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ON LINEAR MAPS APPROXIMATELY PRESERVING THE APPROXIMATE POINT SPECTRUM OR THE SURJECTIVITY SPECTRUM

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Abstract. Let X and Y be superreflexive complex Banach spaces and let $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ be the Banach algebras of all bounded linear operators on X and Y, respectively. We describe a linear map $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ that almost preserves the approximate point spectrum or the surjectivity spectrum. Furthermore, in the case where $X = Y$ is a separable complex Hilbert space, we show that such a map is a small perturbation of an automorphism or an anti-automorphism.

1. Introduction

Many authors are interested in describing additive or linear maps that preserve, compress or depress some distinguished parts of the spectrum of an operator acting between Banach spaces (see, among others [2–4, 9]). Among these parts, the approximate point spectrum and the surjectivity spectrum are of special interest.

Recently, in [1], linear maps on $\mathcal{L}(X)$, which almost preserve or almost compress the spectrum are studied. Motivated by the approximate versions of preserving and compressing the spectrum discussed in [1], we identify in this note the approximately multiplicative or anti-multiplicative linear maps among all linear maps $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ that almost preserve or almost compress the approximate point spectrum or the surjectivity spectrum.

2. Notations and preliminaries

Let X and Y be two complex Banach spaces and let $\mathcal{L}(X, Y)$ be the Banach space of all bounded operators from X into Y. As usual, we abbreviate $\mathcal{L}(X, X)$ to $\mathcal{L}(X)$. Let dist_H denote the Hausdorff distance (on the set of compact subsets of $\mathbb{C}$) and B_X the closed unit ball of X. We write $\mathrm{D} = \{z \in \mathbb{C} :| z |< 1\}$.

Recall that the minimum modulus and the surjectivity modulus of an operator $T \in \mathcal{L}(X, Y)$ are defined respectively, see [7], by

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$$\mathrm{m}(T) = \inf\{\|Tx\| : x \in X, \|x\| = 1\} \text{ and } \mathrm{q}(T) = \sup\{r \geq 0 : rB_Y \subset TB_X\}.$$

Note that $\mathrm{m}(T) > 0$ if and only if T is bounded below, i.e., T is injective and has closed range, and $\mathrm{q}(T) > 0$ if and only if T is surjective. The approximate point spectrum and the surjectivity spectrum of T are given respectively by $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \mathrm{m}(T - \lambda) = 0\}$ and $\sigma_{su}(T) = \{\lambda \in \mathbb{C} : \mathrm{q}(T - \lambda) = 0\}$. Recall also that $\mathrm{m}(T^*) = \mathrm{q}(T)$ and $\mathrm{q}(T^*) = \mathrm{m}(T)$ where $T^* \in \mathcal{L}(Y^*, X^*)$ is the adjoint of T acting between the dual spaces of Y and X.

Let $T \in \mathcal{L}(X, Y)$. We introduce the two following subsets of $\mathbb{C}$ denoted $\sigma^{\epsilon}_{ap}(T)$ and $\sigma^{\epsilon}_{su}(T)$ and defined by

$$\sigma^{\epsilon}_{ap}(T) := \{\lambda \in \mathbb{C} : \mathrm{m}(T - \lambda) < \epsilon\}$$

and

$$\sigma^{\epsilon}_{su}(T) := \{\lambda \in \mathbb{C} : \mathrm{q}(T - \lambda) < \epsilon\}$$

for $\epsilon > 0$. We use the terms *pseudo approximate point spectrum* and *pseudo surjectivity spectrum* to designate them respectively. It is clear that $\sigma_{ap}(T) \subset \sigma^{\epsilon}_{ap}(T)$ and $\sigma_{su}(T) \subset \sigma^{\epsilon}_{su}(T)$.

Throughout this paper $\sigma_*(T)$ denotes $\sigma_{ap}(T)$ or $\sigma_{su}(T)$ and $\sigma^{\epsilon}_*(T)$ denotes $\sigma^{\epsilon}_{ap}(T)$ or $\sigma^{\epsilon}_{su}(T)$. Let ω denote the minimum modulus if $* = ap$ and let it denote the surjectivity modulus if $* = su$.

We will make an extensive use of the following result.

Lemma 2.1. *Let $T \in \mathcal{L}(X)$. Then the following assertions hold.*

(i) $\sigma_*(T) = \bigcap_{\epsilon>0} \sigma^{\epsilon}_*(T)$.

(ii) $\sigma^{\epsilon_1}_*(T) \subset \sigma^{\epsilon_2}_*(T)$ *for all* $0 < \epsilon_1 < \epsilon_2$.

(iii) $\alpha\sigma^{\epsilon}_*(T) = \sigma^{|\alpha|\epsilon}_*(\alpha T)$ *for all* $\alpha \neq 0$ *and* $\epsilon > 0$.

(iv) $\sigma_*(T) + \epsilon \mathrm{D} \subset \sigma^{\epsilon}_*(T)$ *for all* $\epsilon > 0$.

(v) $\sigma^{\epsilon}_*(T + S) \subset \sigma^{\epsilon+\|S\|}_*(T)$ *for all* $\epsilon > 0$ *and* $S \in \mathcal{L}(X)$.

(vi) $\sigma_*(T + S) \subset \sigma^{\epsilon}_*(T)$ *for all* $\epsilon > 0$ *and* $S \in \mathcal{L}(X)$ *with* $\|S\| < \epsilon$.

