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COMMON FIXED POINTS OF TWO MULTIVALUED NONEXPANSIVE MAPS BY A ONE-STEP IMPLICIT ALGORITHM IN HYPERBOLIC SPACES

S. H. Khan, H. Fukhar-ud-din and A. Kalsoom

Abstract. In this paper, we construct an implicit algorithm for two multivalued nonexpansive maps in a hyperbolic space and use it to approximate common fixed points of these maps through $\triangle$-convergence and strong convergence.

1. Introduction and preliminaries

A subset K of a metric space X is proximinal if for each $x \in X$, there exists an element $k \in K$ such that

$$d(x, K) = \inf\{d(x, y) : y \in K\} = d(x, k).$$

Let $CB(K)$, $C(K)$ and $P(K)$ be the families of closed and bounded subsets, compact subsets and proximinal bounded subsets of K, respectively. Let H be the Hausdorff metric induced by the metric d of X, that is,

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}$$

for every $A, B \in CB(X)$. A multivalued map $T : K \to CB(X)$ is nonexpansive if

$$H(Tx, Ty) \le d(x, y)$$

for all $x, y \in K$. A point $x \in K$ is a fixed point of T if $x \in Tx$. Denote the set of all fixed points of T by $F(T)$ and $P_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}$.

We consider the following definition of a hyperbolic space introduced by Kohlenbach [8].

Definition 1. A metric space (X, d) is a hyperbolic space if there exists a map $W : X^2 \times [0, 1] \to X$ satisfying

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(i) $d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1-\alpha)\, d(u, y)$,

(ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta|\, d(x, y)$,

(iii) $W(x, y, \alpha) = W(y, x, (1-\alpha))$,

(iv) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq \alpha d(x, y) + (1-\alpha)\, d(z, w)$

for all $x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$.

An important example of a hyperbolic space is a $CAT(0)$ space. It is nonlinear in nature and its brief introduction is as under.

A metric space (X, d) is a *length space* if any two points of X are joined by a rectifiable path (that is, a path of finite length) and the distance between any two points of X is taken to be the infimum of the lengths of all rectifiable paths joining them. In this case, d is known as a *length metric* (otherwise an *inner metric* or *intrinsic metric*). In case that no rectifiable path joins two points of the space, the distance between them is taken to be ∞.

A *geodesic path* joining $x \in X$ to $y \in X$ is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and yt. We say that X is: (i) a *geodesic space* if any two points of X are joined by a geodesic path and (ii) *uniquely geodesic* if there is exactly one geodesic path denoted by $\alpha x \oplus (1-\alpha)y$ joining x and y for each $x, y \in X$. The set $\{\alpha x \oplus (1-\alpha)y : \alpha \in [0, 1]\}$ will be denoted by $[x, y]$, called the segment joining x to y. A subset C of a geodesic space X is convex if for any $x, y \in C$, we have $[x, y] \subset C$.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) is defined to be a collection of three points in X (the *vertices* of Δ) and three geodesic segments between each pair of vertices (the *edges* of Δ). A *comparison triangle* for geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$ and such a triangle always exists(see [1]).

A geodesic metric space is a $CAT(0)$ space if all geodesic triangles Δ in X with a comparison triangle $\overline{\Delta} \subset \mathbb{R}^2$ satisfy the $CAT(0)$ inequality

$$d(x, y) \leq d(\bar{x}, \bar{y})$$

for all $x, y \in \Delta$ and for all comparison points $\bar{x}, \bar{y} \in \overline{\Delta}$.

Let X be a $CAT(0)$ space. Define $W : X^2 \times [0, 1] \to X$ by $W(x, y, \alpha) = \alpha x \oplus (1-\alpha)y$. Then W satisfies the four properties of a hyperbolic space. Also if X is a Banach space and $W(x, y, \alpha) = \alpha x + (1-\alpha)y$, then X is a hyperbolic space. Therefore, our hyperbolic space represents a unified approach for both linear and nonlinear structures simultaneously.

To elaborate that there are hyperbolic spaces which are not imbedded in any Banach space, we give the following example.

