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THE (CO)SHAPE AND (CO)HOMOLOGICAL PROPERTIES OF CONTINUOUS MAPS

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Dedicated to Academician Georg Chogoshvili
on the occasion of 100-th anniversary of his birthday

Abstract. The purpose of this paper is to investigate continuous maps from the standpoint of geometric topology and algebraic topology. Using a direct system approach and an inverse system approach of continuous maps, we study the (co)shape and (co)homological properties of continuous maps. Applications of the obtained results include the constructions of long exact sequences of continuous maps for the (co)homology pro-groups and (co)homology inj-groups, spectral Čech (co)homology groups and spectral singular (co)homology groups.

1. Introduction

One of the original ideas of geometric topology, in particular (co)shape theory, is to describe the properties of continuous maps of general topological spaces by using the expansions of continuous maps into direct and inverse systems of continuous maps between the spaces which behave well locally. This idea has many interesting applications in different branches of algebraic topology and geometric topology and it was successfully used by various mathematicians (see [1–3, 7, 12–14, 17, 24, 25]).

The present paper is devoted to the study of the following natural question: can we use a direct system approach and an inverse system approach of continuous maps to define a long exact spectral homology and cohomology sequences of continuous maps?

Note that the long exact sequences of continuous maps for homotopy pro-groups and shape groups, i.e. spectral homotopy groups, were first investigated in [7]. Using the geometric realizations of the nerves and Vietoris complexes of open coverings of spaces D. A. Edwards and P. T. McAuley defined two functors from

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the category of continuous maps to the pro-category for the category of continuous maps of CW-spaces and constructed the long exact sequences connecting the pro-homotopy groups of continuous maps and their domains and ranges to one another [5, Theorem IV.1.1]. Besides, they found conditions, having which, continuous maps between compact metric spaces induce to the long exact sequences, relating the shape groups of maps and spaces [7, Theorem IV.1.2].

The goal of our paper is different from the goal of paper [7]. In particular, we are interested in the (co)homological properties of continuous maps. To this end, here we use the K. Morita [18, 19], T. Porter [20] and G. Chogoshvili [5] approximations of spaces and define Edwards-McAuley type functors based only on the locally finite normal open coverings and finite subcomplexes of singular complexes of spaces and construct the long exact sequences connecting the pro-(co)homology groups, inj-(co)homology groups, spectral Čech (co)homology groups and spectral singular (co)homology groups of arbitrary continuous maps and their domains and ranges to one another, respectively.

The results obtained in this paper are based on the achievements of (co)shape theory [3, 4, 10, 13] and the above-mentioned methods of approach, and their applications include:

i) the definitions of shape and coshape concepts of continuous maps;

ii) the constructions of covariant and contravariant functors from the category of continuous maps of topological spaces to the category of long exact sequences of (co)homology pro-groups, (co)homology inj-groups, spectral Čech (co)homology groups [8, 13, 19] and spectral singular (co)homology groups [5];

iii) the definitions of spectral (co)homology groups of continuous maps and the constructions of long exact spectral Čech and singular (co)homology sequences of arbitrary pairs of spaces (cf. [13, 19, 26]).

We assume that the reader is familiar with shape and homology theories. Without any specification we use the notions and terminology from the books [8] and [13].

Throughout the paper we assume that all spaces and maps are topological spaces and continuous maps, respectively.

We denote by the symbol $\mathrm{cov}_N(X)$ the set of all locally finite normal coverings of the space X. If α and β are the coverings of the space X, then the symbol $\alpha < \beta$ means that β is a refinement of α.

Let the symbol **Top** denote the category of all spaces and maps. Denote by **CW** the full subcategory of **Top** consisting of all CW-complexes. The symbols **S** and **SC** denote the categories of simplicial complexes and simplicial maps and semisimlicial complexes and semisimplicial maps, respectively [11, 16, 22]. Finally, by $\mathrm{R_S} : \mathbf{S} \to \mathbf{CW}$ [22] and $\mathrm{R}_{\mathbf{SC}} : \mathbf{SC} \to \mathbf{CW}$ [9, 11, 15] denote the functors of geometric realizations.

Write **HTop** for the homotopy category of **Top** and **HCW** for the homotopy category of **CW**. Denote by the symbol **KS** the extended homotopy category associated with **S** [7, 15].

Let **Ab** denote the category of all abelian groups and homomorphisms.

Now we give some facts which we need below.

