Mirvakili and B. Davvaz**

**Abstract.** In this paper, the class of $(m, n)$-hyperrings is introduced and several properties are found and examples are presented. $(m, n)$-hyperrings are a generalization of hyperrings. We define the fundamental relation $\Gamma^*$ on an $(m, n)$-hyperring $R$ such that $R/\Gamma^*$ is the smallest $(m, n)$-ring, and then some related properties are investigated.

## 1. Introduction

The notion of an $n$-ary group is a natural generalization of the notion of a group and has many applications in different branches. The idea of investigations of such groups seems to be going back to E. Kasner's lecture at the fifty-third annual meeting of the American Association for the Advancement of Science in 1904. But the first paper concerning the theory of $n$-ary groups was written (under inspiration of Emmy Noether) by W. Dörnte in 1928 [9] and was studied extensively by many mathematicians, for example see [14]. Such and similar $n$-ary systems have many applications in different branches. For example, in the theory of automata, $n$-ary semigroups and $n$-ary groups are used, some $n$-ary groupoids are applied in the theory of quantum groups. Different applications of ternary structures in physics are described. In Physics, also such structures as $n$-ary Filippov algebras and $n$-Lie algebras are used. In some papers several authors generalize the study of ordinary rings to the case where the ring operations are respectively $m$-ary and $n$-ary, see [4].

Since 1934, when Marty [12] introduced for the first time the notion of a hypergroup, the Hyperstructure Theory had applications to several domains, for instance non Euclidean geometry, graphs and hypergraphs, binary relations, lattices, automata, cryptography, codes, artificial intelligence, probabilities etc. (see [2, 3, 5, 16]). 70 years later, a nice generalization of a hypergroup, called an $n$-hypergroup has been introduced and studied by Davvaz and Vougiouklis (see [7]) and studied by Leoreanu and Davvaz [10, 11] and Davvaz et al. [6]. An $n$-ary hypergroup is a nice generalization of the notion of a group, a hypergroup in the sense of Marty and

*2010 Mathematics Subject Classification*: 16Y99
*Keywords and phrases*: $(m, n)$-hyperring; $n$-ary operation; $n$-ary hyperoperation; $n$-ary hypergroup; fundamental equivalence relation.

an $n$-ary group, too. In [1, 13], Davvaz et al. studied Krasner $(m,n)$-hyperrings. In this paper we study $(m,n)$-hyperrings in a general form.

## 2. Basic definitions

Let $H$ be a non-empty set and $f$ be a mapping $f\colon H\times H\to\mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ denotes the set of all non-empty subsets of $H$. Then, $f$ is called a *binary hyperoperation* on $H$. In general, a mapping $f\colon H\times\cdots\times H\to\mathcal{P}^*(H)$, where $H$ appears $n$ times, is called an *$n$-ary hyperoperation* and $n$ is called the *arity* of this hyperoperation. If $f$ is an $n$-ary hyperoperation defined on $H$, then $(H,f)$ is called an *$n$-ary hypergroupoid*. If for all $x_1^n\in H$ the set $f(x_1^n)$ is a singleton, then $f$ is called an *$n$-ary operation* and $(H,f)$ is called an *$n$-ary groupoid*. Since we identify the set $\{x\}$ with the element $x$, any $n$-ary (binary) groupoid is an $n$-ary (binary) hypergroupoid.

We shall use the following abbreviated notation: the sequence $x_i,x_{i+1},\ldots,x_j$ will be denoted by $x_i^j$. For $j<i$, $x_i^j$ is the empty symbol. In this convention

$$f(x_1,\ldots,x_i,y_{i+1},\ldots,y_j,z_{j+1},\ldots,z_n)$$

will be written as $f(x_1^i,y_{i+1}^j,z_{j+1}^n)$. In the case when $y_{i+1}=\cdots=y_j=y$ the last expression will be write in the form $f(x_1^i,\overset{(j-i)}{y},z_{j+1}^n)$.

Similarly, for non-empty subsets $A_1,\ldots,A_n$ of $H$ we define

$$f(A_1^n)=f(A_1,\ldots,A_n)=\bigcup\{f(x_1^n)\,|\,x_i\in A_i,\ i=1,\ldots n\}.$$

If $m=k(n-1)+1$, then $m$-ary hyperoperation $h$ given by

$$h(x_1^{k(n-1)+1})=\underbrace{f(f(\ldots(f(f}_{k}(x_1^n),x_{n+1}^{2n-1}),\ldots),x_{(k-1)(n-1)+2}^{k(n-1)+1})$$

will be denoted by $f_{(k)}$.

An $n$-ary hyperoperation $f$ is called *associative* if

$$f(x_1^{i-1},f(x_i^{n+i-1}),x_{n+i}^{2n-1})=f(x_1^{j-1},f(x_j^{n+j-1}),x_{n+j}^{2n-1})$$

holds for every $i,j\in\{1,\ldots,n\}$ and all $x_1,x_2,\ldots,x_{2n-1}\in H$. An $n$-ary hypergroupoid with the associative hyperoperation is called an *$n$-ary hypersemigroup*.

An $n$-ary hypersemigroup $(H,f)$ in which the equation

$$b\in f(a_1^{i-1}x_i,a_{i+1}^n)\qquad(*)$$

has a solution $x_i\in H$ for every $a_1^{i-1},a_{i+1}^n,b\in H$ and $1\le i\le n$, is called an *$n$-ary hypergroup*. This condition can be formulated as $f(a_1^{i-1},H,a_{i+1}^n)=H$. If $f$ is an $n$-ary operation and $(H,f)$ is an $n$-ary semigroup, then the equation $(*)$ is as follows:

$$b=f(a_1^{i-1},x_i,a_{i+1}^n),$$

and in this case $(H,f)$ is an $n$-ary group. An $n$-ary hypersemigroup $(H,f)$ is *cancellative*, if

$$f(a_2^i, x, a_{i+1}^n) = f(a_2^i, y, a_{i+1}^n) \quad \text{implies} \quad x = y$$

holds for all $x, y, a_2^n \in H$ and for all $i = 1, 2, \ldots, n$.

An $n$-ary hypergroupoid $(H,f)$ is *commutative* if for all $\sigma \in \mathbb{S}_n$ and for every $a_1^n \in H$ we have

$$f(a_1, \ldots, a_n) = f(a_{\sigma(1)}, \ldots, a_{\sigma(n)}).$$

If $a_1^n \in H$ we denote $a_{\sigma(1)}^{\sigma(n)}$ as $a_{\sigma(1)}, \ldots, a_{\sigma(n)}$.

An element $e$ of $H$ is called a *(scalar) identity element* if

$$(x = f(\underbrace{e, \ldots, e}_{i-1}, x, \underbrace{e, \ldots, e}_{n-i})) \quad x \in f(\underbrace{e, \ldots, e}_{i-1}, x, \underbrace{e, \ldots, e}_{n-i})$$

for all $x \in H$ and all $1 \le i \le n$.

## 3. Strongly compatible relations

Strongly compatible equivalence relations play in $n$-ary hypersystem theory a role analogous to congruences in $n$-ary system theory.

Let $\rho$ be an equivalence relation on an $n$-ary hypersemigroup $(H,f)$. We denote by $\overline{\overline{\rho}}$ the relation defined on $\mathcal{P}^*(H)$ as follows. If $A, B \in \mathcal{P}^*(H)$, then

$$A \mathrel{\overline{\overline{\rho}}} B \iff a \mathrel{\rho} b \ \text{ for all } \ a \in A, b \in B.$$

It follows immediately that $\overline{\overline{\rho}}$ is symmetric and transitive. In general, $\overline{\overline{\rho}}$ is not reflexive. Indeed, let us take, for example, the equality relation on $A$, denoted here by $\delta_A$. The relation $\overline{\overline{\delta_A}}$ is reflexive if and only if $|A| = 1$.