EXAMPLE 1. Let B be the open unit ball in complex Hilbert space with respect to the Poincare metric (also called "Poincare distance")

$$d_B(x,y) = \arg\tanh\left|\frac{x-y}{1-x\overline{y}}\right| = \arg\tanh\left(1-\sigma(x,y)\right)^{\frac{1}{2}},$$

where

$$\sigma(x,y) = \frac{(1-|x|^2)(1-|y|^2)}{|1-x\overline{y}|^2} \text{ for all } x,y \in B.$$

Then B is a hyperbolic space which is not imbedded in any Banach space

A metric space (X,d) is called a convex metric space introduced by Takahashi [28] if it satisfies only (i). A subset K of a hyperbolic space X is convex if $W(x,y,\alpha) \in K$ for all $x,y \in K$ and $\alpha \in [0,1]$.

A hyperbolic space (X,d,W) is uniformly convex [27] if for any $u,x,y \in X$, $r > 0$ and $\varepsilon \in (0,2]$, there exists a $\delta \in (0,1]$ such that $d\left(W(x,y,\frac{1}{2}),u\right) \leq (1-\delta)r$ whenever $d(x,u) \leq r, d(y,u) \leq r$ and $d(x,y) \geq \varepsilon r$. A map $\eta : (0,\infty) \times (0,2] \to (0,1]$ which provides such a $\delta = \eta(r,\epsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$, is known as modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ε).

Different notions of hyperbolic space can be found in the literature (see [6, 8, 13, 22] to compare these). The hyperbolic space introduced by Kohlenbach [8] is slightly restrictive than the space of hyperbolic type [4] but more general than hyperbolic space of [22]. Moreover, this class of hyperbolic spaces also contains Hadamard manifolds, Hilbert balls equipped with the hyperbolic metric [6], $\mathbb{R}$-trees and Cartesian products of Hilbert balls as special cases.

Markin [19] initiated the study of fixed points for multivalued nonexpansive maps using Hausdorff metric (see also [9, 20]). The existence of fixed points for multivalued nonexpansive maps in convex metric spaces has been proved by Shimizu and Takahashi [27]. Actually, they obtained:

THEOREM ST ([27]). *Let (X,d) be a bounded, complete and uniformly convex metric space. Then every multivalued map $T : X \to C(X)$ (the family of all compact subsets of X) has a fixed point.*

The existence of common fixed points of two multivalued mappings is elaborated in the following example.

EXAMPLE 2. Let $K = [0,1]$ be endowed with the Euclidean metric. Let $S,T : K \to CB(K)$ be defined by $Sx = \left[0,\frac{x}{4}\right]$ and $Tx = \left[0,\frac{x}{2}\right]$. It is easy to see that for any $x,y \in K$,

$$H(Tx,Ty) = \max\left\{\left|\frac{x}{2}-\frac{y}{2}\right|,0\right\} = \left|\frac{x}{2}-\frac{y}{2}\right| \leq |x-y|.$$

In a similar way, we obtain

$$H(Sx,Sy) = \max\left\{\left|\frac{x}{4}-\frac{y}{4}\right|,0\right\} = \left|\frac{x}{4}-\frac{y}{4}\right| \leq |x-y|,$$

showing that T and S are multivalued nonexpansive maps. Obviously, $F(T) \cap F(S) = \{0\}$.

An interesting and rich fixed point theory for multivalued maps has been developed which has applications in control theory, convex optimization, differential inclusion and economics (see [5] and references cited therein). Some authors have published papers on the existence and convergence of fixed points for multivalued nonexpansive maps in convex metric spaces (see [3, 27]).

The theory of multivalued nonexpansive maps is harder than the corresponding theory of single valued nonexpansive maps. Different iterative algorithms have been used to approximate the fixed points of multivalued nonexpansive maps. Sastry and Babu [23] considered Mann and Ishikawa type iterative algorithms. The results of Sastry and Babu [23] were later generalized by Panyanak [21].

Before we move further, we need to state the following useful lemma due to Nadler [20].