For a category $\mathcal{C}$ by $\mathbf{LEG}(\mathcal{C})$ we denote the category of long exact sequences in $\mathcal{C}$.

An important way of giving another category from a given category $\mathcal{C}$ is to form the category of morphisms of $\mathcal{C}$, denoted by the symbol $\mathbf{Mor}_{\mathcal{C}}$ [22]. The objects of $\mathbf{Mor}_{\mathcal{C}}$ are morphisms $f : X \to Y$ of $\mathcal{C}$ and the morphisms of $\mathbf{Mor}_{\mathcal{C}}$ are commutative diagrams of $\mathcal{C}$, i.e., a pair $\xi = (\xi^1, \xi^2)$ is a morphism from f into $f' : X' \to Y'$, if $\xi^1 : X \to X'$ and $\xi^2 : Y \to Y'$ are morphisms of $\mathcal{C}$ and $f' \cdot \xi^1 = \xi^2 \cdot f$. Hence, we have categories $\mathbf{Mor}_{\mathbf{Top}}$, $\mathbf{Mor}_{\mathbf{CW}}$, $\mathbf{Mor}_{\mathbf{S}}$, $\mathbf{Mor}_{\mathbf{SC}}$, $\mathbf{Mor}_{\mathbf{HTop}}$, $\mathbf{Mor}_{\mathbf{HCW}}$ and $\mathbf{Mor}_{\mathbf{KS}}$. In the category $\mathbf{Mor}_{\mathcal{C}}$ the notion of homotopy of morphisms is defined. Let $\xi = (\xi^1, \xi^2)$, $\eta = (\eta^1, \eta^2) : f \to f'$ be morphisms from f to f'. Then a homotopy from ξ to η is a morphism $F = (F^1, F^2)$ from $f \times 1_I : X \times I \to X' \times I$ to f' such that F^1 is a homotopy from ξ^1 to η^1 and F^2 is a homotopy from ξ^2 to η^2 (see [22]). Denote by the symbol $\mathbf{H}(\mathbf{Mor}_{\mathcal{C}})$ the homotopy category of the category $\mathbf{Mor}_{\mathcal{C}}$. Hence, there exist the following categories $\mathbf{H}(\mathbf{Mor}_{\mathbf{Top}})$, $\mathbf{H}(\mathbf{Mor}_{\mathbf{CW}})$, $\mathbf{K}(\mathbf{Mor}_{\mathbf{S}})$ and functors in diagrams [7]:

$$\mathbf{Mor}_{\mathbf{CW}} \longrightarrow \mathbf{H}(\mathbf{Mor}_{\mathbf{CW}}) \longrightarrow \mathbf{Mor}_{\mathbf{HCW}},$$
$$\mathbf{Mor}_{\mathbf{S}} \longrightarrow \mathbf{K}(\mathbf{Mor}_{\mathbf{S}}) \longrightarrow \mathbf{Mor}_{\mathbf{KS}},$$
$$\mathbf{Mor}_{\mathbf{SC}} \longrightarrow \mathbf{H}(\mathbf{Mor}_{\mathbf{CW}}) \longrightarrow \mathbf{Mor}_{\mathbf{HCW}}.$$

Let pro : $\mathbf{inv}\text{-}\mathcal{C} \to \mathbf{pro}\text{-}\mathcal{C}$ and inj : $\mathbf{dir}\text{-}\mathcal{C} \to \mathbf{inj}\text{-}\mathcal{C}$ be natural functors from the category of inverse systems $\mathbf{inv}\text{-}\mathcal{C}$ to the quotient category $\mathbf{pro}\text{-}\mathcal{C} = \mathbf{inv}\text{-}\mathcal{C}/\sim$ [13] and from the category of direct systems $\mathbf{dir}\text{-}\mathcal{C}$ to the quotient category $\mathbf{inj}\text{-}\mathcal{C} = \mathbf{dir}\text{-}\mathcal{C}/\sim$ [2], respectively. The functors pro and inj induce functors in the following diagrams:

$$\mathbf{pro}\text{-}\mathbf{Mor}_{\mathbf{CW}} \longrightarrow \mathbf{pro}\text{-}\mathbf{H}(\mathbf{Mor}_{\mathbf{CW}}) \longrightarrow \mathbf{pro}\text{-}\mathbf{Mor}_{\mathbf{HCW}},$$
$$\mathbf{inj}\text{-}\mathbf{Mor}_{\mathbf{CW}} \longrightarrow \mathbf{inj}\text{-}\mathbf{H}(\mathbf{Mor}_{\mathbf{CW}}) \longrightarrow \mathbf{inj}\text{-}\mathbf{Mor}_{\mathbf{HCW}},$$
$$\mathbf{pro}\text{-}\mathbf{Mor}_{\mathbf{S}} \longrightarrow \mathbf{pro}\text{-}\mathbf{K}(\mathbf{Mor}_{\mathbf{S}}) \longrightarrow \mathbf{pro}\text{-}\mathbf{Mor}_{\mathbf{KS}},$$
$$\mathbf{inj}\text{-}\mathbf{Mor}_{\mathbf{S}} \longrightarrow \mathbf{inj}\text{-}\mathbf{K}(\mathbf{Mor}_{\mathbf{S}}) \longrightarrow \mathbf{inj}\text{-}\mathbf{Mor}_{\mathbf{KS}},$$
$$\mathbf{pro}\text{-}\mathbf{Mor}_{\mathbf{SC}} \longrightarrow \mathbf{pro}\text{-}\mathbf{H}(\mathbf{Mor}_{\mathbf{CW}}) \longrightarrow \mathbf{pro}\text{-}\mathbf{Mor}_{\mathbf{HCW}},$$
$$\mathbf{inj}\text{-}\mathbf{Mor}_{\mathbf{SC}} \longrightarrow \mathbf{inj}\text{-}\mathbf{H}(\mathbf{Mor}_{\mathbf{CW}}) \longrightarrow \mathbf{inj}\text{-}\mathbf{Mor}_{\mathbf{HCW}}.$$

2. The concepts of (co)shapes of maps

The (co)shape theory of maps is a spectral homotopy theory of maps [1]. The well-known Čech, Vietoris and Chogoshvili constructions yield the functors Č : $\mathbf{Top} \to \mathbf{pro}\text{-}\mathbf{KS}$, V : $\mathbf{Top} \to \mathbf{pro}\text{-}\mathbf{S}$ and CH : $\mathbf{Top} \to \mathbf{inj}\text{-}\mathbf{SC}$ (see [3, 5, 7, 8, 20]). The compositions of functors contained in the sequences

$$\mathbf{Top} \longrightarrow \mathbf{pro}\text{-}\mathbf{KS} \longrightarrow \mathbf{pro}\text{-}\mathbf{CW},$$
$$\mathbf{Top} \longrightarrow \mathbf{pro}\text{-}\mathbf{S} \longrightarrow \mathbf{pro}\text{-}\mathbf{CW},$$
$$\mathbf{Top} \longrightarrow \mathbf{inj}\text{-}\mathbf{SC} \longrightarrow \mathbf{pro}\text{-}\mathbf{CW},$$

for a simplicity, again denote by Č, V and CH, respectively. There exist functors

$$\begin{aligned}&\mathrm{Mor}_{\check{\mathrm{C}}}:\mathbf{Mor_{Top}}\longrightarrow \mathbf{pro\text{-}K}(\mathbf{Mor_S}),\\ &\mathrm{Mor}_{\mathrm{V}}:\mathbf{Mor_{Top}}\longrightarrow \mathbf{pro\text{-}Mor_S},\\ &\mathrm{Mor}_{\mathrm{CH}}:\mathbf{Mor_{Top}}\longrightarrow \mathbf{inj\text{-}Mor_{SC}},\end{aligned}$$

which are called Čech, Vietoris and Chogoshvili mapping functors, respectively.

By the definition of functor $\mathrm{Mor}_{\check{\mathrm{C}}}$ (cf. [7]), for each object $(f: X\to Y)\in \mathrm{Ob}(\mathbf{Mor_{Top}})$ we have an inverse system $\mathrm{Mor}_{\check{\mathrm{C}}}(f)=\{f_\lambda,(p^1_{\lambda\lambda'},p^2_{\lambda\lambda'}),\ \check{\mathrm{C}}\mathrm{ov}_N(f)\}$, where

$$\check{\mathrm{C}}\mathrm{ov}_N(f)=\big\{\lambda=(\alpha,\beta,\nu)\mid \alpha\in \mathrm{cov}_N(Y),\ \ \beta\in\mathrm{cov}_N(X),\ \ \nu:\beta>f^{-1}(\alpha)\big\}$$

is a set of triples λ containing a refining map $\nu:\beta>f^{-1}(\alpha)$ of coverings β into the inverse image $f^{-1}(\alpha)$ of the covering α, $f_\lambda=f_{\alpha\beta\nu}:N(\beta)\to N(\alpha)$ is a simplicial map of nerves defined by the formula