Definition 3.1. Let $(H,f)$ be an $n$-ary hypersemigroup and $\rho$ be an equivalence relation on $H$. Then, $\rho$ is a *strongly compatible relation* if

$$a_i \mathrel{\rho} b_i \quad \text{for all} \quad 1 \le i \le n \text{ then, } f(a_1, \ldots, a_n) \mathrel{\overline{\overline{\rho}}} f(b_1, \ldots, b_n).$$

Theorem 3.2. *Let $(H,f)$ be an $n$-ary hypersemigroup and let $\rho$ be an equivalence relation on $H$. The following conditions are equivalent.*

(1) *The relation $\rho$ is strongly compatible.*

(2) *If $x_1^n, a, b \in H$ and $a \mathrel{\rho} b$ then for every $i \in \{1, \ldots, n\}$ we have*

$$f(x_1^{i-1}, a, x_{i+1}^n) \mathrel{\overline{\overline{\rho}}} f(x_1^{i-1}, b, x_{i+1}^n).$$

(3) *The quotient $(H/\rho, f/\rho)$ is an $n$-ary semigroup.*

By Theorem 3.2, if $\rho$ is a strongly compatible relation on an $n$-ary hypersemigroup $(H,f)$ then the quotient $(H/\rho, f/\rho)$ is an $n$-ary semigroup such that

$$f/\rho(\rho(a_1), \ldots, \rho(a_n)) = \rho(x) \ \text{ for all } \ x \in f(a_1, \ldots, a_n)$$

where $a_1, \dots a_n \in H$. If $(H, f)$ is an $n$-ary hypergroup and $\rho$ is a strongly compatible relation then the quotient $(H/\rho, f/\rho)$ is an $n$-ary group. Also, by Theorem 3.2, we obtain:

THEOREM 3.3. [7] *Let $(H, f)$ be an $n$-ary hypergroup and let $\rho$ be an equivalence relation on $H$. Then, the relation $\rho$ is strongly compatible if and only if the quotient $(H/\rho, f/\rho)$ is an $n$-ary group.*

Davvaz and Vougiouklis showed in [7] that the relation $\beta_f^*$ on an $n$-ary hypersemigroup $(H, f)$ is the transitive closure of the relation $\beta_f = \bigcup_{k\geq 1} \beta_k$, where $\beta_1$ is the diagonal relation and, for every integer $k > 1, \beta_k$ is the relation defined as follows:

$$x \ \beta_k \ y \iff \exists z_1^m \in H : \{x, y\} \subseteq f_{(k)}(z_1^n), \text{ where } \ m = k(n-1) + 1.$$

It is well known that $\beta_f^*$ is the smallest strongly compatible equivalence relation on an $n$-ary hypersemigroup $(H, f)$. Leoreanu and Davvaz [11] showed that the relation $\beta_f$ is transitive. The relation $\beta_f^*$ on an $n$-ary hypersemigroup (hypergroup) is called the *fundamental relation* and $(H/\beta_f^*, f/\beta_f^*)$ is called *fundamental $n$-ary semigroup (group)*. Thus, we have

THEOREM 3.4. *Let $(H, f)$ be an $n$-ary hypersemigroup. Then,*

(1) *$(H/\beta_f^*, f/\beta_f^*)$ is an $n$-ary semigroup.*

(2) *If $(H, f)$ is an $n$-ary hypergroup, then $(H/\beta_f^*, f/\beta_f^*)$ is an $n$-ary group and the relation $\beta_f$ is an equivalence relation.*

## 4. $(m, n)$-hyperrings

A recent book [5] is devoted especially to the study of hyperring theory. It begins with some basic results concerning ring theory and algebraic hyperstructures, which represent the most general algebraic context, in which the reality can be modeled. Several kinds of hyperrings are introduced and analyzed in the following chapters: Krasner hyperrings, multiplicative hyperrings, general hyperrings. Now, in this section the class of $(m, n)$-hyperrings is introduced and several properties are found and examples are presented.

DEFINITION 4.1. An *$(m, n)$-hyperring* is an algebraic hyperstructure $(R, f, g)$, which satisfies the following axioms:

(1) $(R, f)$ is an $m$-ary hypergroup,

(2) $(R, g)$ is an $n$-ary hypersemigroup,

(3) the $n$-ary hyperoperation $g$ is distributive with respect to the $m$-ary hyperoperation $f$, i.e., for every $a_1^{i-1}, a_{i+1}^n, x_1^m \in R, \ 1 \leq i \leq n$,

$$g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), \dots, g(a_1^{i-1}, x_m, a_{i+1}^n)).$$

$(R, f, g)$ is called an *$n$-ary hyperring* if $n = m$.

If $(R,f)$ is an $m$-ary hypersemigroup, then $(R,f,g)$ is called an $(m,n)$-*hypersemiring*.

In $(m,n)$-hyperring $(R,f,g)$, if $f$ is an $m$-ary operation then $(R,f,g)$ is called an $(m,n)$-multiplicative hyperring and if $g$ be an $n$-ary operation then $(R,f,g)$ is called an additive $(m,n)$-hyperring. A multiplicative and additive $(m,n)$-hyperring is called an $(m,n)$-*ring*.

A non-empty subset $S\subseteq R$ is called an $(m,n)$-*subhyperring* if $(S,f,g)$ is an $(m,n)$-hyperring.

Let $i\in\{1,\dots,n\}$. An $i$-*hyperideal* $I$ of $R$ is an $(m,n)$-subhyperring of $R$ such that for every $r_1^n\in R$, $g(r_1^{i-1},I,r_{i+1}^n)\subseteq I$.

If $I$ is an $i$-hyperideal and for every $r_1^n\in R$, $g(r_1^{i-1},I,r_{i+1}^n)=I$, then $I$ called a *strong* $i$-*hyperideal*.

A non-empty subset $I$ of $R$ is called (a) a *(strong)* $(m,n)$-*hyperideal* if $I$ is (a) an (strong) $i$-hyperideal of $R$ for every $i\in\{1,\dots,n\}$.

LEMMA 4.2. *For any $(m,n)$-hyperring $(R,f,g)$ and $I\subseteq R$ the following conditions are equivalent:*

(1) *$I$ is a strong $(m,n)$-hyperideal of $R$.*

(2) *$I$ is a strong $i$-hyperideal of $R$ for $i=1$ and $i=n$.*

(3) *$I$ is a strong $i$-hyperideal of $R$ for some $1<i<n$.*

*Proof.* (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) Let $1<i<n$ and $r_1^n\in R$. Then

$$\begin{aligned}g(r_1^{i-1},I,r_1^n)&=g(r_1^{i-1},g(\overset{(n)}{I}),r_{i+1}^n)=g(g(r_1^{i-1},\overset{(n-i+1)}{I}),\overset{(i-1)}{I},r_{i+1}^n)\\&=g(\overset{(i)}{I},r_{i+1}^n)=I.\end{aligned}$$

(3) $\Rightarrow$ (1) Let for some $1<i<n$ and for every $r_1^n\in R$, $g(r_1^{i-1},I,r_{i+1}^n)=I$. Thus, for every $s_2^n\in R$ we have:

$$\begin{aligned}g(I,s_2^n)&=g(g(\overset{(n)}{I}),s_2^n)=g(\overset{(n-i-1)}{I},g(\overset{(i+1)}{I},s_2^{n-i}),s_{n-i+1}^n)\\&=g(\overset{(n-i)}{I},r_{n-i+1}^n).\end{aligned}$$

If we repeat the above process, then we obtain $g(I,s_2^n)=I$ and so $I$ is a strong 1-hyperideal. By the similar way, $I$ is a strong $n$-hyperideal of $R$. Thus, we conclude that $I$ is a strong $i$-hyperideal for every $1\le i\le n$ and so $I$ is a strong $(m,n)$-hyperideal. ■

An element $o$ is called a *(scalar) zero* of $(R,f,g)$ if it is a (scalar) identity of $(R,f)$ and for every $x_2^n\in R$ we have

$$\begin{gathered}(o=f(o,x_2^n)=f(x_2,o,x_3^n)=\cdots=f(x_2^n,o))\\ o\in f(o,x_2^n)\cap f(x_2,o,x_3^n)\cap\cdots\cap f(x_2^n,o).\end{gathered}$$

EXAMPLE 1. Let $(G,+)$ be a commutative group of the exponent $n-1$ (for example $\mathbb{Z}_{n-1}$) and $H$ be a subgroup of $G$. Then, we have $f(x_1^n)=\sum_{i=1}^n x_i+H$. $f$ is an associative $n$-ary hyperoperation and $(G,f)$ is a commutative $n$-ary hypergroup. Also, $(G,f,f)$ is an $n$-hyperring.