Lemma 1. *Let $A, B \in CB(E)$ and $a \in A$. If $\eta > 0$, then there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \eta$.*

Based on the above lemma, Song and Wang [25] modified the iterative algorithm due to Panyanak [2] and improved the results presented therein. Their algorithm is:

Let K be a nonempty and convex subset of a Banach space E. Choose $x_1 \in K, z_1 \in Tx_1$ and define $\{x_n\}$ as

$$\begin{cases} y_n = (1-b_n)x_n + b_n z_n \\ x_{n+1} = (1-a_n)x_n + a_n u_n \end{cases} \tag{1.1}$$

where $z_n \in Tx_n$, $u_n \in Ty_n$ are such that $\|z_n - u_n\| \leq H(Tx_n, Ty_n) + \eta_n$ and $\|z_{n+1} - u_n\| \leq H(Tx_{n+1}, Ty_n) + \eta_n$ hold, $\{a_n\}$ and $\{b_n\}$ are in $(0, 1)$ satisfying $\lim_{n\to\infty} b_n = 0$ and $\sum a_n b_n = \infty$, and $\eta_n \in (0, \infty)$ is such that $\lim_{n\to\infty} \eta_n = 0$.

It is to be noted that Song and Wang [25] needed the condition $Tp = \{p\}$ in order to prove their main theorem. Actually, Panyanak [2] proved some results using Ishikawa type iteration process without this condition. Song and Wang [25] showed that without this condition his process was not well-defined. They reconstructed the process using the condition $Tp = \{p\}$ which made it well-defined.

Recently, Shahzad and Zegeye [26] showed their concerns on the work of Song and Wang [25]. In particular, they pointed out that the assumption $Tp = \{p\}$ for any $p \in F(T)$ in [25] is quite strong. In order to get rid of the condition $Tp = \{p\}$ for any $p \in F(T)$, they used $P_T(x) := \{y \in Tx : \|x - y\| = d(x, Tx)\}$ for a multivalued map $T : K \to P(K)$ and proved some strong convergence results using Mann and Ishikawa type iterative algorithms. Song and Cho [24] improved the results of [26] where as Khan and Yildirim [12] used an iterative algorithm independent but faster than Ishikawa algorithm to further generalize the results of [24].

Recently, Khan et al. [11] considered the following implicit iterative algorithm for single-valued maps:

$$x_n = \alpha_n x_{n-1} + \beta_n S x_n + \gamma_n T x_n, \quad n \in \mathbb{N} \tag{1.2}$$

where $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$ with $\alpha_n + \beta_n + \gamma_n = 1$.

In this paper, we first give a multivalued version of the algorithm (1.2) in hyperbolic spaces and use $P_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}$ instead of $Tp = \{p\}$ for any $p \in F(T)$ to approximate common fixed points of two multivalued nonexpansive maps. We confirm some simple but useful properties of P_T in hyperbolic spaces basically proved in Banach spaces by Song and Cho [24]. Moreover, we use the method of direct construction of Cauchy sequence as indicated by Song and Cho [24] (and opposed to [26]) but also used by many other authors including [9, 10–12]. The algorithm we use in this paper is as under.

Let K be a nonempty convex subset of a hyperbolic space X. Let $S, T : K \to C(K)$ be two multivalued maps and $P_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}$. Choose $x_0 \in K$. Define $\{x_n\}$ as

$$x_n = W\left(z_n, W\left(x_{n-1}, y_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right) \tag{1.3}$$

where $y_n \in P_S(x_n), z_n \in P_T(x_n)$, and $\alpha_n, \beta_n \in (0,1)$ such that $\alpha_n + \beta_n < 1$.