$$f_\lambda(u)=(f\circ\nu)(u),\quad u\in N(\beta),\quad u\in\beta,$$

and $(p^1_{\lambda\lambda'},p^2_{\lambda\lambda'})$ is a class of morphisms $(\mu^1_*,\mu^2_*):f_{\lambda'}\to f_\lambda$ in $\mathbf{Mor_S}$ induced by the refining map $(\mu^1,\mu^2):\lambda'=(\alpha',\beta',\nu')\to\lambda=(\alpha,\beta,\nu)$, where $\mu^1:\beta'\to\beta$ and $\mu^2: f^{-1}\alpha'\to f^{-1}\alpha$ are the refinements satisfying the condition $\mu^2\cdot\nu'=\nu\cdot\mu^1$.

For each morphism $\big(\eta=(\eta^1,\eta^2):f'\to f\big)\in \mathrm{Mor}_{\mathbf{Mor_{Top}}}(f',f)$ define a morphism $\mathrm{Mor}_{\check{\mathrm{C}}}(\eta)=(\eta_\lambda,\varphi):\mathrm{Mor}_{\check{\mathrm{C}}}(f')\to\mathrm{Mor}_{\check{\mathrm{C}}}(f)$, where $\varphi:\check{\mathrm{C}}\mathrm{ov}_N(f)\to\check{\mathrm{C}}\mathrm{ov}_N(f')$ is a function defined by the formulas

$$\begin{aligned}\varphi(\lambda)=\big((\eta^2)^{-1}(\alpha),(\eta^1)^{-1}(\beta),\eta_*(\nu)\big)\,,\quad \lambda=(\alpha,\beta,\nu)\in\check{\mathrm{C}}\mathrm{ov}_N(f),\\ \eta_*(\nu)(w)=(f')^{-1}(\eta^2)^{-1}f\nu\eta^1(w),\quad w\in(\eta^1)^{-1}(\beta)\end{aligned}$$

and $\eta_\lambda; f'_{\varphi(\lambda)}\to f_\lambda$ is a morphism defined by the formula

$$\eta_\lambda=\big((\eta^1)_\beta,(\eta^2)_\alpha\big)\,,\quad \lambda=(\alpha,\beta,\nu)\in\check{\mathrm{C}}\mathrm{ov}_N(f).$$

By the definition of functor $\mathrm{Mor}_{\mathrm{V}}$ (cf. [7]), for each object $(f: X\to Y)\in \mathrm{Ob}(\mathbf{Mor_{Top}})$ we have the inverse system $\mathrm{Mor}_{\mathrm{V}}(f)=\{f_\lambda,(p^1_{\lambda\lambda'},p^2_{\lambda\lambda'}),\ \check{\mathrm{C}}\mathrm{ov}_N(f)\}$, where $\check{\mathrm{C}}\mathrm{ov}_N(f)$ is the set

$$\{\lambda=(\alpha,\beta)\mid\alpha\in\mathrm{cov}_N(Y),\ \ \beta\in\mathrm{cov}_N(X),\ \ \beta>f^{-1}(\alpha)\},$$

$f_\lambda=f_{\alpha\beta}:\mathrm{V}(\beta)\to\mathrm{V}(\alpha)$ is a simplicial map of Vietoris complexes defined by the formula

$$f_\lambda(x)=f(x),\quad x\in\mathrm{V}(\beta),\quad x\in X,$$

and $(p^1_{\lambda\lambda'},p^2_{\lambda\lambda'}):f_{\lambda'}\to f_\lambda$ is a morphism given by $p^1_{\lambda\lambda'}(x)=x$ for a vertex x of $\mathrm{V}(\beta')$ and by $p^2_{\lambda\lambda'}(y)=y$ for a vertex y of $\mathrm{V}(\alpha')$.