EXAMPLE 2. Let $R=\{a,b,c\}$ and $f$ and $g$ be defined by the following tables:

| $f$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $\{a,b\}$ | $\{a,b\}$ | $c$ |
| $b$ | $\{a,b\}$ | $\{a,b\}$ | $c$ |
| $c$ | $c$ | $c$ | $\{a,b\}$ |

| $g$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $\{a,b\}$ | $\{a,b\}$ | $\{a,b\}$ |
| $b$ | $\{a,b\}$ | $\{a,b\}$ | $\{a,b\}$ |
| $c$ | $\{a,b\}$ | $\{a,b\}$ | $c$ |

Then $(R,f,g)$ is a $(2,2)$-hyperring and $a,b$ are two zero of $R$.

LEMMA 4.3. *For any $(m,n)$-hypersemiring $(R,f,g)$ and $o\in R$ the following conditions are equivalent:*

(1) *$o$ is a scalar zero of $R$,*

(2) *$o$ is a scalar $i$-zero for some $1<i<n$, i.e., for every $x_1^n\in R$, $g(x_1^{i-1},o,x_{i+1}^n)=o$,*

(3) *$o$ is a scalar $i$-zero for $i=1$ and $i=n$, i.e., for every $x_1^n\in R$,*

$$g(o,x_2^n)=o=g(x_1^{n-1},o)$$

*Proof.* (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3) Let $x_2^n\in R$, we have

$$\begin{aligned} g(o,x_2^n) &= g(g(\overset{(n)}{o}),x_1^n)=g(\overset{(n-i)}{o},g(\overset{(i)}{o},x_2^{n-i+1}),x_{n-i+2}^n) \\ &= g(\overset{(n-i+1)}{o},x_{n-i+2}^n). \end{aligned}$$

If we repeat the above process, then we obtain $g(o,x_2^n)=o$. In a similar way for every $x_1^n\in R$ we have $g(x_1^{n-1},o)=o$.

(3) $\Rightarrow$ (1) Let $1<i<n$ and $x_1^n\in R$. Then, we have

$$\begin{aligned} g(x_1^{i-1},o,x_{i+1}^n) &= g(x_1^{i-1},g(\overset{(n)}{o}),x_{i+1}^n)=g(g(x_1^{i-1},\overset{(n-i+1)}{o}),\overset{(i-1)}{o},x_{i+1}^n) \\ &= g(\overset{(i)}{o},x_{i+1}^n)=o. \quad \blacksquare \end{aligned}$$

DEFINITION 4.4. Let $(R_1,f_1,g_1)$ and $(R_2,f_2,g_2)$ be two $(m,n)$-hyperrings. A *homomorphism* from $R_1$ to $R_2$ is a mapping $\phi:R_1\to R_2$ such that

$$\phi(f_1(a_1^m))=f_2(\phi(a_1),\dots,\phi(a_m)) \quad \text{and} \quad \phi(g_1(b_1^n))=g_2(\phi(b_1),\dots,\phi(b_n))$$

hold for all $a_1^n,b_1^m\in R_1$. If $\phi$ is injective, then it is called an *embedding*. The map $\phi$ is an *isomorphism* if $\phi$ is injective and onto. We say that $R_1$ is *isomorphic* to $R_2$, denoted by $R_1\cong R_2$, if there is an isomorphism from $R_1$ to $R_2$. Let $\phi:R_1\to R_2$ be a homomorphism and $S_1$ be an $(m,n)$-subhyperring of $R_1$ and $S_2$ be an $(m,n)$-subhyperring of $R_2$, then $\phi(S_1)$ is an $(m,n)$-subhyperring of $R_2$ and if $\phi^{-1}(S_2)$ is non-empty, then $\phi^{-1}(S_2)$ is an $(m,n)$-subhyperring of $R_1$.

Let $\phi : R_1 \to R_2$ be a homomorphism. Then the kernel $\phi$, is defined by

$$\ker \phi = \{(a, b) \in R_1 \times R_1 \mid \phi(a) = \phi(b)\}.$$

EXAMPLE 3. Every hyperring is a $(2, 2)$-hyperring and every hypergroup is a $(2, 0)$-hyperring.

EXAMPLE 4. Let $(G, \circ)$ be an ordinary group. We consider $g = \circ$ and define the 3-ary hyperoperation $f : G^3 \to G$ such that for every $x_1^3 \in G$, $f(x_1^3) = x_1 \circ x_2^{-1} \circ x_3$. We have

$$\begin{aligned}
f(f(x_1^3), x_4^5) &= f(x_1 \circ x_2^{-1} \circ x_3, x_4^5) \\
&= x_1 \circ x_2^{-1} \circ x_3 \circ x_4^{-1} \circ x_5, \\
f(x_1, f(x_2^4), x_5) &= f(x_1, x_2 \circ x_3^{-1} \circ x_4, x_5) \\
&= x_1 \circ (x_2 \circ x_3^{-1} \circ x_4)^{-1} \circ x_5 \\
&= x_1 \circ x_2^{-1} \circ x_3 \circ x_4^{-1} \circ x_5, \\
f(x_1^2, f(x_3^5)) &= f(x_1^2, x_3 \circ x_4^{-1} \circ x_5) \\
&= x_1 \circ x_2^{-1} \circ x_3 \circ x_4^{-1} \circ x_5.
\end{aligned}$$

Also,

$$\begin{aligned}
g(x_1, f(y_1^3)) &= x_1 \circ y_1 \circ y_2^{-1} \circ y_3 \\
&= x_1 \circ y_1 \circ y_2^{-1} \circ x_1^{-1} \circ x_1 \circ y_3 \\
&= g(x_1, y_1) \circ g(x_1, y_2)^{-1} \circ g(x_1, y_2) \\
&= f(g(x_1, y_1), g(x_1, y_2), g(x_1, y_2)).
\end{aligned}$$

In the same way we obtain $g(f(y_1^3), x_2) = f(g(y_1, x_2), g(y_2, x_2), g(y_3, x_2))$. Thus, $(R, f, g)$ is a $(3, 2)$-ring.

If $b, c \in R$ then we say that an $(m, n)$-hyperringoid $(R, f, g)$ is $(b, c)$-*derived* from a hyperringoid $(R, +, \cdot)$ and denote this fact by $(R, f, g) = der_c^b(R, +, \cdot)$ if two $m$-ary hyperoperation and $n$-ary hyperoperation $f$ and $g$ respectively, have the form

$$f(x_1^m) = \sum_{i=1}^{m} x_i + b, \quad \forall\ x_1^m \in R,$$

and

$$g(x_1^n) = \prod_{j=1}^{n} y_j \cdot c, \quad \forall\ y_1^n \in R.$$

In this case, when $b$ is a zero scalar of $(R, +)$ and $c$ is an identity scalar of $(R, \circ)$ we say that $(R, f, g)$ is *derived* from $(R, +, \cdot)$ and denote this fact by $(R, f, g) = der(R, +, \cdot)$. It is clear that if $b$ belongs to the center of a hypersemigroup $(R, +)$ and $c$ belongs to the center of a hypersemigroup $(R, \cdot)$ then two $m$-ary hyperoperation and $n$-ary hyperoperations $f$ and $g$ are associative and $(R, f)$ and $(R, g)$ are an $m$-ary hypersemigroup and an $n$-ary hypersemigroup, respectively. Now, if $b$ is the zero scalar and $f$ is defined by $f(x_1^m)=\sum_{i=1}^m x_i$ then we denote $(R,f,g)=der_c(R,+,\cdot)$ and say that $(R,f,g)$ is $c$-derived from $(R,+,\cdot)$. If $(R,+,\cdot)$ is a hyperring and $c\in Z(R,\cdot)$ then, the $c$-derived $(R,f,g)$ is an $(m,n)$-hyperring.