(vii) $\sigma^{\epsilon}_*(T) \subset \bigcup\{\sigma_*(T + S) : S \in \mathcal{L}(X), \|S\| < \epsilon\}$ *for all* $\epsilon > 0$.

Proof. It is immediate to check the assertions (i), (ii) and (iii).

Let $T \in \mathcal{L}(X)$. It is easy to see that $\omega(T + S) \geq \omega(T) - \|S\|$ for all $S \in \mathcal{L}(X)$.

Let $\lambda \in \sigma_*(T)$ and let $\alpha \in \mathbb{C}$ such that $|\alpha| < \epsilon$. It turns out that $\omega(T - \lambda - \alpha) - |\alpha| \leq \omega(T - \lambda) = 0$ and so $\omega(T - \lambda - \alpha) < \epsilon$ which yields (iv).

In order to check (v), let $S \in \mathcal{L}(X)$ and assume that $\lambda \notin \sigma^{\epsilon+\|S\|}_*(T)$. Then we have $\omega(T - \lambda) \geq \epsilon + \|S\|$. Therefore we get $\omega(T + S - \lambda) \geq \omega(T - \lambda) - \|S\| \geq \epsilon$. Thus $\lambda \notin \sigma^{\epsilon}_*(T + S)$.

Now, let $S \in \mathcal{L}(X)$ with $\|S\| < \epsilon$ and $\lambda \notin \sigma^{\epsilon}_*(T)$. Then $\omega(T + S - \lambda) \geq \omega(T - \lambda) - \|S\| \geq \epsilon - \|S\| > 0$. This completes the proof of the assertion (vi).

If $\lambda \notin \sigma_*(T + S)$ for all $S \in \mathcal{L}(X)$ with $\|S\| < \epsilon$, then $\omega(T + S - \lambda) > 0$ for all $S \in \mathcal{L}(X)$ with $\|S\| < \epsilon$. Observe that $\omega(T) = \sup\{r > 0, \omega(T - S) > 0$ for all $S \in \mathcal{L}(X, Y), \|S\| < r\}$, see [7, Proposition II.9.10]. Hence we get that $\omega(T - \lambda) \geq \epsilon$ and so (vii) holds. ■

Let us give another basic tool that will be used later in this paper.

Let $\mathcal{U} \subset \mathcal{P}(\mathbb{N})$ be a free ultrafilter on $\mathbb{N}$ and denote by $\mu_{\mathcal{U}}$ the finitely additive $\{0,1\}$-valued measure on $\mathbb{N}$, given by $\mu_{\mathcal{U}}(A) = 1$ if $A \in \mathcal{U}$.

We consider the Banach space $\ell^\infty(X)$ of all bounded sequences (x_n) with $x_n \in X$ for all $n \in \mathbb{N}$, equipped with the norm $\|(x_n)\| := \sup_n \|x_n\|$. Then $\mathcal{N}_{\mathcal{U}}(X) := \{(x_n) \in \ell^\infty(X) : \lim_{\mathcal{U}} \|x_n\| = 0\}$ is a closed linear subspace of $\ell^\infty(X)$. The quotient Banach space $X^{\mathcal{U}} := \ell^\infty(X)/\mathcal{N}_{\mathcal{U}}(X)$ is called the *ultrapower* of X with respect to $\mathcal{U}$. We continue to denote the equivalence class of (x_n) also by (x_n). It should cause no confusion if we denote $(x_n)_{n\in M}$ by $\hat{x}$ where $\mu_{\mathcal{U}}(M) = 1$ and $x_n \in X$ for all $n \in M$. The norm on $X^{\mathcal{U}}$ is given by

$$\|\hat{x}\| = \lim_{\mathcal{U}} \|x_n\| \quad \text{where} \quad \hat{x} = (x_n) \in X^{\mathcal{U}}.$$

The ultrapower $\mathcal{L}(X)^{\mathcal{U}}$ is a Banach algebra with respect to the product

$$\hat{T}\hat{S} = (T_n S_n) \quad \text{where} \quad \hat{T} = (T_n), \quad \hat{S} = (S_n) \in \mathcal{L}(X)^{\mathcal{U}}.$$

There exists a canonical isometric linear map $\mathcal{L}(X,Y)^{\mathcal{U}} \to \mathcal{L}(X^{\mathcal{U}}, Y^{\mathcal{U}})$ which is defined by

$$\hat{T}(\hat{x}) = (T_n x_n) \quad \text{where} \quad \hat{T} = (T_n) \in \mathcal{L}(X,Y)^{\mathcal{U}} \quad \text{and} \quad \hat{x} = (x_n) \in X^{\mathcal{U}}.$$

We consider $\mathcal{L}(X,Y)^{\mathcal{U}}$ as being a closed subspace of $\mathcal{L}(X^{\mathcal{U}}, Y^{\mathcal{U}})$. For more details on ultrapowers, we refer the reader to [10].

Lemma 2.2. *Let X and Y be complex Banach spaces and $\hat{T} = (T_n) \in \mathcal{L}(X,Y)^{\mathcal{U}} \subset \mathcal{L}(X^{\mathcal{U}}, Y^{\mathcal{U}})$. Then:*

(i) $\mathrm{m}(\hat{T}) = \lim_{\mathcal{U}} \mathrm{m}(T_n)$.