For each morphism $\big(\eta=(\eta^1,\eta^2):f'\to f\big)\in \mathrm{Mor}_{\mathbf{Mor_{Top}}}(f',f)$ define a morphism $\mathrm{Mor}_{\mathrm{V}}(\eta)=(\eta_\lambda,\varphi):\mathrm{Mor}_{\mathrm{V}}(f')\to\mathrm{Mor}_{\mathrm{V}}(f)$, where $\varphi:\check{\mathrm{C}}\mathrm{ov}_N(f)\to\check{\mathrm{C}}\mathrm{ov}_N(f')$

is a function given by the formula $\varphi(\lambda) = \big((\eta^2)^{-1}(\alpha), (\eta^1)^{-1}(\beta)\big)$, $\lambda = (\alpha, \beta) \in \check{\mathrm{C}}\mathrm{ov}_N(f)$ and $\eta_\lambda = ((\eta^1)_\lambda, (\eta^2)_\lambda)) : f'_{\varphi(\lambda)} \to f_\lambda$ is a morphism, where $(\eta^1)_\lambda(x) = \eta^1(x)$ for a vertex $x \in \mathrm{V}(\beta)$ and $(\eta^2)_\lambda(y) = \eta^2(y)$ for a vertex $y \in \mathrm{V}(\alpha)$.

As in [7, 20], we can prove that the composition of functors of the sequence

$$\mathbf{Mor_{Top}} \longrightarrow \mathbf{pro\text{-}K}(\mathbf{Mor_S}) \longrightarrow \mathbf{pro\text{-}Mor_{KS}}$$

is naturally equivalent to the composition of functors of the sequence

$$\mathbf{Mor_{Top}} \longrightarrow \mathbf{pro\text{-}Mor_S} \longrightarrow \mathbf{pro\text{-}K}(\mathbf{Mor_S}) \longrightarrow \mathbf{pro\text{-}Mor_{KS}}\,.$$

The geometric realization functor $\mathrm{R_S} : \mathbf{S} \to \mathbf{CW}$ induces a functor $\mathrm{Mor}_{\mathrm{R_S}} : \mathbf{Mor_S} \to \mathbf{Mor_{CW}}$, which to each simplicial map assigns its geometric realization and, consequently, defines the functor

$$\mathrm{pro\text{-}Mor}_{\mathrm{R_S}} : \mathbf{pro\text{-}Mor_S} \longrightarrow \mathbf{pro\text{-}Mor_{CW}},$$

whose compositions with the above-given functors yield functors from $\mathbf{Mor_{Top}}$ into $\mathbf{pro\text{-}Mor_{CW}}$, $\mathbf{pro\text{-}H}(\mathbf{Mor_{CW}})$ and $\mathbf{pro\text{-}Mor_{HCW}}$, which, for a simplicity, are again denoted by $\mathrm{Mor}_{\check{\mathrm{C}}}$ and $\mathrm{Mor}_{\mathrm{V}}$.

As a space X can be approximated by a direct system of topological spaces having the homotopy type of finite CW-complexes, a map of spaces $f : X \to Y$ can also be approximated by a direct system of maps of spaces having the homotopy type of finite CW-complexes.

Let $f : X \to Y$ be a map and let $S(f) : S(X) \to S(Y)$ be a semisimplicial map of semisimplicial singular complexes $S(X)$ and $S(Y)$ induced by f [11]. By the definition of $S(f)$ we have

$$S(f)(\sigma_n) = f \cdot \sigma_n, \quad \sigma_n : \Delta^n \to X.$$

Let $\{X_\alpha\}_{\alpha \in A}$ and $\{Y_\beta\}_{\beta \in B}$ be the families of all finite subcomplexes of $S(X)$ and $S(Y)$, respectively. By $i_{\alpha\alpha'} : X_\alpha \to X_{\alpha'}$ and $j_{\beta\beta'} : Y_\beta \to Y_{\beta'}$ we denote the inclusion semisimplicial maps. The set of all pairs (α, β), for which $S(f)(X_\alpha) \subseteq Y_\beta$, forms a directed set $(M, \leq)$, where

$$(\alpha, \beta) \leq (\alpha', \beta') \Leftrightarrow \alpha \leq \alpha', \ \ \beta \leq \beta'.$$

Let $f_{(\alpha,\beta)} = S(f)_{|X_\alpha} : X_\alpha \to Y_\beta$. The pair $\pi_{(\alpha,\beta)(\alpha',\beta')} = (i_{\alpha\alpha'}, j_{\beta\beta'})$, $(\alpha, \beta) \leq (\alpha', \beta')$, is a morphism of $f_{(\alpha,\beta)}$ to $f_{(\alpha',\beta')}$ since

$$j_{\beta\beta'} \cdot f_{(\alpha,\beta)} = j_{\beta\beta'} \cdot S(f)_{|X_\alpha} = S(f)_{|X_{\alpha'}} \cdot i_{\alpha\alpha'} = f_{(\alpha',\beta')} \cdot i_{\alpha\alpha'}.$$