Example 5. Let $(R,+,\cdot)$ be a commutative ring and $S$ be a subring of $R$. Then, we can define an additive $(m,n)$-hyperring $(R,f,g)$ as follows:

$$f(x_1^m)=S+\sum_{i=1}^m x_i, \quad \forall \ x_1^m\in R^m,$$

$$g(x_1^n)=\prod_{i=1}^n x_i, \qquad \forall \ x_1^n\in R^n.$$

Example 6. Let $R=(\mathbb{Z}_2,+,\cdot)$. If $(R,f,g)$ is a $(3,3)$-hyperring derived of $(R,+,\cdot)$ then for every $x_1^3\in R$ and $y_1^3\in R$ we have $f(x_1^3)=x_1+x_2+x_3$ and $g(y_1^3)=y_1\cdot y_2\cdot y_3$. So for every $x\in R$ we have that $x$ is a neutral element but only 0 is zero element. In fact $(R,f,g)=der(\mathbb{Z}_2,+,\cdot)$.

Theorem 4.5. *Let $(R,f,g)$ be an $(m,n)$-hyperring and the relation $\rho$ be a strongly compatible relation on both $m$-ary hypergroup $(R,f)$ and $n$-ary hypersemigroup $(R,g)$. Then, the quotient $(R/\rho,f/\rho,g/\rho)$ is an $(m,n)$-ring.*

*Proof.* By Theorem 3.3, the quotient $(R/\rho,f/\rho)$ is an $m$-ary group. Also, by Theorem 3.2, $(R/\rho,g/\rho)$ is an $n$-ary semigroup. We show that the $n$-ary operation $g/\rho$ is distributive with respect to the $m$-ary operation $f/\rho$, i.e., for every $a_1^{i-1},a_{i+1}^n,x_1^m\in R,\ 1\le i\le n$.

$g/\rho(\rho(a_1),\dots,\rho(a_{i-1}),f/\rho(\rho(x_1),\dots,\rho(x_m)),\rho(a_{i+1}),\dots,\rho(a_n))=f/\rho(u_1,\dots,u_m)$
where for every $j=1,\dots,m$, $u_j=g/\rho(\rho(a_1),\dots,\rho(a_{i-1}),\rho(x_j),\rho(a_{i+1}),\dots\rho(a_n))$. Since the $n$-ary hyperoperation $g$ is distributive with respect to the $m$-ary hyperoperation $f$, thus

$$g(a_1^{i-1},f(x_1^m),a_{i+1}^n)=f(g(a_1^{i-1},x_1,a_{i+1}^n),\dots,g(a_1^{i-1},x_m,a_{i+1}^n))$$

and so

$$\rho(g(a_1^{i-1},f(x_1^m),a_{i+1}^n))=\rho(f(g(a_1^{i-1},x_1,a_{i+1}^n),\dots,g(a_1^{i-1},x_m,a_{i+1}^n))).$$

Thus, we have

$$\begin{aligned}&\rho(g(a_1^{i-1},f(x_1^m),a_{i+1}^n))\\ &\quad=g/\rho(\rho(a_1),\dots,\rho(a_{i-1}),\rho(f(x_1^m)),\rho(a_{i+1}),\dots,\rho(a_n))\\ &\quad=g/\rho(\rho(a_1),\dots,\rho(a_{i-1}),f/\rho(\rho(x_1),\dots,\rho(x_m)),\rho(a_{i+1}),\dots,\rho(a_n)).\end{aligned}$$

And

$$\begin{aligned}&\rho(f(g(a_1^{i-1},x_1,a_{i+1}^m),\dots,g(a_1^{i-1},x_m,a_{i+1}^n)))\\ &\quad=f/\rho(\rho(g(a_1^{i-1},x_1,a_{i+1}^m)),\dots,\rho(g(a_1^{i-1},x_m,a_{i+1}^n))))\\ &\quad=f/\rho(u_1,\dots,u_m),\end{aligned}$$

Where for every $j=1,\dots,m$, $u_j=g/\rho(\rho(a_1),\dots,\rho(a_{i-1}),\rho(x_j),\rho(a_{i+1}),\dots\rho(a_m))$.

Therefore, the $n$-ary operation $g/\rho$ is distributive with respect to the $m$-ary operation $f/\rho$, and so the quotient $(R/\rho,f/\rho,g/\rho)$ is an $(m,n)$-ring. ∎

## 5. Fundamental relations on $(m, n)$-hyperrings

The fundamental relation $\Gamma^*$ was introduced on hyperrings by Vougiouklis [17]. Also, commutative fundamental equivalence relation $\alpha^*$ was studied on hyperrings by Davvaz and Vougiouklis [8]. Now, we consider the notion of fondamental relation on $(m, n)$-hyperrings.

Definition 5.1. Let $(R, f, g)$ be an $(m, n)$-hyperring. For every $k \in \mathbb{N}^*$ and $l_1^s \in \mathbb{N}$, where $s = k(m-1)+1$, we define a relation $\Gamma_{k;l_1^s}$, as follows: $x\ \Gamma_{k;l_1^s}\ y$ if and only if there exist $x_{i1}^{it_i} \in R$, where $t_i = l_i(n-1)+1$, $i = 1, \dots, s$ such that

$$\{x, y\} \subseteq f_{(k)}(u_1, \dots, u_s)$$

where for every $i = 1, \dots, s$, $u_i = g_{(l_i)}(x_{i1}^{it_i})$.

Now, set $\Gamma_k = \bigcup_{l_1^s \in \mathbb{N}} \Gamma_{k;l_1^s}$ and $\Gamma = \bigcup_{k \in \mathbb{N}^*} \Gamma_k$. Then, the relation $\Gamma$ is reflexive and symmetric. Let $\Gamma^*$ be the transitive closure of relation $\Gamma$.

It easy to see that $\beta_f \subseteq \Gamma$, $\beta_f^* \subseteq \Gamma^*$, $\beta_g \subseteq \Gamma$ and $\beta_g^* \subseteq \Gamma^*$.

Remark 1. Pelea and Purdea defined in [15] a relation $\alpha$ on a multialgebra. Indeed, the relation $\Gamma$ introduced in Definition 5.1 is a particular case of the relation $\alpha$ in [15] context to $(m, n)$-hyperrings.

Remark 2. Vougiouklis defined in [17] the fundamental relation $\gamma^*$ on a hyperring $R$ as the smallest equivalence relation on $R$ such that the quotient $R/\gamma^*$ is a fundamental ring. In fact, in $(2,2)$-hyperring (hyperring) we have $\Gamma^* = \gamma^*$ and in any $(2,0)$-hyperring (hypergroup) $\Gamma^* = \beta^*$. So $\Gamma^*$-relation is a natural generalization of the $\gamma^*$-relation and $\beta^*$-relation.

Lemma 5.2. *Let $(R, f, g)$ be an $(m, n)$-hyperring, then for every $k \in \mathbb{N}^*$ we have $\Gamma_k \subseteq \Gamma_{k+1}$.*

*Proof.* Let $x\ \Gamma_k\ y$. Then there exist $l_1^s \in \mathbb{N}^*$ and $x_{i1}^{it_i} \in R$, where $s = k(m-1)+1$, $t_i = l_i(n-1)+1$ and $i = 1, \dots, s$ such that

$$\{x, y\} \subseteq f_{(k)}(u_1, \dots, u_s)$$

where for every $i = 1, \dots, s$, $u_i = g_{(l_i)}(x_{i1}^{it_i})$. Thus,

$$\{x, y\} \subseteq f_{(k)}(g_{(l_1)}(x_{11}^{1t_1}), \dots, g_{(l_s)}(x_{s1}^{st_s}))$$

Now, for every $i = 1, \dots, s$, there exist $x_{it_i+1}, \dots, x_{ir_i} \in R$ such that $x_{it_i} \in f(x_{it_i}, x_{it_i+1}, \dots, x_{ir_i})$.