(ii) $\mathrm{q}(\hat{T}) = \lim_{\mathcal{U}} \mathrm{q}(T_n)$.

Proof. (i) According to [7, Theorem II.9.11], we have

$$\mathrm{m}(T) = \inf\{\|TS\|, S \in \mathcal{L}(Y), \|S\| = 1\}.$$

Let $\epsilon > 0$. Then for each $n \in \mathbb{N}$ there exists $S_n \in \mathcal{L}(Y)$ with $\|S_n\| = 1$ and

$$\|T_n S_n\| < \mathrm{m}(T_n) + \epsilon.$$

Let $\hat{S} = (S_n) \in \mathcal{L}(Y)^{\mathcal{U}}$. Since $\|\hat{S}\| = 1$, it turns out that

$$\mathrm{m}(\hat{T}) \le \|\hat{T}\hat{S}\| = \lim_{\mathcal{U}} \|T_n S_n\| \le \lim_{\mathcal{U}} \mathrm{m}(T_n) + \epsilon$$

which gives $\mathrm{m}(\hat{T}) \le \lim_{\mathcal{U}} \mathrm{m}(T_n)$.

Let $\hat{T} = (T_n) \in \mathcal{L}(X,Y)^{\mathcal{U}} \subset \mathcal{L}(X^{\mathcal{U}}, Y^{\mathcal{U}})$. Let $\hat{x} = (x_n) \in X^{\mathcal{U}}$. We have

$$\|\hat{T}\hat{x}\| = \lim_{\mathcal{U}} \|T_n x_n\| \ge \lim_{\mathcal{U}} \mathrm{m}(T_n)\|x_n\| = \lim_{\mathcal{U}} \mathrm{m}(T_n)\|\hat{x}\|$$

and so $\mathrm{m}(\hat{T}) \ge \lim_{\mathcal{U}} \mathrm{m}(T_n)$.

(ii) See [1, Lemma 2.5]. ∎

Lemma 2.3. *Let X be a complex Banach space, and let $\hat{S} = (S_n), \hat{T} = (T_n) \in \mathcal{L}(X)^{\mathcal{U}} \subset \mathcal{L}(X^{\mathcal{U}})$. Suppose that there are bounded sequences of positive numbers (ϵ_n) and (δ_n) such that $\sigma_*^{\epsilon_n}(S_n) \subset \sigma_*^{\delta_n}(T_n)$ almost everywhere on $\mathbb{N}$. Then $\sigma_*^{\epsilon}(\hat{S}) \subset \sigma_*^{\delta}(\hat{T})$ whenever $\epsilon, \delta > 0$ are such that $\epsilon < \lim_{\mathcal{U}} \epsilon_n$ and $\delta > \lim_{\mathcal{U}} \delta_n$.*

Proof. Let $0 < \epsilon < \epsilon' < \lim_{\mathcal{U}} \epsilon_n$ and $\lim_{\mathcal{U}} \delta_n < \delta' < \delta$. So $\epsilon' < \epsilon_n$ and $\delta_n < \delta'$ almost everywhere. Set $^\circ\sigma_*^{\epsilon'}((S_n)) := \{\lim_{\mathcal{U}} \lambda_n : \lambda_n \in \sigma_*^{\epsilon'}(S_n)\ \ \mu_{\mathcal{U}}\text{-almost everywhere}\ \}$. First we establish that $\sigma_*^{\epsilon}(\hat{S}) \subset {^\circ\sigma_*^{\epsilon'}}((S_n))$. Let $\lambda \notin {^\circ\sigma_*^{\epsilon'}}((S_n))$, so $\lambda \notin \sigma_*^{\epsilon'}(S_n)$ $\mu_{\mathcal{U}}$-almost everywhere and hence $\omega(S_n - \lambda) \geq \epsilon'$ $\mu_{\mathcal{U}}$-almost everywhere. Therefore $\omega(\hat{S} - \lambda) = \lim_{\mathcal{U}} \omega(S_n - \lambda) \geq \epsilon' > \epsilon$, i.e., $\lambda \notin \sigma_*^{\epsilon}(\hat{S})$.

Now we show that $^\circ\sigma_*^{\epsilon'}((S_n)) \subset \sigma_*^{\delta}(\hat{T})$. Let $\lambda \in {^\circ\sigma_*^{\epsilon'}}((S_n))$, i.e., $\lambda = \lim_{\mathcal{U}} \lambda_n$ where $\lambda_n \in \sigma_*^{\epsilon'}(S_n)$ $\mu_{\mathcal{U}}$-almost everywhere. According to Lemma 2.1 (ii) and the hypothesis of this Lemma, we get

$$\lambda_n \in \sigma_*^{\epsilon'}(S_n) \subset \sigma_*^{\epsilon_n}(S_n) \subset \sigma_*^{\delta_n}(T_n) \subset \sigma_*^{\delta'}(T_n) \quad \text{almost everywhere on } \mathbb{N}.$$

Clearly, $\hat{T} - \lambda = (T_n - \lambda_n)$. We have so $\omega(\hat{T} - \lambda) = \lim_{\mathcal{U}} \omega(T_n - \lambda_n) \leq \delta' < \delta$. This implies that $\lambda \in \sigma_*^{\delta}(\hat{T})$. ∎

The two following lemmas are derived from [1], and adapted to pseudo approximate point spectrum and pseudo surjectivity spectrum.