It is clear that the family $\mathrm{CH}(f) = \{f_{(\alpha,\beta)}, \pi_{(\alpha,\beta)(\alpha',\beta')}, M\}$ is a direct system of the category $\mathbf{Mor_{SC}}$. A morphism $\varphi = (\varphi^1, \varphi^2) : f \to f'$ of the category $\mathbf{Mor_{Top}}$ induces a morphism

$$\mathrm{CH}(\varphi) : \mathrm{CH}(f) = \{f_{(\alpha,\beta)}, \pi_{(\alpha,\beta)(\alpha',\beta')}, M\} \to \mathrm{CH}(f') = \{f'_{(\gamma,\delta)}, \pi'_{(\gamma,\delta)(\gamma',\delta')}, M\}$$

in an obvious way. Assume that $S(\varphi^1)(X_\alpha) = X'_\gamma$ and $S(\varphi^2)(Y_\beta) = Y'_\delta$. Let $\theta : M \to M'$ be a map given by $\theta(\alpha, \beta) = (\gamma, \delta)$. A pair $\varphi_{(\alpha,\beta)} = (\varphi^1_{(\alpha,\beta)}, \varphi^2_{(\alpha,\beta)})$, where $\varphi^1_{(\alpha,\beta)} = S(\varphi^1)_{|X_\alpha} : X_\alpha \to X'_\gamma$ and $\varphi^2_{(\alpha,\beta)} = S(\varphi^2)_{|Y_\beta} : Y_\beta \to Y'_\delta$, is a morphism of $f_{(\alpha,\beta)}$ to $f'_{(\gamma,\delta)}$. It is easy to see that the family $(\varphi_{(\alpha,\beta)}, \theta)$ is the desired morphism $\mathrm{CH}(\varphi)$. For a simplicity, we put $\lambda = (\alpha,\beta)$, $X_\lambda = X_\alpha$, $Y_\lambda = Y_\beta$ for each $(\alpha,\beta) \in M = \Lambda$ and $p_{\lambda\lambda'} = i_{\alpha\alpha'}$, $q_{\lambda\lambda'} = j_{\beta\beta'}$ for each $\lambda = (\alpha,\beta) \le \lambda' = (\alpha',\beta')$. Consequently, we have obtained the functor

$$\mathrm{CH} : \mathbf{Mor_{Top}} \to \mathbf{inj\text{-}Mor_{SC}},$$

which to each object $f : X \to Y$ and morphism $\varphi = (\varphi^1, \varphi^2) : f \to f'$ of the category $\mathbf{Mor_{Top}}$ assigns a direct system $\mathrm{CH}(f) = \{f_\lambda, \pi_{\lambda\lambda'}, \Lambda\}$ and morphism $\mathrm{CH}(\varphi) = (\varphi_\lambda, \theta)$, respectively. Also note that $\boldsymbol{X} = (X_\lambda, [p_{\lambda\lambda'}], \Lambda)$ and $\boldsymbol{Y} = (Y_\lambda, [q_{\lambda\lambda'}], \Lambda)$ are the direct systems coassociated [2] with X and Y, respectively.

The functor of geometric realization $\mathrm{R_{SC}} : \mathbf{SC} \to \mathbf{CW}$ induces the functor

$$\mathrm{Mor_{R_{SC}}} : \mathbf{Mor_{SC}} \to \mathbf{Mor_{CW}},$$

which to each semisimplicial map f assigns its geometric realization $|f|$ and, consequently, defines functor

$$\mathrm{inj\text{-}Mor_{R_{SC}}} : \mathbf{inj\text{-}Mor_{SC}} \to \mathbf{inj\text{-}Mor_{CW}},$$

whose compositions with the above given functors yield functors from $\mathbf{Mor_{Top}}$ into $\mathbf{inj\text{-}Mor_{CW}}$, $\mathbf{inj\text{-}H}(\mathbf{Mor_{CW}})$ and $\mathbf{inj\text{-}Mor_{HCW}}$, which, for a simplicity, we again denote by the symbol $\mathrm{Mor_{CH}}$.

3. Spectral (co)homology sequences of maps

The spectral Čech cohomology theory defined for the category of pairs of spaces with normally embedded subspaces and based on the locally finite normal coverings is useful in studying the covering dimensions of spaces and homotopy classes of maps [13, 18, 19]. This cohomology theory is shape invariant and satisfies all the Eilenberg–Steenrod axioms, the relative homomorphism axiom, the wedge axiom and coincides with the classical Čech cohomology theory on the category of pairs of paracompact spaces (see [13, 18, 19, 26]).