$$\begin{aligned}\{x, y\} &\subseteq f_{(k)}(g_{(l_1)}(x_{11}^{1t_1}), \dots, g_{(l_s)}(x_{s1}^{st_s})) \\ &\subseteq f_{(k)}(g_{(l_1)}(x_{11}^{1t_1-1}, f(x_{1t_1}, x_{1t_1+1}, \dots, x_{1r_1})), \dots, \\ &\qquad g_{(l_s)}(x_{s1}^{st_s-1}, f(x_{st_s}, x_{st_s+1}, \dots, x_{sr_s}))) \\ &= f_{(k)}(f(g_{(l_1)}(x_{11}^{1r_1}), \dots, g_{(l_s)}(x_{s1}^{sr_s}))) \\ &= f_{(k+1)}(g_{(l_1)}(x_{11}^{1r_1}), \dots, g_{(l_s)}(x_{s1}^{sr_s}))\end{aligned}$$

This means $x\ \Gamma_{k+1}\ y$ and so $\Gamma_k \subseteq \Gamma_{k+1}$. ∎

Corollary 5.3. *Let $(R, f, g)$ be an $(m, n)$-hyperring. Then, for every $k \in \mathbb{N}^*$ we have $\Gamma_k^* \subseteq \Gamma_{k+1}^*$.*

Example 7. Lemma 5.2 and Corollary 5.3 are not valid for hypersemirings. Let $(R, f, g)$ be an $(m, n)$-hyperring as follows: Let $R = \{1, \ldots, 6\}$ and 2-ary hyperoperation $f$ on $H$ defined as follows:

| $f$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | $\{1,2,3\}$ | $\{1,2\}$ | $\{1,3\}$ | $\{1,2\}$ | $\{1,3\}$ | $\{1,3\}$ |
| 2 | $\{1,2\}$ | $\{1,2,3\}$ | $\{2,3\}$ | $\{2,3\}$ | $\{1,2\}$ | $\{1,2\}$ |
| 3 | $\{1,3\}$ | $\{2,3\}$ | $\{1,2,3\}$ | $\{1,3\}$ | $\{2,3\}$ | $\{2,3\}$ |
| 4 | $\{1,2\}$ | $\{2,3\}$ | $\{1,3\}$ | $\{1,2,3\}$ | $\{1,2,3\}$ | $\{1,2,3\}$ |
| 5 | $\{1,3\}$ | $\{1,2\}$ | $\{2,3\}$ | $\{1,2,3\}$ | $\{1,2,3\}$ | $\{1,2,3\}$ |
| 6 | $\{1,3\}$ | $\{1,2\}$ | $\{2,3\}$ | $\{1,2,3\}$ | $\{1,2,3\}$ | $\{4,5\}$ |

For every $x, y, z \in R$ we have $f(x, f(y, z)) = f(f(x, y), z) = \{1, 2, 3\}$. Thus, $(R, f)$ is a 2-ary hypersemigroup. Also, we define the $n$-ary hyperoperation $g(x_1^n) = \{1, 2, 3\}$ and we have

$$g(x_1^{i-1}, f(y_1, y_2), x_{i+1}^n) = \{1, 2, 3\} = f(g(x_1^{i-1}, y_1, x_{i+1}^n), g(x_1^{i-1}, y_2, x_{i+1}^n)).$$

Therefore, $(R, f, g)$ is a $(2, n)$-ary hypersemiring. We have $4\ \Gamma_2\ 5$ but $4\ \not\Gamma_3\ 5$ or $5\Gamma_1 5$ but $5\ \not\Gamma_2 5$. Also $4\ \Gamma_2^*\ 5$ but $4\ \not\Gamma_3^*\ 5$.

Lemma 5.4. *The relation $\Gamma^*$ is a strongly compatible relation on both $m$-ary hypergroup $(R, f)$ and $n$-ary hypersemigroup $(R, g)$.*

*Proof.* Since the relation $\Gamma$ is reflexive and symmetric, so $\Gamma^*$ is an equivalence relation. It is enough to show that if $x\ \Gamma\ y$ then for every $x_1^m, y_1^n \in R$, $i = 1, \ldots, m$ and $j = 1, \ldots, n$

$$f(x_1^{i-1}, x, x_{i+1}^m)\ \overline{\overline{\Gamma}}\ f(x_1^{i-1}, y, x_{i+1}^m),$$
$$g(y_1^{j-1}, x, y_{j+1}^n)\ \overline{\overline{\Gamma}}\ g(y_1^{j-1}, y, y_{j+1}^n).$$

Let $x\ \Gamma\ y$. Then there exist $k \in \mathbb{N}^*$, $l_1^s \in \mathbb{N}^*$ and $x_{i1}^{it_i} \in R$, where $s = k(m-1)+1$ and $t_i = l_i(n-1)+1$, $i = 1, \ldots, s$ such that

$$\{x, y\} \subseteq f_{(k)}(u_1, \ldots, u_s),$$

where for every $i = 1, \ldots, s$, $u_i = g_{(l_i)}(x_{i1}^{it_i})$. Thus,

$$\{x, y\} \subseteq f_{(k)}(g_{(l_1)}(x_{11}^{1t_1}), \ldots, g_{(l_s)}(x_{s1}^{st_s})).$$

Now, let $x_1^m \in R$. Then for every $i = 1, \ldots, m$ we obtain:

$$\begin{aligned}\{f(x_1^{i-1}, x, x_{i+1}^m), f(x_1^{i-1}, y, x_{i+1}^m)\} &\subseteq f(x_1^{i-1}, f_{(k)}(g_{(l_1)}(x_{11}^{1t_1}), \ldots, g_{(l_s)}(x_{s1}^{st_s})), x_{i+1}^m) \\ &= f_{(k+1)}(x_1^{i-1}, g_{(l_1)}(x_{11}^{1t_1}), \ldots, g_{(l_s)}(x_{s1}^{st_s}), x_{i+1}^m).\end{aligned}$$

Set
$$\begin{cases} u'_j = g_{(0)}(x_j), & \text{if } j = 1,\dots,i-1 \\ u'_j = g_{(l_{j-i+1})}(x_{(j-i+1)1}^{(j-i+1)t_{(j-i+1)}}), & \text{if } j = i,\dots,s+i-1 \\ u'_j = g_{(0)}(x_{j-s+1}), & \text{if } j = s+i,\dots,s+m-1. \end{cases}$$
Therefore, for every $v \in f(x_1^{i-1}, x, x_{i+1}^m)$ and $w \in f(x_1^{i-1}, y, x_{i+1}^m)$ we have
$$\{v, w\} \subseteq f_{(k+1)}(u'_1, \dots, u'_{s+m-1})$$
and so $v\ \Gamma\ w$. Thus, $f(x_1^{i-1}, x, x_{i+1}^m)\ \overline{\overline{\Gamma}}\ f(x_1^{i-1}, y, x_{i+1}^m)$ and it is easy to see that
$$f(x_1^{i-1}, x, x_{i+1}^m)\ \overline{\overline{\Gamma^*}}\ f(x_1^{i-1}, y, x_{i+1}^m).$$
Now, note that for every $y_1^n \in R$ and $j = 1,\dots,n$ we have
$$\{g(y_1^{j-1}, x, y_{j+1}^n), g(y_1^{j-1}, y, y_{j+1}^n)\} \subseteq g(y_1^{j-1}, f_{(k)}(g_{(l_1)}(x_{11}^{1t_1}), \dots, g_{(l_s)}(x_{s1}^{st_s})), y_{j+1}^n).$$
Since the $n$-ary hyperoperation $g$ is distributive with respect to the $m$-ary hyperoperation $f$, then
$$g(y_1^{j-1}, f_{(k)}(g_{(l_1)}(x_{11}^{1t_1}), \dots, g_{(l_s)}(x_{s1}^{st_s})), y_{j+1}^n) = f_{(k)}(u''_1, \dots, u''_s),$$
where $u''_i = g_{(l_i+1)}(y_i^{j-1}, x_{i1}^{it_i}, y_{j+1}^n)$, for every $i = 1,\dots,s$. Therefore,
$$\{g(y_1^{j-1}, x, y_{j+1}^n), g(y_1^{j-1}, y, y_{j+1}^n)\} \subseteq g(u''_1, \dots, u''_s)$$
and so for every $t \in g(y_1^{j-1}, x, y_{j+1}^n)$ and $z \in g(y_1^{j-1}, y, y_{j+1}^n)$, we obtain $t\ \Gamma\ z$. Thus,
$$g(y_1^{j-1}, x, y_{j+1}^n)\ \overline{\overline{\Gamma}}\ g(y_1^{j-1}, y, y_{j+1}^n).$$
and we conclude that
$$g(y_1^{j-1}, x, y_{j+1}^n)\ \overline{\overline{\Gamma^*}}\ g(y_1^{j-1}, y, y_{j+1}^n). \quad \blacksquare$$