Lemma 2.4. *Let X and Y be complex Banach spaces and $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a surjective linear map such that*

$$\sigma_*(\phi(T)) \subset \sigma_*^{\delta}(T) \quad \text{for all} \quad T \in \mathcal{L}(X), \|T\| = 1$$

and some $\delta > 0$. Then $\mathrm{q}(\phi) \leq 1 + \delta$ and

$$\sigma_*^{\epsilon}(\phi(T)) \subset \sigma_*^{\delta(\|T\| + \frac{\epsilon}{k}) + \frac{\epsilon}{k}}(T) \quad \text{for all} \quad T \in \mathcal{L}(X),$$

$\epsilon > 0$ and $0 < k < \mathrm{q}(\phi)$.

Proof. Let $T \in \mathcal{L}(X), \epsilon > 0$ and $0 < k < \mathrm{q}(\phi)$. Let then $k < \tau < \mathrm{q}(\phi)$. Let $\lambda \in \sigma_*^{\epsilon}(\phi(T))$. According to Lemma 2.1 (vii), there exists $S \in \mathcal{L}(Y)$ with$\|S\| < \epsilon$ such that $\lambda \in \sigma_*(\phi(T) + S)$. It is clear that $S = \phi(R)$ for some $R \in \mathcal{L}(X)$ such that $\|R\| < \frac{\epsilon}{\tau}$. Indeed, using the definition of $\mathrm{q}(\phi)$ and the fact that $\|\frac{1}{\epsilon} S\| < 1$, there is $R' \in \mathcal{L}(X)$ with $\|R'\| \leq 1$ such that $\mathrm{q}(\phi)\frac{1}{\epsilon} S = \phi(R')$, i.e., $S = \phi(\frac{\epsilon}{\mathrm{q}(\phi)} R')$, then we may take $R = \frac{\epsilon}{\mathrm{q}(\phi)} R'$.

Now, let $0 < \rho < (1 + \delta)(\frac{\epsilon}{k} - \frac{\epsilon}{\tau})$. We first treat the case where $T + R \neq 0$. By Lemma 2.1 (ii),(iii),(v) and our hypothesis, we have

$$\begin{aligned}
\lambda &\in \sigma_*(\phi(T) + S) = \sigma_*(\phi(T + R)) \\
&= \|T + R\| \sigma_* \left(\phi \left(\frac{T + R}{\|T + R\|} \right) \right) \\
&\subset \|T + R\| \sigma_*^{\delta} \left(\frac{T + R}{\|T + R\|} \right) = \sigma_*^{\delta\|T + R\|}(T + R) \subset \sigma_*^{\delta\|T + R\| + \rho}(T + R) \\
&\subset \sigma_*^{\delta(\|T\| + \frac{\epsilon}{\tau}) + \rho}(T + R) \subset \sigma_*^{\delta(\|T\| + \frac{\epsilon}{\tau}) + \rho + \frac{\epsilon}{\tau}}(T) \subset \sigma_*^{\delta(\|T\| + \frac{\epsilon}{k}) + \frac{\epsilon}{k}}(T).
\end{aligned}$$

If $T+R=0$, the inclusion $\sigma_*(\phi(T+R)) \subset \sigma_*^{\delta(\|T\|+\frac{\epsilon}{\tau})+\rho}(T+R)$ is obvious. The rest of inclusions can be checked as in the precedent case.

Since $\epsilon \mathrm{D} = \sigma_*^{\epsilon}(\phi(0)) \subset \sigma_*^{\delta(\|0\|+\frac{\epsilon}{k})+\frac{\epsilon}{k}}(0) = \frac{1+\delta}{k}\epsilon\mathrm{D}$ for all $\epsilon>0$ and $0<k<\mathrm{q}(\phi)$, then $\mathrm{q}(\phi) \leq 1+\delta$. ■

Lemma 2.5. *Let X and Y be complex Banach spaces and $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a continuous linear map such that*

$$\sigma_*(T) \subset \sigma_*^{\delta}(\phi(T)) \quad \textit{for all} \quad T \in \mathcal{L}(X), \|T\| = 1$$

and some $\delta > 0$. Then $1-\delta \leq \|\phi\|$ and

$$\sigma_*^{\epsilon}(T) \subset \sigma_*^{\delta(\|T\|+\epsilon)+\nu\epsilon}(\phi(T)) \quad \textit{for all} \quad T \in \mathcal{L}(X),$$

$\epsilon > 0$ and $\nu > \|\phi\|$.

Proof. Let $T \in \mathcal{L}(X), \epsilon > 0$ and $\nu > \|\phi\|$. Let then $k < \rho < (\nu - \|\phi\|)\epsilon$. Let $\lambda \in \sigma_*^{\epsilon}(T)$. According to Lemma 2.1 (vii), there exists $S \in \mathcal{L}(Y)$ with$\|S\| \leq \epsilon$ such that $\lambda \in \sigma_*(T+S)$.