The purpose of the present section is to study the exactness of spectral Čech (co)homology sequences of maps, in particular, of pairs of spaces with arbitrary subspaces.

The spectral cohomology functor $\check{\mathrm{H}}^*(-;G) : \mathbf{Top} \to \mathbf{Ab}$ with coefficients in abelian group G is the composition of functors $\check{\mathrm{C}} : \mathbf{Top} \to \mathbf{pro\text{-}KS}$, $\mathrm{inj\text{-}H^*} : \mathbf{pro\text{-}KS} \to \mathbf{inj\text{-}Ab}$ and $\varinjlim : \mathbf{inj\text{-}Ab} \to \mathbf{Ab}$, where H^* is a formal cohomology functor of the category of simplicial complexes [8]. The value of composition $(\mathrm{inj\text{-}H^*}) \circ \check{\mathrm{C}} : \mathbf{Top} \to \mathbf{inj\text{-}Ab}$ on the space X is called an n-dimensional spectral Čech cohomology inj-group of X and is denoted by $\mathrm{inj}\text{-}H^n(X;G)$. By definition, an n-dimensional spectral Čech cohomology group of the space X is the value of composition $\varinjlim \circ (\mathrm{inj\text{-}H^*}) \circ \check{\mathrm{C}}$ for the space X, i.e.,

$$\check{H}^n(X;G) = \left(\varinjlim \circ (\mathrm{inj\text{-}H}^n) \circ \check{\mathrm{C}}\right)(X), \quad X \in \mathrm{Ob}(\mathbf{Top}).$$

For a simplicial map $f : K \to K'$ in $\mathbf{S}$, the cochain map $f^* : C^*(K';G) \to C^*(K;G)$ for $\varphi \in C^n(K';G)$ is given by the formula

$$f^n(\varphi)(v_0, v_1, \ldots, v_n) = \varphi(f(v_0), f(v_1), \ldots, f(v_n)),$$

where $(v_0, v_1, \ldots, v_n)$ is a simplex of a simplicial complex K [8]. There exists a long exact sequence of the cochain map f^* (see [23])

$$\cdots \longrightarrow H^n(K';G) \longrightarrow H^n(f^*;G) \longrightarrow H^{n+1}(K;G) \longrightarrow H^{n+1}(K';G) \longrightarrow \cdots .$$

For contiguous simplicial maps $f, g : K \to K'$, the induced cochain maps $f^*, g^* : C^*(K';G) \to C^*(K;G)$ are cochain homotopic. Hence the long exact spectral Čech cohomology sequences associated to f and g are isomorphic.

The association of the considered cohomological long exact sequence to the simplicial map $f : K \to K'$ induces a functor $\mathrm{E} : \mathbf{K}(\mathbf{Mor_S}) \to \mathbf{LES}(\mathbf{Ab})$. Thus we have the functor

$$\text{inj-E} : \mathbf{pro\text{-}K}(\mathbf{Mor_S}) \longrightarrow \mathbf{inj\text{-}LES}(\mathbf{Ab}).$$

The composition of functors $\mathrm{Mor}_{\check{\mathrm{C}}}$ and inj-E yields the functor

$$(\text{inj-E}) \circ \mathrm{Mor}_{\check{\mathrm{C}}} : \mathbf{Mor_{Top}} \longrightarrow \mathbf{inj\text{-}LES}(\mathbf{Ab}).$$

Now we state the facts which we will use later. Let $\{t_\alpha\}_{\alpha \in A} \in \mathbf{inj\text{-}LES}(\mathbf{Ab})$ be an inj-object consisting of the following exact sequences

$$t_\alpha : \cdots \longrightarrow G_\alpha^{n+1} \longrightarrow G_\alpha^n \longrightarrow G_\alpha^{n-1} \longrightarrow \cdots, \quad n \in Z, \quad \alpha \in A.$$

Then $\delta(\{t_\alpha\}_{\alpha \in A})$ is a sequence

$$\cdots \longrightarrow \{G_\alpha^{n+1}\}_{\alpha \in A} \longrightarrow \{G_\alpha^n\}_{\alpha \in A} \xrightarrow{\mathbf{h}^n} \{G_\alpha^{n-1}\}_{\alpha \in A} \longrightarrow \cdots,$$

where $\mathbf{h}^n = (\{h_\alpha^n\}_{\alpha \in A})$. By Theorem 2.2 of [2] this sequence is exact. Consequently, there exists a functor

$$\delta : \mathbf{inj\text{-}LES}(\mathbf{Ab}) \longrightarrow \mathbf{LES}(\mathbf{inj\text{-}Ab}).$$