In order to verify that the algorithm (1.3) is well-defined, define $f : K \to K$ by $f(x) = W\left(z, W\left(x_0, y, \frac{\beta_1}{1-\alpha_1}\right), \alpha_1\right)$ for some $y \in P_S(x)$ and for some $z \in P_T(x)$. For a given $x_0 \in K$, the existence of x_1 is guaranteed if f has a fixed point. Now, for any $u, v \in K$, let $y \in P_S(u), y' \in P_S(v)$, $z \in P_T(u)$, $z' \in P_T(v)$. On using (iv) of Definition 1, we have

$$\begin{aligned}
&d(f(u), f(v)) \\
&\leq (1-\alpha_1)\, d\left(W\left(x_0, y, \frac{\beta_1}{1-\alpha_1}\right), W\left(x_0, y', \frac{\beta_1}{1-\alpha_1}\right)\right) + \alpha_1 d(z, z') \\
&\leq \alpha_1 d(z, z') + \beta_1 d(y, y') \\
&\leq \alpha_1 d(z, P_T(v)) + \beta_1 d(y, P_S(v)) \\
&\leq \alpha_1 H(P_T(u), P_T(v)) + \beta_1 H(P_S(u), P_S(v)) \\
&\leq \alpha_1 d(u, v) + \beta_1 d(u, v) \\
&\leq (\alpha_1 + \beta_1)\, d(u, v).
\end{aligned}$$

Since $\alpha_1 + \beta_1 \in (0,1)$, therefore f is a contraction. By Banach contraction principle, f has a unique fixed point. Thus the existence of x_1 is established. Continuing in this way, the existence of $x_2, x_3, \ldots$ is guaranteed. Hence the above algorithm is well-defined.

In 1976, Lim [18] introduced the concept $\triangle$-convergence in metric spaces. In 2008, Kirk and Panyanak [14] specialized Lim's concept to $CAT(0)$ spaces and proved a number of results involving weak convergence in Banach spaces. Since then the notion of $\triangle$-convergence has been widely studied and a number of articles have appeared (e.g., [2, 3, 14–16]). To reach the definition of $\triangle$-convergence, we first recall the notions of asymptotic radius and asymptotic center as under.

Let $\{x_n\}$ be a bounded sequence in a metric space X. For $x \in X$, define a continuous functional $r(x, \{x_n\})$ by:

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

Then:

(i) $r_K(\{x_n\}) = \inf\{r\,(x, \{x_n\}) : x \in K\}$ of $\{x_n\}$ is called the asymptotic radius of $\{x_n\}$ with respect to $K \subset X$;

(ii) for any $y \in K$, the set $A_K(\{x_n\}) = \{x \in X : r(x, \{x_n\} \leq r(y, \{x_n\})\}$ is called the asymptotic center of $\{x_n\}$ with respect to $K \subset X$.

If the asymptotic radius and the asymptotic center are taken with respect to X, then these are simply denoted by $r(\{x_n\})$ and $A(\{x_n\})$, respectively. In general, $A(\{x_n\})$ may be empty or may even contain infinitely many points. It is known that a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity enjoys the property that bounded sequences have unique asymptotic center with respect to closed convex subsets [17].

A sequence $\{x_n\}$ in X is said to $\triangle$-converge to $x \in X$ if x is the unique asymptotic center of $\{x_{n_i}\}$ for every subsequence $\{x_{n_i}\}$ of $\{x_n\}$. In this case, we call x as $\triangle$-limit of $\{x_n\}$ and write $\triangle\text{-}\lim_n x_n = x$.

For the development of our main results, some key results are listed in the form of lemmas:

Lemma 1.1 [7]. *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space and $\{x_n\}$ a bounded sequence in K such that $A(\{x_n\}) = \{y\}$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = r\,(y, \{x_n\})$, then $\lim_{m\to\infty} y_m = y$.*

Lemma 1.2 [7]. *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \leq r$, $\limsup_{n\to\infty} d(y_n, x) \leq r$ and $\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = r$ for some $r \geq 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

2. Main results

Song and Cho [24] proved that the sets of fixed points of multivalued maps T and P_T are equal in Banach spaces. We confirm the same in hyperbolic spaces in the following lemma. Note that the first lemma remains true even in metric spaces.

Lemma 2.1. *Let K be a nonempty subset of a metric space X and $T : K \to C(K)$ be a multivalued map. Then the followings are equivalent:*

(i) $x \in F\,(T)$, that is, $x \in Tx$;

(ii) $P_T\,(x) = \{x\}$, that is, $x = y$ for each $y \in P_T\,(x)$;

(iii) $x \in F\,(P_T)$, that is, $x \in P_T\,(x)$.