We proceed as in the proof of Lemma 2.4. If $T+S \neq 0$, we get, by Lemma 2.1 (ii), (vi), that

$$\begin{aligned}
\lambda \in \sigma_*(T+S) &= \|T+S\|\sigma_*\left(\frac{T+S}{\|T+S\|}\right) \\
&\subset \|T+S\|\sigma_*^{\delta}\left(\phi\left(\frac{T+S}{\|T+S\|}\right)\right) = \sigma_*^{\delta\|T+S\|}(\phi(T+S)) \subset \sigma_*^{\delta\|T+S\|+\rho}(\phi(T+S)) \\
&\subset \sigma_*^{\delta(\|T\|+\epsilon)+\rho}(\phi(T)+\phi(S)) \subset \sigma_*^{\delta(\|T\|+\epsilon)+\rho+\|\phi\|\epsilon}(\phi(T)) \subset \sigma_*^{\delta(\|T\|+\epsilon)+\nu\epsilon}(\phi(T))
\end{aligned}$$

If $T+S=0$, obviously, $\sigma_*(T+S) \subset \sigma_*^{\delta(\|T+S\|+\rho}(\phi(T+S))$ and it follows, similarly to the precedent case, the desired inclusion. By taking $T=0$ in the inclusion checked, we have

$$\epsilon\mathrm{D} = \sigma_*^{\epsilon}(0) \subset \sigma_*^{\delta(\|0\|+\epsilon)+\nu\epsilon}(\phi(0)) = (\delta+\nu)\epsilon\mathrm{D}$$

for all $\epsilon > 0$ and $\nu > \|\phi\|$. Then $1-\delta \leq \|\phi\|$. ■

The following result (see for instance [2, 3, 4], will be important in the sequel.

Lemma 2.6. *Let X and Y be complex Banach spaces and let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on X and Y, respectively. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a linear map. Suppose that either of the following conditions hold:*

(1) *$\phi : \mathcal{A} \to \mathcal{B}$ is surjective and $\sigma_*(\phi(T)) = \sigma_*(T)$ for all $T \in \mathcal{A}$ or*

(2) *$\phi : \mathcal{A} \to \mathcal{B}$ is bijective and $\sigma_*(\phi(T)) \subset \sigma_*(T)$ for all $T \in \mathcal{A}$ or*

(3) *$\phi : \mathcal{A} \to \mathcal{B}$ is bijective and $\sigma_*(T) \subset \sigma_*(\phi(T))$ for all $T \in \mathcal{A}$.*

Then either there exists an invertible operator $A \in \mathcal{L}(X,Y)$ such that $\phi(T) = ATA^{-1}$ for all $T \in \mathcal{A}$ or there exists an invertible operator $A \in \mathcal{L}(X^*,Y)$ such that $\phi(T) = AT^*A^{-1}$ for all $T \in \mathcal{A}$. In the last case, X and Y are reflexive.

3. Main results

Before formulating our results, we introduce the following quantities (see [6]) :

$$\text{mult}(\phi) := \sup\{\|\phi(TS) - \phi(T)\phi(S)\| : T, S \in \mathcal{L}(X), \|T\| = \|S\| = 1\},$$
$$\text{amult}(\phi) := \sup\{\|\phi(TS) - \phi(S)\phi(T)\| : T, S \in \mathcal{L}(X), \|T\| = \|S\| = 1\}$$

which allow to measure respectively the *multiplicativity* and the *anti-multiplicativity* of ϕ.

The following theorems are given for superreflexive Banach spaces. For details on this type of spaces, see for instance [5, 11]. Recall that if X is a superreflexive Banach space then the Banach algebra $\mathcal{L}(X)^{\mathcal{U}}$ is an unital standard operator algebra on $X^{\mathcal{U}}$ (see [1, Lemma 2.2].

Proposition 3.1. *Let X and Y be complex Banach spaces. Let (ϕ_n) be a sequence of surjective linear maps from $\mathcal{L}(X)$ onto $\mathcal{L}(Y)$ and let $\hat{\phi}$ be the linear map (ϕ_n) from $\mathcal{L}(X)^{\mathcal{U}} \subset \mathcal{L}(X^{\mathcal{U}})$ into $\mathcal{L}(Y)^{\mathcal{U}} \subset \mathcal{L}(Y^{\mathcal{U}})$. The following assertions hold.*

(i) *If there exist $k, K > 0$ and a sequence of positive numbers (ϵ_n) tending to 0 such that*

$$\sigma_*(\phi_n(T)) \subset \sigma_*^{\epsilon_n}(T) \quad \textit{for all} \quad T \in \mathcal{L}(X), \|T\| = 1,$$
$$q(\phi_n) > k \ \textit{and} \ \|\phi_n\| < K$$

for each $n \in \mathbb{N}$, then

$$\sigma_*(\hat{\phi}(\hat{T})) \subset \sigma_*(\hat{T}) \quad \textit{for all} \quad \hat{T} = (T_n) \in \mathcal{L}(X)^{\mathcal{U}}.$$

(ii) *If there exist $K > 0$ and a sequence of positive numbers (ϵ_n) tending to 0 such that*

$$\sigma_*(T) \subset \sigma_*^{\epsilon_n}(\phi_n(T)) \quad \textit{for all} \quad T \in \mathcal{L}(X), \|T\| = 1$$
$$\textit{and} \ \|\phi_n\| < K$$

for each $n \in \mathbb{N}$, then

$$\sigma_*(\hat{T}) \subset \sigma_*(\hat{\phi}(\hat{T})) \quad \textit{for all} \quad \hat{T} = (T_n) \in \mathcal{L}(X)^{\mathcal{U}}.$$