THEOREM 5.5. *The quotient* $(R/\Gamma^*, f/\Gamma^*, g/\Gamma^*)$ *is an* $(m,n)$*-ring.*

*Proof.* Since by Lemma 5.4, the relation $\Gamma^*$ is a strongly compatible relation on both $m$-ary hypergroup $(R, f)$ and $n$-ary hypersemigroup $(R, g)$, then by Theorem 4.5, the quotient $(R/\Gamma^*, f/\Gamma^*, g/\Gamma^*)$ is an $(m,n)$-ring. $\blacksquare$

THEOREM 5.6. *The relation* $\Gamma^*$ *is the smallest equivalence relation such that the quotient* $(R/\Gamma^*, f/\Gamma^*, g/\Gamma^*)$ *is an* $(m,n)$*-ring.*

*Proof.* Let $\theta$ be the smallest equivalence relation such that $(R/\theta, f/\theta, g/\theta)$ is an $(m,n)$-ring. We prove that $\Gamma^* = \theta$. Since $(R/\Gamma^*, f/\Gamma^*, g/\Gamma^*)$ is an $(m,n)$-ring so $\theta \subseteq \Gamma^*$. If $x\ \Gamma\ y$, then there exist $k \in \mathbb{N}^*$, $l_1^s \in \mathbb{N}^*$ and $x_{i1}^{it_i} \in R$, where $s = k(m-1)+1$ and $t_i = l_i(n-1)+1$, $i = 1,\dots,s$ such that
$$\{x, y\} \subseteq f_{(k)}(u_1, \dots, u_s),$$

where for every $i = 1, \ldots, s$, $u_i = g_{(l_i)}(x_{i1}^{it_i})$. Thus, we have

$$\{\theta(x), \theta(y)\} \subseteq (f/\theta)_{(k)}(U_1, \ldots, U_s),$$

where for every $i = 1, \ldots, s$, $U_i = (g/\theta)_{(l_i)}(\theta(x_{i1}), \ldots, \theta(x_{it_i}))$. But $R/\theta$ is an $(m,n)$-ring and it follows that $\theta(x) = \theta(y)$. Thus, $x\ \Gamma\ y$ implies that $x\ \theta\ y$. Finally, let $x\ \Gamma^*\ y$. Thus, there exist $h \in \mathbb{N}$ and $w_1^h \in R$ such that

$$x = w_0\ \Gamma\ w_1\ \ldots\ w_h \Gamma\ w_{h+1} = y.$$

Since $\theta$ is transitively closed and $\Gamma \subseteq \theta$ we obtain

$$x = w_0\ \theta\ w_1\ \ldots\ w_h \theta\ w_{h+1} = y.$$

Therefore, $x\ \theta^*\ y$ and so $\Gamma^* \subseteq \theta$. ■

THEOREM 5.7. *For all additive $(m,n)$-hyperrings, we have $\Gamma^* = \beta_f^*$.*

*Proof.* Since $(R/\Gamma^*, f/\Gamma^*, g/\Gamma^*)$ is an $(m,n)$-ring, then $(R/\Gamma^*, f/\Gamma^*)$ is an $m$-ary group, and so $\beta_f^* \subseteq \Gamma^*$.

Conversely, let $x\ \Gamma\ y$. Then, there exist $k \in \mathbb{N}^*$, $l_1^s \in \mathbb{N}^*$ and $x_{i1}^{it_i} \in R$, where $s = k(m-1)+1$ and $t_i = l_i(n-1)+1$, $i = 1, \ldots, s$ such that

$$\{x, y\} \subseteq f_{(k)}(u_1, \ldots, u_s),$$

where for every $i = 1, \ldots, s$, $u_i = g_{(l_i)}(x_{i1}^{it_i})$. But, $g$ is an $n$-ary operation and so $u_i$ is singleton. Therefore, $x\ \beta_f\ y$. Now, if $x\ \Gamma^*\ y$ then there exist $w_1, w_2, \ldots, w_{n-1} \in R$ such that $x = w_0\ \Gamma\ w_1\ \ldots\ w_{n-1}\ \Gamma\ w_n = y$. Thus, we obtain $x = w_0\ \beta_f\ w_1\ \ldots\ w_{n-1}\ \beta_f\ w_n = y$ and $x\ \beta_f^*\ y$. ■

COROLLARY 5.8. *For every additive $(m,n)$-hyperring, the relation $\Gamma$ is an equivalence relation, i.e. $\Gamma = \Gamma^*$.*

*Proof.* By Theorem 3.4, we have $\beta_f = \beta_f^*$. Also, by Theorem 5.7, $\beta_f^* = \Gamma^*$. Now, we conclude that

$$\Gamma \subseteq \Gamma^* = \beta_f^* = \beta_f \subseteq \Gamma.$$

So $\Gamma = \Gamma^*$. ■

THEOREM 5.9. *Let $(R, f, g)$ be an $(m,n)$-hyperring. Then,*

(1) *$(R/\beta_f^*, f/\beta_f^*, g/\beta_f^*)$ is an $(m,n)$-multiplicative hyperring,*

(2) *$(R/\beta_g^*, f/\beta_g^*, g/\beta_g^*)$ is an additive $(m,n)$-hyperring.*

*Proof.* (1) For every $x_1^m \in R$ we have

$$f/\beta_f^*(\beta_f^*(x_1), \ldots, \beta_f^*(x_m)) = \beta_f^*(z) \quad \text{for all} \quad z \in f(\beta_f^*(x_1), \ldots, \beta_f^*(x_m)).$$

By Theorem 3.4, $(R/\beta_f^*, f/\beta_f^*)$ is an $m$-ary group. We define an $n$-ary hyperoperation $g/\beta_f^*$ on $R/\beta_f^*$ as follows:

$$g/\beta_f^*(\beta_f^*(x_1), \ldots, \beta_f^*(x_n)) = \{\beta_f^*(z) \mid z \in g(\beta_f^*(x_1), \ldots, \beta_f^*(x_n))\}.$$

We prove that $g/\beta_f^*$ is associative. Let $x_1^{2n-1} \in R$. Then,

$$\begin{aligned}
&g/\beta_f^*(\beta_f^*(x_1),\ldots,\beta_f^*(x_{i-1}),g/\beta_f^*(\beta_f^*(x_i),\ldots,\beta_f^*(x_{m+i-1})),\\
&\quad \beta_f^*(x_{m+i}),\ldots,\beta_f^*(x_{2m-1}))\\
&= \{\beta_f^*(z) | z \in g(\beta_f^*(x_1),\ldots,\beta_f^*(x_{i-1}),g(\beta_f^*(x_i),\ldots,\beta_f^*(x_{m+i-1})),\\
&\quad \beta_f^*(x_{m+i}),\ldots,\beta_f^*(x_{2m-1}))\}\\
&= \{\beta_f^*(z) | z \in g(\beta_f^*(x_1),\ldots,\beta_f^*(x_{j-1}),g(\beta_f^*(x_j),\ldots,\beta_f^*(x_{m+j-1})),\\
&\quad \beta_f^*(x_{m+j}),\ldots,\beta_f^*(x_{2m-1}))\}\\
&= g/\beta_f^*(\beta_f^*(x_1),\ldots,\beta_f^*(x_{j-1}),g/\beta_f^*((\beta_f^*(x_j),\ldots,\beta_f^*(x_{m+j-1})),\\
&\quad \beta_f^*(x_{m+j}),\ldots,\beta_f^*(x_{2m-1})).
\end{aligned}$$

Thus, $(R/\beta_f^*, g/\beta_f^*)$ is an $n$-ary hypersemigroup. It is easi to see that $n$-ary hyperoperation $g/\beta_f^*$ is distributive with respect to the $m$-ary operation $f/\beta_f^*$.