Moreover, $F\,(T) = F\,(P_T)$.

Proof. (i) implies (ii). Since $x \in Tx$, then $d(x, Tx) = 0$. Therefore, for any $y \in P_T(x)$, $d(x, y) = 0$. That is $x = y$. This proves that $P_T(x) = \{x\}$.

Clearly, (iii) is an immediate consequence of (ii).

(iii) implies (i). Since $x \in P_T(x), d(x, Tx) = d(x, x) = 0$ and so $x \in Tx$ by the closedness of Tx. ■

In the sequel, $F = F(S) \cap F(T)$ is the set of all common fixed points of the multivalued maps S and T.

LEMMA 2.2. *Let K be a nonempty closed convex subset of a hyperbolic space X and let $S, T : K \to C(K)$ be two multivalued maps such that P_T and P_S are nonexpansive maps and $F \neq \emptyset$. Then for the sequence $\{x_n\}$ in (1.3), $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$.*

Proof. Let $p \in F$. Then $p \in P_T(p) = \{p\}$ and $p \in P_S(p) = \{p\}$. Using (1.3), we have

$$\begin{aligned}
d(x_n, p) &= d\left(W\left(z_n, W\left(x_{n-1}, y_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) \\
&\le \alpha_n d(z_n, p) + (1-\alpha_n) d\left(W\left(x_{n-1}, y_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \\
&\le \alpha_n d(z_n, p) + \beta_n d(x_{n-1}, p) + (1-\alpha_n-\beta_n) d(y_n, p) \\
&= \alpha_n d(z_n, P_T(p)) + \beta_n d(x_{n-1}, p) + (1-\alpha_n-\beta_n) d(y_n, P_S(p)) \\
&\le \alpha_n H(P_T(x_n), P_T(p)) + \beta_n d(x_{n-1}, p) + (1-\alpha_n-\beta_n) H(P_S(x_n), P_S(p)) \\
&\le \alpha_n d(x_n, p) + \beta_n d(x_{n-1}, p) + (1-\alpha_n-\beta_n) d(x_n, p).
\end{aligned}$$

This implies $\beta_n d(x_n, p) \le \beta_n d(x_{n-1}, p)$ so that

$$d(x_n, p) \le d(x_{n-1}, p).$$

This means that $\{d(x_n, p)\}$ is decreasing and bounded below. Therefore $\lim\limits_{n\to\infty} d(x_n, p)$ exists. ■

LEMMA 2.3. *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space X and let $S, T : K \to C(K)$ be two multivalued maps such that P_T and P_S are nonexpansive and $F \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy $0 < a \le \alpha_n, \beta_n \le b < 1$. Then for the sequence $\{x_n\}$ in (1.3) we have $\lim_{n\to\infty} d(x_n, P_S(x_n)) = 0 = \lim_{n\to\infty} d(x_n, P_T(x_n))$.*

Proof. It follows from Lemma 2.2 that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. Assume that $\lim_{n\to\infty} d(x_n, p) = c$ for some $c \ge 0$. For $c = 0$, the result is trivial. Let us proceed for $c > 0$.

An equivalent form of $\lim_{n\to\infty} d(x_n, p) = c$ is

$$\lim_{n\to\infty} d\left(W\left(z_n, W\left(x_{n-1}, y_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right) = c. \tag{2.1}$$

Since P_T is nonexpansive and $F \neq \emptyset$, we have

$$\begin{aligned} d(y_n, p) &\leq d(y_n, P_S(p)) \leq H(P_S(x_n), P_S(p)) \\ &\leq d(x_n, p) \leq d(x_{n-1}, p) \text{ for each } p \in F. \end{aligned}$$

The above inequality after taking lim sup becomes

$$\limsup_{n\to\infty} d(y_n, p) \leq c. \tag{2.2}$$

Similarly, we can show that

$$\limsup_{n\to\infty} d(z_n, p) \leq c. \tag{2.3}$$

Since $d(y_n, p) \leq d(x_{n-1}, p)$, therefore

$$\begin{aligned} d\left(W\left(x_{n-1}, y_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) &\leq \frac{\beta_n}{1-\alpha_n} d(x_{n-1}, p) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(y_n, p) \\ &\leq \frac{\beta_n}{1-\alpha_n} d(x_{n-1}, p) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(x_{n-1}, p) \\ &= d(x_{n-1}, p). \end{aligned}$$