Proof. (i) Let $\hat{T} = (T_n) \in \mathcal{L}(X)^{\mathcal{U}}$. Let $\epsilon > 0$ and let ρ such that $\epsilon < \rho < (k+1)\epsilon$. Applying Lemma 2.4, we obtain

$$\sigma_*^{\frac{\rho k}{k+1}}(\phi_n(T_n)) \subset \sigma_*^{\epsilon_n(\|T_n\| + \frac{\rho}{k+1}) + \frac{\rho}{k+1}}(T_n) \quad \text{for all} \quad n \in \mathbb{N}.$$

Since $\lim_{\mathcal{U}} \epsilon_n(\|T_n\| + \frac{\rho}{k+1}) + \frac{\rho}{k+1} \le \lim_{\mathcal{U}} \epsilon_n(K + \frac{\rho}{k+1}) + \frac{\rho}{k+1} = \frac{\rho}{k+1} < \epsilon$, it follows by Lemma 2.3, that

$$\sigma_*^{\frac{\epsilon k}{k+1}}(\hat{\phi}(\hat{T})) \subset \sigma_*^{\epsilon}(\hat{T}).$$

Consequently, by Lemma 2.1 (i), it turns out that

$$\sigma_*(\hat{\phi}(\hat{T})) = \cap_{\epsilon>0}\sigma_*^{\frac{\epsilon k}{k+1}}(\hat{\phi}(\hat{T})) \subset \cap_{\epsilon>0}\sigma_*^{\epsilon}(\hat{T}) = \sigma_*(\hat{T}),$$

as desired.

(ii) Let $\epsilon > 0$ and let ρ such that $\epsilon < \rho < (K^{-1}+1)\epsilon$. Applying Lemma 2.5 instead of Lemma 2.4, and using the same technique as in the proof of (i), we conclude the proof of (ii). ∎

In the following theorem, we describe linear maps that almost compress the approximate point spectrum or the surjectivity spectrum.

THEOREM 3.2. *Let X and Y be superreflexive Banach spaces. Then for each $K, \epsilon > 0$ there is $\delta > 0$ such that if $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ is a bijective linear map with*

$$\sigma_*(\phi(T)) \subset \sigma_*^{\delta}(T) \quad \textit{for all} \quad T \in \mathcal{L}(X), \|T\| = 1$$

and $\|\phi\|, \|\phi^{-1}\| < K$, then

$$\min\{\mathrm{mult}(\phi), \mathrm{amult}(\phi)\} < \epsilon.$$

Proof. Suppose that there exist $K, \tau > 0$ and a sequence (ϕ_n) of bijective linear maps from $\mathcal{L}(X)$ onto $\mathcal{L}(Y)$ verifying

$$\sigma_*(\phi_n(T)) \subset \sigma_*^{\frac{1}{n}}(T) \quad \text{for all} \quad T \in \mathcal{L}(X), \|T\| = 1,$$
$$\|\phi_n\|, \|\phi_n^{-1}\| < K$$

and

$$\min\{\mathrm{mult}(\phi_n), \mathrm{amult}(\phi_n)\} \geq \tau$$

for each $n \in \mathbb{N}$. We consider the map

$$\hat{\phi} = (\phi_n) : \mathcal{L}(X)^{\mathcal{U}} \subset \mathcal{L}(X^{\mathcal{U}}) \to \mathcal{L}(Y)^{\mathcal{U}} \subset \mathcal{L}(Y^{\mathcal{U}}).$$

The linear map $\hat{\phi}$ is continuous and by [8, Lemma 2.1] it is bijective with inverse given by $\hat{\phi}^{-1} = (\phi_n^{-1})$.

Observe that $q(\phi_n) = \|\phi_n^{-1}\|^{-1} > K^{-1}$ for each $n \in \mathbb{N}$. Let $\hat{T} = (T_n) \in \mathcal{L}(X)^{\mathcal{U}}$. Using Proposition 3.1 (i), we obtain that

$$\sigma_*(\hat{\phi}(\hat{T})) \subset \sigma_*(\hat{T}).$$

Since $\mathcal{L}(X)^{\mathcal{U}}$ and $\mathcal{L}(Y)^{\mathcal{U}}$ are unital standard operator algebras on $X^{\mathcal{U}}$ and $Y^{\mathcal{U}}$ respectively, we get, by Lemma 2.6, that $\hat{\phi}$ is either a homomorphism or an anti-homomorphism. Since $\mathrm{mult}(\hat{\phi}) = \lim_{\mathcal{U}} \mathrm{mult}(\phi_n)$ and $\mathrm{amult}(\hat{\phi}) = \lim_{\mathcal{U}} \mathrm{amult}(\phi_n)$ (see [1, Lemma 3.4]), we have

$$\begin{aligned}\lim_{\mathcal{U}} \min\{\mathrm{mult}(\phi_n), \mathrm{amult}(\phi_n)\} &= \min\{\lim_{\mathcal{U}} \mathrm{mult}(\phi_n), \lim_{\mathcal{U}} \mathrm{amult}(\phi_n)\} \\ &= \min\{\mathrm{mult}(\hat{\phi}), \mathrm{amult}(\hat{\phi})\} = 0\end{aligned}$$

which yields a contradiction. ∎

Note that a Hilbert space is supereflexive. As an application of the above theorem in the context of a Hilbert space, we give the following result.