(2) The proof of (2) is similar to (1). ■

THEOREM 5.10 *Let $(R,f,g)$ be an $(m,n)$-hyperring. Then,*

$$R/\Gamma^* \cong (R/\beta_g^*)/\beta^*_{f/\beta_g^*}$$

*Proof.* Let $\Psi : R \to (R/\beta_g^*)/\beta^*_{f/\beta_g^*}$ be the canonical projection. We denote the equivalence relation associated to $\varphi$ by $\theta$. For every $a \in R$ we have $\Gamma^*(a) \subseteq \theta(a)$. On the other hand, since $\beta_g^*(x) \subseteq \Gamma^*(x)$ for all $x \in R$, we have

$$\begin{aligned}
\bigcup_{\beta_g^*(z)\in f/\beta_g^*(\beta_g^*(x_1),\ldots,\beta_g^*(x_m))} \beta_g^*(z) &= \bigcup_{z\in f(\beta_g^*(x_1),\ldots,\beta_g^*(x_m))} \beta_g^*(z)\\
&\subseteq \bigcup_{z\in f(\Gamma^*(x_1),\ldots,\Gamma^*(x_m))} \Gamma^*(z) = \Gamma^*(w) \text{ for all } w \in f(x_1^m).
\end{aligned}$$

Consequently, we obtain

$$\bigcup_{z\in f_{(k)}(\beta_g^*(x_1),\ldots,\beta_g^*(x_l))} \beta_g^*(z) \subseteq \Gamma^*(w),$$

where $w \in f_{(k)}(x_1^l)$, $l = k(m-1)+1$ and $x_1^l \in R$. Moreover, since $\Gamma^*$ is transitive, we have

$$\theta(a) = \bigcup_{\{z \ | \ \beta_g^*(z) \ \beta^*_{f/\beta_g^*} \ \beta_g^*(a)\}} \beta_g^*(z) \subseteq \Gamma^*(a) \text{ for all } a \in R.$$

Therefore, $\theta = \Gamma^*$. ■

Let $(R_1,f,g)$ and $(R_2,f,g)$ be two $(m,n)$-hypersemirings. We define $(f_1,f_2) : (A\times B)^m \to \mathcal{P}^*(A\times B)$ by

$$(f,g)((a_1,b_1),\ldots,(a_n,b_n)) = \{(a,b) \mid a \in f(a_1,\ldots,a_n), b \in g(b_1,\ldots,b_n)\}.$$

We defined $(g_1, g_2)$ in a similar way. Clearly $(R_1 \times R_2, (f_1, f_2), (g_1, g_2))$ is an $(m, n)$-hypersemiring and we call this $(m, n)$-hypersemiring the *direct hyperproduct* of $R_1$ and $R_2$.

LEMMA 5.11. *Let $(R_1, f, g)$ and $(R_2, f, g)$ be two $(m, n)$-hypersemirings, $a, c \in R_1$ and $b, d \in R_2$. If $\Gamma^*_{R_1}$, $\Gamma^*_{R_2}$ and $\Gamma^*_{R_1 \times R_2}$ are the $\Gamma^*$-relations on $R_1$, $R_2$ and $R_1 \times R_2$ respectively, then $(a, b)\ \Gamma^*_{R_1 \times R_2}\ (c, d)$ implies $a\ \Gamma^*_{R_1}\ c$ and $b\ \Gamma^*_{R_2}\ d$.*

*Proof.* Let $(a, b)\ \Gamma^*_{R_1 \times R_2}\ (c, d)$. Then, there exist $k \in \mathbb{N}^*$, $l_1^s \in \mathbb{N}^*$ and $x_{i1}^{it_i} \in R_1, y_{i1}^{it_i} \in R_2$, where $s = k(m-1)+1$ and $t_i = l_i(n-1)+1$, $i = 1, \ldots, s$ such that

$$\{(a, b), (c, d)\} \subseteq f_{(k)}(u_1, \ldots, u_s),$$

where for every $i = 1, \ldots, s$, $u_i = g_{(l_i)}(x_{i1}^{it_i}, y_{i1}^{it_i})$. Thus, we have

$$\begin{aligned}(a, b), (c, d) &\in f_{(k)}(g_{(l_1)}(x_{11}^{1t_1}, y_{11}^{1t_1}), \ldots, g_{(l_s)}(x_{s1}^{st_s}, y_{s1}^{st_s})) \\ &= (f_{(k)}(g_{(l_1)}(x_{11}^{1t_1}), \ldots, g_{(l_s)}(x_{s1}^{st_s})), f_{(k)}(g_{(l_1)}(y_{11}^{1t_1}), \ldots, g_{(l_s)}(y_{s1}^{st_s})))\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}\{a, c\} &\subseteq f_{(k)}(g_{(l_1)}(x_{11}^{1t_1}), \ldots, g_{(l_s)}(x_{s1}^{st_s})) \quad \text{and} \\ \{b, d\} &\subseteq f_{(k)}(g_{(l_1)}(y_{11}^{1t_1}), \ldots, g_{(l_s)}(y_{s1}^{st_s})).\end{aligned}$$

This implies that $a\ \Gamma^*_{R_1}\ c$ and $b\ \Gamma^*_{R_2}\ d$. ■

THEOREM 5.12. *Let $(R_1, f, g)$ and $(R_2, f, g)$ be two $(m, n)$-hyperrings, $a, c \in R_1$ and $b, d \in R_2$. If $\Gamma^*_{R_1}$, $\Gamma^*_{R_2}$ and $\Gamma^*_{R_1 \times R_2}$ are $\Gamma^*$-relations on $R_1$, $R_2$ and $R_1 \times R_2$ respectively, then*

$$(a, b)\ \Gamma^*_{R_1 \times R_2}\ (c, d) \quad \text{if and only if} \quad a\ \Gamma^*_{R_1}\ c \text{ and } \ b\ \Gamma^*_{R_2}\ d.$$

*Proof.* By Lemma 5.11, we need to prove that $a\ \Gamma^*_{R_1}\ c$ and $b\ \Gamma^*_{R_2}\ d$ imply $(a, b)\ \Gamma^*_{R_1 \times R_2}\ (c, d)$. Thus, there exist $p, q \in \mathbb{N}^*$, $w_0^p \in R_1$, $u_0^q \in R_2$ and $k_1^p, l_1^q \in \mathbb{N}^*$ such that

$$a = w_0 \Gamma_{k_1} w_1 \Gamma_{k_2} \ldots \Gamma_{k_p} w_p = c \text{ and } b = u_0 \Gamma_{l_1} u_1 \Gamma_{l_2} \ldots \Gamma_{l_q} u_q = d.$$

Let $p \leq q$. Then $a = w_0 \Gamma_{k_1} w_1 \Gamma_{k_2} \ldots \Gamma_{k_p} w_p \Gamma_{k_{p+1}} w_{p+1} \Gamma_{k_{p+2}} \ldots \Gamma_{k_q} w_q = c$ where $w_p = w_{p+1} = \cdots = w_q$ and $k_{p+1} = \cdots = k_q = 0$. By Corollary 5.3, if $k = \max_{1 \leq i \leq q}\{k_i, l_i\}$ then

$$a = w_0 \Gamma_k w_1 \Gamma_k \ldots \Gamma_k w_q = c \text{ and } b = u_0 \Gamma_k u_1 \Gamma_k \ldots \Gamma_k u_q = d.$$

So for every $1 \leq i \leq q$ we have $w_{i-1} \Gamma_k w_i$ and $u_{i-1} \Gamma_k u_i$. Therefore, $w_{i-1}, w_i \in f_{(k)}(g_{(l_1)}(x_{i11}^{i1t_1}), \ldots, g_{(l_s)}(x_{is1}^{ist_s}))$ and $u_{i-1}, u_i \in f_{(k)}(g_{(l_1)}(z_{i11}^{i1t_1}), \ldots, g_{(l_s)}(z_{is1}^{ist_s}))$. Thus,

$$\{(w_{i-1}, u_{i-1}), (w_i, u_i)\} \subset f_{(k)}(g_{(l_1)}((x, z)_{i11}^{i1t_1}), \ldots, g_{(l_s)}((x, z)_{is1}^{ist_s})).$$