Taking lim sup on both sides in the above estimate, we have

$$\limsup_{n\to\infty} d\left(W\left(x_{n-1}, y_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \leq c. \tag{2.4}$$

From (2.1), (2.3), (2.4) and Lemma 1.2, it follows that

$$\lim_{n\to\infty} d\left(W\left(x_{n-1}, y_n, \frac{\beta_n}{1-\alpha_n}\right), z_n\right) = 0. \tag{2.5}$$

The estimate

$$\begin{aligned} d(x_n, z_n) &\leq d\left(W\left(z_n, W\left(x_{n-1}, y_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), z_n\right) \\ &\leq \alpha_n d(z_n, z_n) + (1-\alpha_n) d\left(W\left(x_{n-1}, y_n, \frac{\beta_n}{1-\alpha_n}\right), z_n\right) \\ &= (1-\alpha_n) d\left(W\left(x_{n-1}, y_n, \frac{\beta_n}{1-\alpha_n}\right), z_n\right) \\ &\leq (1-a) d\left(W\left(x_{n-1}, y_n, \frac{\beta_n}{1-\alpha_n}\right), z_n\right) \end{aligned}$$

and (2.5) give that

$$\lim_{n\to\infty} d(x_n, z_n) = 0. \tag{2.6}$$

Now

$$d(x_n, p) = d\left(W\left(z_n, W\left(x_{n-1}, y_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), p\right)$$

$$\leq \alpha_n d(z_n, p) + (1-\alpha_n) d\left(W\left(x_{n-1}, y_n, \frac{\beta_n}{1-\alpha_n}\right), p\right)$$
$$\leq \alpha_n d(x_n, p) + (1-\alpha_n) d\left(W\left(x_{n-1}, y_n, \frac{\beta_n}{1-\alpha_n}\right), p\right)$$

means $(1-\alpha_n)d(x_n,p) \leq (1-\alpha_n)d\left(W\left(x_{n-1}, y_n, \frac{\beta_n}{1-\alpha_n}\right), p\right)$ or $d(x_n,p) \leq d\left(W\left(x_{n-1}, y_n, \frac{\beta_n}{1-\alpha_n}\right), p\right)$. Hence

$$c \leq \liminf_{n\to\infty} d\left(W\left(x_{n-1}, y_n, \frac{\beta_n}{1-\alpha_n}\right), p\right). \tag{2.7}$$

Combining (2.4) and (2.7), we get

$$\lim_{n\to\infty} d\left(W\left(x_{n-1}, y_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) = c. \tag{2.8}$$

On utilization of Lemma 1.2 for (2.2), (2.8), we get

$$\lim_{n\to\infty} d(x_{n-1}, y_n) = 0. \tag{2.9}$$

Next, we consider

$$\begin{aligned}
d(x_n, x_{n-1}) &\leq d\left(W\left(z_n, W\left(x_{n-1}, y_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), x_{n-1}\right)\\
&\leq \alpha_n d(z_n, x_{n-1}) + (1-\alpha_n)\, d\left(W\left(x_{n-1}, y_n, \frac{\beta_n}{1-\alpha_n}\right), x_{n-1}\right)\\
&\leq \alpha_n \left\{d\left(z_n, x_n\right) + d\left(x_n, x_{n-1}\right)\right\} + \left(1-\alpha_n-\beta_n\right) d\left(x_{n-1}, y_n\right)\\
&\leq \frac{\alpha_n}{1-\alpha_n} d\left(z_n, x_n\right) + \left(\frac{1-\alpha_n-\beta_n}{1-\alpha_n}\right) d\left(x_{n-1}, y_n\right)\\
&\leq \frac{b}{1-b} d\left(z_n, x_n\right) + \frac{1-2a}{1-b} d\left(x_{n-1}, y_n\right).
\end{aligned}$$