Corollary 3.3. *Let H be a separable Hilbert space. Then for each $K, \epsilon > 0$ there is $\delta > 0$ such that if $\phi : \mathcal{L}(H) \to \mathcal{L}(H)$ is a bijective linear map with*

$$\sigma_*(\phi(T)) \subset \sigma_*^{\delta}(T) \quad \textit{for all} \quad T \in \mathcal{L}(X), \|T\| = 1$$

and $\|\phi\|, \|\phi^{-1}\| < K$, then $\|\phi - \psi\| < \epsilon$ for some automorphism or anti-automorphism $\psi : \mathcal{L}(H) \to \mathcal{L}(H)$.

Proof. The hypothesis of this Corollary and Theorem 3.2 give immediately for each $K, \delta' > 0$ that $\min\{\mathrm{mult}(\phi), \mathrm{amult}(\phi)\} < \delta'$. It is well known that $\mathrm{jmult}(\phi) \leq \min\{\mathrm{mult}(\phi), \mathrm{amult}(\phi)\}$ where $\mathrm{jmult}(\phi) := \sup\{\|\phi(T^2) - \phi(T)^2\| : T \in \mathcal{L}(H), \|T\| = 1\}$ (see [6]). Therefore $\mathrm{jmult}(\phi) < \delta'$.

Let $\epsilon > 0$ and let $\epsilon' = \min\{\epsilon, \|\phi^{-1}\|^{-1}\}$. Clearly $\mathrm{q}(\phi) = \|\phi^{-1}\|^{-1} > K^{-1}$, we obtain, by [1, Corollary 3.10], that $\|\phi - \psi\| < \epsilon'$ for some epimorphism or anti-epimorphism $\psi : \mathcal{L}(H) \to \mathcal{L}(H)$. Since ϕ is invertible and $\|\phi - \psi\| < \|\phi^{-1}\|^{-1}$, then ψ is invertible. Consequently ψ is either an automorphism or an anti-automorphism. ∎

If we replace $\phi(T)$ by T and T by $\phi(T)$ in Theorem 3.2 we obtain, by using Lemma 2.5 instead of Lemma 2.4, the following theorem.

Theorem 3.4. *Let X and Y be superreflexive Banach spaces. Then for each $K, \epsilon > 0$ there is $\delta > 0$ such that if $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ is a bijective linear map with*

$$\sigma_*(T) \subset \sigma_*^{\delta}(\phi(T)) \quad \textit{for all} \quad T \in \mathcal{L}(X), \|T\| = 1$$

and $\|\phi\|, \|\phi^{-1}\| < K$, then

$$\min\{\mathrm{mult}(\phi), \mathrm{amult}(\phi)\} < \epsilon.$$

Using Theorem 3.4 and the same technique as in Corollary 3.3, we get the following corollary.

Corollary 3.5. *Let H be a separable Hilbert space. Then for each $K, \epsilon > 0$ there is $\delta > 0$ such that if $\phi : \mathcal{L}(H) \to \mathcal{L}(H)$ is a bijective linear map with*

$$\sigma_*(T) \subset \sigma_*^{\delta}(\phi(T)) \quad \textit{for all} \quad T \in \mathcal{L}(H), \|T\| = 1$$

and $\|\phi\|, \|\phi^{-1}\| < K$, then $\|\phi - \psi\| < \epsilon$ for some automorphism or anti-automorphism $\psi : \mathcal{L}(H) \to \mathcal{L}(H)$.

The following theorem gives a description of linear maps which almost preserve the approximate point spectrum or the surjectivity spectrum.

Theorem 3.6. *Let X and Y be superreflexive Banach spaces. Then for each $k, K, \epsilon > 0$ there is $\delta > 0$ such that if $\phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ is a surjective linear map with*

$$\mathrm{dist}_H(\sigma_*(\phi(T)), \sigma_*(T)) < \delta \quad \textit{for all} \quad T \in \mathcal{L}(X), \|T\| = 1,$$

$\mathrm{q}(\phi) > k$ *and* $\|\phi\| < K$, *then*

$$\phi \quad \text{is injective and} \quad \min\{\mathrm{mult}(\phi), \mathrm{amult}(\phi)\} < \epsilon.$$

Proof. Suppose that there exist $k, K, \tau > 0$ and a sequence (ϕ_n) of surjective linear maps from $\mathcal{L}(X)$ onto $\mathcal{L}(Y)$ satisfying

$$\sup_{\|T\|=1} \mathrm{dist}_H(\sigma_*(\phi_n(T)), \sigma_*(T)) \to 0, \quad \mathrm{q}(\phi_n) > k, \quad \|\phi_n\| < K$$

and

$$\phi_n \text{ is not injective or } \min\{\mathrm{mult}(\phi_n), \mathrm{amult}(\phi_n)\} \geq \tau$$

for each $n \in \mathbb{N}$. Let ϵ_n be a sequence of positive numbers such that

$$\lim \epsilon_n = 0 \quad \text{and} \quad \sup_{\|T\|=1} \mathrm{dist}_H(\sigma_*(\phi_n(T)), \sigma_*(T)) < \epsilon_n \quad \text{for all} \quad n \in \mathbb{N}.$$