Hence, we obtain

$$(a, b) = (w_0, u_0)\Gamma_{R_1 \times R_2}(w_1, u_1)\Gamma_{R_1 \times R_2} \dots \Gamma_{R_1 \times R_2}(w_q, u_q) = (c, d),$$

and so $(a, b)\ \Gamma^*_{R_1 \times R_2}\ (c, d)$. ■

THEOREM 5.13. *Let $R_1$ and $R_2$ be two $(m, n)$-hyperrings, $\Gamma^*_{R_1}$, $\Gamma^*_{R_2}$ and $\Gamma^*_{R_1 \times R_2}$ be the $\Gamma^*$-relations on $R_1$, $R_2$ and $R_1 \times R_2$ respectively. Then,*

$$(R_1 \times R_2)/\Gamma^*_{R_1 \times R_2} \cong R_1/\Gamma^*_{R_1}\ \times\ R_2/\Gamma^*_{R_2}.$$

EXAMPLE 8. We show that Theorem 5.13 for $(m, n)$-hypersemirings does not hold. Let $R = \{1, 2, 3, 4\}$. We define $m$-ary hyperoperation $f$ on $R$ as follows:

$f(1, \dots, 1) = \{3, 4\}$,

$f(x_1, \dots, x_m) = \{2, 4\}, \forall (x_1, \dots, x_m) \neq (1, \dots, 1)$.

$(R, f)$ is an $m$-ary hypersemigroup, since for every $z_1, \dots, z_m \in R$, we have $1 \notin f(z_1^m)$ and so:

$$f(x_1^{i-1}, f(x_i^{m+i-1}), x_{i+m}^{2m-1}) = \{2, 4\} = f(x_1^{j-1}, f(x_j^{m+j-1}), x_{j+m}^{2m-1}),\ \forall x_1^{2m-1} \in R.$$

$(R, f)$ is not an $m$-ary hypergroup, since for every $z_1, \dots, z_m \in R$, we have $1 \notin f(z_1^m)$. We define an $n$-ary hyperoperation $g$ for all $x_1^n \in R$, by $g(x_1^m) = \{2, 4\}$. Now, $(R, g)$ is an $m$-ary hypersemigroup. For every $a_1^n, x_1^m \in R,\ 1 \le i \le n$,

$$g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = \{2, 4\} = f(g(a_1^{i-1}, x_1, a_{i+1}^n), \dots, g(a_1^{i-1}, x_m, a_{i+1}^n)).$$

Therefore, $(R, f, g)$ is an $(m, n)$-hypersemiring. We obtain $\Gamma^*_R(1) = \{1\}$ and $\Gamma^*_R(2) = \{2, 3, 4\} = \Gamma^*_R(3) = \Gamma^*_R(4)$. Let $R \times R$ be the direct hyperproduct of $R$ and $R$. Then, we have:

$$(f, f)((x_1, y_1), \dots, (x_m, y_m)) = \{(a, b) \mid a \in f(x_1, \dots, x_m), b \in f(y_1, \dots, y_m)\}.$$

and

$$(g, g)((u_1, v_1), \dots, (u_n, v_n)) = \{(c, d) \mid c \in g(u_1, \dots, u_n), d \in g(v_1, \dots, v_n)\}.$$

Therefore, we obtain:

$(f, f)((1, 1), \dots, (1, 1)) = \{3, 4\} \times \{3, 4\}$,

$(f, f)((1, y_1), \dots, (1, y_m)) = \{3, 4\} \times \{2, 4\},\ \forall (y_1^m) \neq (\overset{(m)}{1})$,

$(f, f)((x_1, 1), \dots, (x_m, 1)) = \{2, 4\} \times \{3, 4\},\ \forall (x_1^m) \neq (\overset{(m)}{1})$,

$(f, f)((x_1, y_1), \dots, (x_m, y_m)) = \{2, 4\} \times \{2, 4\},\ \forall (y_1^m), (x_1^m) \neq (\overset{(m)}{1})$,

$(g, g)((x_1, y_1), \dots, (x_n, y_n)) = \{2, 4\} \times \{2, 4\},\ \forall y_1^n, x_1^n \in R$.

We have $(R \times R, (f, f), (g, g))$ is an $(m, n)$-hypersemiring. It easy to see that

$\Gamma^*_{R \times R}(i, j) = \{2, 3, 4\} \times \{2, 3, 4\}, \quad \text{if } i, j = 2, 3, 4.$

$\Gamma^*_{R\times R}(i,j) = \{(i,j)\}, \quad \text{if } i = 1 \text{ or } j = 1.$

Hence, we have $|R \times R/\Gamma^*_{R\times R}| = 8$. But $|R/\Gamma^*_R| = 2$ and so $|R/\Gamma^*_R \times R/\Gamma^*_R| = 4$. Therefore,

$$R \times R/\Gamma^*_{R\times R} \ncong R/\Gamma^*_R \times R/\Gamma^*_R.$$

REFERENCES

[**1**] S. M. Anvariyeh, S. Mirvakili and B. Davvaz, *Fundamental relation on $(m,n)$-ary hypermodules on $(m,n)$-ary hyperring*, ARS Combinatoria, **94** (2010) 273–288.

[**2**] P. Corsini, *Prolegomena of Hypergroup Theory*, Second edition, Aviani editor, 1993.

[**3**] P. Corsini and V. Leoreanu, *Applications of Hyperstructures Theory*, Advances in Mathematics, Kluwer Academic Publisher, 2003.

[**4**] G. Crombez, *On $(n,m)$-rings*, Abh. Math. Semin. Univ. Hamburg, **37** (1972), 180–199.

[**5**] B. Davvaz, V. Leoreanu-Fotea, *Hyperring Theory and Applications*, International Academic Press, USA, 2007.

[**6**] B. Davvaz, W.A. Dudek and S. Mirvakili, *Neutral elements, fundamental relations and $n$-ary hypersemigroups*, Intern. J. Algebra Comput., **19** (4) (2009) 567–583.

[**7**] B. Davvaz and T. Vougiouklis, *$n$-ary hypergroups*, Iranian J. Sci. Techn., Transaction A, **30**(A2) (2006) 165–174.

[**8**] B. Davvaz and T. Vougiouklis, *Commutative rings obtained from hyperrings ($H_v$-rings) with $\alpha^*$-relations*, Comm. Algebra, **35** (11) (2007) 3307–3320.

[**9**] W. Dörnte, *Untersuchungen über einen verallgemeinerten Gruppenbegriff*, Math. Z., **29** (1928), 1–19.

[**10**] V. Leoreanu-Fotea and B. Davvaz, *Join $n$-space and lattices*, J. Multiple-Valued Logic Soft Comput., **15** (2009) 421–432.

[**11**] V. Leoreanu and B. Davvaz, *$n$-hypergroups and binary relations*, European J. Combinatorics, **29** (5) (2008) 1207–1218.

[**12**] F. Marty, *Sur un generalization de la notion de group*, 8*th* Congress Math. Scandenaves, Stockholm, (1934) 45–49.

[**13**] S. Mirvakili and B. Davvaz, *Relations on Krasner $(m,n)$-hyperrings*, European J. Combinatorics, **31** (2010) 790–802.

[**14**] E.L. Post, *Polyadic groups*, Trans. Amer. Math. Soc., **48** (1940) 208–350.

[**15**] C. Pelea and I. Purdea, *Multialgebras, universal algebras and identities*, J. Aust. Math. Soc., **81** (2006) 121–139.

[**16**] T. Vougiouklis, *Hyperstructures and Their Representations*, Hadronic Press, Inc, 115, Palm Harber, USA (1994).

[**17**] T. Vougiouklis, *The fundamental relation in hyperrings. The general hyperfield*, Proc. Fourth Int. Congress on Algebraic Hyperstructures and Applications (AHA 1990), World Scientific, (1991) 203–211.

(received 24.01.2013; in revised form 24.05.2013; available online 01.08.2013)

Department of Mathematics, Yazd University, Yazd, Iran

*E-mail*: davvaz@yazd.ac.ir