Taking $\limsup$ on both sides in the above estimate and then utilizing (2.6) and (2.9), we get

$$\lim_{n\to\infty} d(x_n, x_{n-1}) = 0. \tag{2.10}$$

The inequality $d(x_n, y_n) \leq d(x_n, x_{n-1}) + d(x_{n-1}, y_n)$ together with (2.9) and (2.10) gives that

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Since $d(x, P_S(x)) = \inf_{z\in P_S(x)} d(x, z)$, therefore

$$d(x_n, P_S(x_n)) \leq d(x_n, y_n) \to 0 \text{ as } n \to \infty.$$

Similarly

$$d(x_n, P_T(x_n)) \leq d(x_n, z_n) \to 0 \text{ as } n \to \infty. \quad \blacksquare$$

Using above lemma, we prove $\triangle$-convergence of the algorithm (1.3).

THEOREM 2.4. *Let K be a nonempty, closed and convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and S, T, P_T, P_S and $\{x_n\}$ be as in Lemma 2.3. Then $\{x_n\}$ $\triangle$-converges to a common fixed point of S and T (or P_S and P_T).*

Proof. Note that $\{x_n\}$ has a unique asymptotic center because $\{x_n\}$ is bounded (by Lemma 2.2). That is, $A(\{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Then by Lemma 2.3, we have $\lim_{n\to\infty} d(u_n, P_T(u_n)) = 0 = \lim_{n\to\infty} d(u_n, P_S(u_n))$. We claim that u is a common fixed point of P_S and P_T.

To prove this, take $\{z_m\}$ in $P_T(u)$. Then

$$\begin{aligned} r(z_m, \{u_n\}) &= \limsup_{n\to\infty} d(z_m, u_n) \\ &\leq \limsup_{n\to\infty} \{d(z_m, P_T(u_n)) + d(P_T(u_n), u_n)\} \\ &\leq \limsup_{n\to\infty} H(P_T(u), P_T(u_n)) \\ &\leq \limsup_{n\to\infty} d(u, u_n) \\ &= r(u, \{u_n\}). \end{aligned}$$

This implies that $|r(z_m, \{u_n\}) - r(u, \{u_n\})| \to 0$ as $m \to \infty$. It follows from Lemma 1.1 that $\lim_{m\to\infty} z_m = u$. Since $P_T(K)$ is a closed and bounded subset of K, therefore $P_T(u)$ is closed and consequently $\lim_{m\to\infty} z_m = u \in P_T(u)$. Hence $u \in F(P_T)$. Similarly, $u \in F(P_T)$. Hence $u \in F$. Since $\lim_{n\to\infty} d(x_n, u)$ exists (by Lemma 2.2), therefore by the uniqueness of asymptotic centers, we have:

$$\begin{aligned} \limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x) \leq \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, u) = \limsup_{n\to\infty} d(u_n, u), \end{aligned}$$

a contradiction. Hence $x = u$.

Thus $A(\{u_n\}) = \{u\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ $\triangle$-converges to a common fixed point of S and T (or P_S and P_T). ■

Next, we give a necessary and sufficient condition for the strong convergence of the algorithm (1.3).

THEOREM 2.5. *Let K be a nonempty, closed and convex subset of a complete hyperbolic space X and S, T, P_T, P_S and $\{x_n\}$ be as in Lemma 2.2. Then the sequence $\{x_n\}$ converges strongly to $p \in F$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Proof. If $\{x_n\}$ converges to $p \in F$, then $\lim_{n\to\infty} d(x_n, p) = 0$. Since $0 \leq d(x_n, F) \leq d(x_n, p)$, we have $\liminf_{n\to\infty} d(x_n, F) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. From Lemma 2.2, we have

$$d(x_{n+1}, F) \leq d(x_n, F).$$

Therefore $\lim_{n\to\infty} d(x_n, F)$ exists. By hypothesis $\liminf_{n\to\infty} d(x_n, F) = 0$, so we have $\lim_{n\to\infty} d(x_n, F) = 0$.