It is well known that $\mathrm{dist}_H(\sigma_*(\phi_n(T)), \sigma_*(T)) = \max\{\inf\{\epsilon > 0, \quad \sigma_*(\phi_n(T)) \subset \sigma_*(T) + \epsilon \mathrm{D}\}, \inf\{\epsilon > 0, \quad \sigma_*(T) \subset \sigma_*(\phi_n(T)) + \epsilon \mathrm{D}\}\}$. So, by Lemma 2.1 (iv), we get for all $T \in \mathcal{L}(X), \|T\| = 1$ that

$$\sigma_*(\phi_n(T)) \subset \sigma_*(T) + \epsilon_n \mathrm{D} \subset \sigma_*^{\epsilon_n}(T)$$

and

$$\sigma_*(T) \subset \sigma_*(\phi_n(T)) + \epsilon_n \mathrm{D} \subset \sigma_*^{\epsilon_n}(\phi_n(T))$$

for each $n \in \mathbb{N}$.

Now, we consider the continuous linear operator

$$\hat{\phi} = (\phi_n) : \mathcal{L}(X)^{\mathcal{U}} \subset \mathcal{L}(X^{\mathcal{U}}) \to \mathcal{L}(Y)^{\mathcal{U}} \subset \mathcal{L}(Y^{\mathcal{U}}).$$

Since $\mathrm{q}(\hat{\phi}) = \lim_{\mathcal{U}} \mathrm{q}(\phi_n) \geq k > 0$, so $\hat{\phi}$ is surjective. By Proposition 3.1 (i), (ii), we obtain that for all $\hat{T} = (T_n) \in \mathcal{L}(X)^{\mathcal{U}}$

$$\sigma_*(\hat{\phi}(\hat{T})) = \sigma_*(\hat{T}).$$

Thus, Lemma 2.6 yields that $\hat{\phi}$ is either an isomorphism or an anti-isomorphism.

We have then that $\hat{\phi}$ is bijective and so (ϕ_n) is bijective, thus (ϕ_n) is injective, furthermore we have

$$\begin{aligned} \lim_{\mathcal{U}} \min\{\mathrm{mult}(\phi_n), \mathrm{amult}(\phi_n)\} &= \min\{\lim_{\mathcal{U}} \mathrm{mult}(\phi_n), \lim_{\mathcal{U}} \mathrm{amult}(\phi_n)\} \\ &= \min\{\mathrm{mult}(\hat{\phi}), \mathrm{amult}(\hat{\phi})\} = 0 \end{aligned}$$

which is a contradiction. ∎

Corollary 3.7. *Let H be a separable Hilbert space. Then for each $k, K, \epsilon > 0$ there is $\delta > 0$ such that if $\phi : \mathcal{L}(H) \to \mathcal{L}(H)$ is a surjective linear map with*

$$\mathrm{dist}_H(\sigma_*(\phi(T)), \sigma_*(T)) < \delta \quad \text{for all} \quad T \in \mathcal{L}(H), \|T\| = 1,$$

$\mathrm{q}(\phi) > k$ *and* $\|\phi\| < K$, *then* $\|\phi - \psi\| < \epsilon$ *for some automorphism or anti-automorphism* $\psi : \mathcal{L}(H) \to \mathcal{L}(H)$.

Proof. Using Theorem 3.6 and [1, Corollary 3.10], we proceed as in the proof of Corollary 3.3. ∎

REFERENCES

[**1**] J. Alaminos, J. Extremera, A.R. Villena, *Approximately spectrum-preserving maps*, J. Funct. Anal. **261** (2011) 233–266.

[**2**] J. Cui, J. Hou, *Additive maps on standard operator algebras preserving parts of the spectrum*, J. Math. Anal. Appl. **282** (1) (2003) 266–278.

[**3**] J. Cui, J. Hou, *Linear maps between Banach algebras compressing certain spectral functions*, Rocky Mountain J. Math. **34** (2) (2004) 565–584.

[**4**] J. Hou, L. Huang, *Additive maps between standard operator algebras compressing certain spectral functions*, Acta Math. Sin. **24** (12) (2008) 2041–2048.

[**5**] R. C. James, *Super-reflexive Banach spaces*, Canadian J. Math. **24** (1972), 896–904.

[**6**] B. E. Johnson, *Approximately multiplicative maps between Banach algebras*, J. Lond. Math. Soc. (2) **37** (1988), no. 2, 294–316.

[**7**] V. Müller, *Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras*, second edition, Oper. Theory Adv. Appl., vol. **139**, Birkhäuser Verlag, Basel, 2007.

[**8**] F. Räbiger, M.P.H. Wolff, *On the approximation of positive operators and the behaviour of the spectra of the approximants*, Integral Equations Operator Theory **28** (1) (1997) 72–86.

[**9**] P. Šemrl, *Two characterizations of automorphisms on* $\mathcal{B}(X)$, Studia Math. **105** (2) (1993), 143–149.

[**10**] B. Sims, *Ultra-Techniques in Banach Space Theory*, Queens Papers in Pure and Appl. Math. vol. **60**, Queens University, Kingston, ON, 1982.

[**11**] D. van Dulst, *Reflexive and Superreflexive Banach Spaces.* Amsterdam : Mathematisch Centrum, 1978.

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