Next, we claim that $\{x_n\}$ is a Cauchy sequence in K. In fact, let $m, n \in N$ and assume $m > n$. Then it follows (as done in Lemma 2.2) that

$$d(x_m, p) \leq d(x_n, p) \text{ for all } p \in F.$$

Thus we have

$$d(x_m, x_n) \leq d(x_m, p) + d(x_n, p) \leq 2d(x_n, p).$$

Taking inf on the set F, we have $d(x_m, x_n) \leq d(x_n, F)$. On letting $m \to \infty, n \to \infty$, the inequality $d(x_m, x_n) \leq d(x_n, F)$ gives that $\{x_n\}$ is a Cauchy sequence in K. The completeness of X guarantees that $\{x_n\}$ converges, say to q.

Next, we show that $q \in F$. Indeed, by $d\left(x_n, F\left(P_T\right)\right) = \inf_{y\in F(P_T)} d\left(x_n, y\right)$. So for each $\epsilon > 0$, there exists $p_n^{(\epsilon)} \in F\left(P_T\right)$ such that

$$d\left(x_n, p_n^{(\epsilon)}\right) \leq d(x_n, F\left(P_T\right)) + \frac{\epsilon}{3}.$$

This implies $\lim_{n\to\infty} d\left(x_n, p_n^{(\epsilon)}\right) \leq \frac{\epsilon}{3}$. From the inequality $d\left(p_n^{(\epsilon)}, q\right) \leq d\left(x_n, p_n^{(\epsilon)}\right) + d(x_n, q)$, it follows that

$$\limsup_{n\to\infty} d\left(p_n^{(\epsilon)}, q\right) \leq \frac{\epsilon}{3}.$$

Finally the estimate

$$\begin{aligned} d(P_T\left(q\right), q) &\leq d\left(q, p_n^{(\epsilon)}\right) + d\left(p_n^{(\epsilon)}, P_T\left(q\right)\right) \\ &\leq d\left(q, p_n^{(\epsilon)}\right) + H\left(P_T\left(p_n^{(\epsilon)}\right), P_T\left(q\right)\right) \\ &\leq 2d\left(p_n^{(\epsilon)}, q\right). \end{aligned}$$

gives that $d(P_T\left(q\right), q) < \epsilon$. Since ϵ is arbitrary, therefore $d(P_T\left(q\right), q) = 0$. Similarly we can show that $d(P_S\left(q\right), q) = 0$. Since F is closed, $q \in F$ as required. ■

The following definitions are needed for further use.

(i) A map $T : K \to CB(K)$ is *semi-compact* if any bounded sequence $\{x_n\}$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$, has a convergent subsequence.

(ii) Let f be a nondecreasing selfmap on $[0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ and let $d(x, F) = \inf\{d(x, y) : y \in F\}$. Let $S, T : K \to CB(K)$ be two multi-valued maps with $F \neq \emptyset$. Then the two maps are said to satisfy condition (A) if

$$d\left(x, Tx\right) \geq f(d(x, F)) \text{ or } d\left(x, Sx\right) \geq f(d(x, F)) \text{ for all } x \in K.$$

The following strong convergence results can easily be obtained by applying Lemma 2.3.

Theorem 2.6. *Let K be a nonempty closed convex subset of a complete and uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and S, T, P_T, P_S and $\{x_n\}$ be as in Lemma* 2.3. *Suppose that a pair of maps P_T and P_S satisfy condition* (A), *then the sequence $\{x_n\}$ defined in* (1.3) *converges strongly to $p \in F$.*

Theorem 2.7. *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and S, T, P_T, P_S and $\{x_n\}$ be as in Lemma* 2.3. *Suppose that one of the maps in P_T and P_S is semi-compact, then the sequence $\{x_n\}$ defined in* (1.3) *converges strongly to $p \in F$.*

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S. H. Khan, Department of Mathematics, Statistics & Physics, Qatar University, Doha 2713, Qatar

E-mail: safeerhussain5@yahoo.com, safeer@qu.edu.qa

H. Fukhar-ud-din, Department of Mathematics & Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia and Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur 63100, Pakistan

E-mail: hfdin@kfupm.edu.sa, hfdin@yahoo.com

A. Kalsoom, Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur 63100, Pakistan

E-mail: amna_iub@yahoo.com