Now define a functor from the category $\mathbf{Mor_{Top}}$ to the category $\mathbf{LES}(\mathbf{inj\text{-}Ab})$ as the composition

$$\delta \circ (\text{inj-E}) \circ \mathrm{Mor}_{\check{\mathrm{C}}} : \mathbf{Mor_{Top}} \to \mathbf{LES}(\mathbf{inj\text{-}Ab}).$$

Let $\text{inj-}H^n(f;G) = \{H^n(f_\lambda;G)\}_{\lambda=(\alpha,\beta,\nu)\in \check{\mathrm{C}}\mathrm{ov}_N(f)}$ and $\check{H}^n(f;G) = \varinjlim \text{inj-}H^n(f;G)$. Thus we have

THEOREM 3.1. *For each map $f : X \to Y$ there exists a long exact sequence of* inj-*groups*

$$\cdots \longrightarrow \text{inj-}H^n(X;G) \longrightarrow \text{inj-}H^n(f;G) \longrightarrow \text{inj-}H^{n+1}(Y;G) \longrightarrow \text{inj-}H^{n+1}(X;G) \longrightarrow \cdots .$$

Applying the direct limit functor to this sequence and using Lemma 1 of [13, Ch. II, §3.2] we can prove

Theorem 3.2. *If $f : X \to Y$ is a map, then the sequence*

$$\cdots \longrightarrow \check{H}^n(X;G) \longrightarrow \check{H}^n(f;G) \longrightarrow \check{H}^{n+1}(Y;G) \longrightarrow \check{H}^{n+1}(X;G) \longrightarrow \cdots$$

is exact.

An advantage of the above-given approach is also that it gives us a tool of constructing a long exact spectral Čech cohomology sequence of pairs of spaces with non-normally embeddable subspaces. We have the following

Corollary 3.3. (cf. Theorem 5 of [13. Ch. II, §3.2]) *For each pair (X, A) of a topological space X and an arbitrary subspace A the sequence*

$$\cdots \longrightarrow \check{H}^n(A;G) \longrightarrow \check{H}^n(i;G) \longrightarrow \check{H}^{n+1}(X;G) \longrightarrow \check{H}^{n+1}(A;G) \longrightarrow \cdots ,$$

where $i : A \to X$ is the inclusion map, is exact.

The spectral Čech homology theory based on all locally finite normal coverings on the category of pairs of spaces with normally embedded subspaces is shape invariant and satisfies all the Eilenberg-Steenrod axioms except for the exactness axiom, the relative axiom, the wedge axiom and coincides with the classical Čech homology theory on the category of pairs of paracompact spaces (see [13, 18, 19, 26]).

The spectral Čech homology functor $\check{\mathrm{H}}_*(-;G) : \mathbf{Top} \to \mathbf{Ab}$ is the composition of functors $\check{\mathrm{C}} : \mathbf{Top} \to \mathbf{pro\text{-}KS}$, $\mathrm{pro\text{-}H}_* : \mathbf{pro\text{-}KS} \to \mathbf{pro\text{-}Ab}$ and $\varprojlim : \mathbf{pro\text{-}Ab} \to \mathbf{Ab}$, where H_* is a formal homology functor of the category of simplicial complexes [8]. By definition, for each space $X \in \mathrm{Ob}(\mathbf{Top})$

$$\text{pro-}H_n(X;G) = ((\text{pro-H}_n) \circ \check{\mathrm{C}})(X)$$

and

$$\check{H}_n(X;G) = (\varprojlim : \circ (\text{pro-H}_n) \circ \check{\mathrm{C}}(X).$$

By analogy with the spectral Čech homology pro-groups and groups, the homotopy pro-groups and shape groups based on the locally finite normal open coverings of a space $X \in \mathrm{Ob}(\mathbf{Top})$ are defined [13]. They coincide with Edwards–McAuley spectral homotopy pro-groups and groups based on all open coverings on the category of paracompact spaces and for them almost all the results of paper [7] are true.

For each simplicial map $f : K \to K'$, the chain map $f_* : C_*(K;G) \to C_*(K';G)$ is given by homomorphisms $f_n : C_n(K;G) \to C_n(K';G)$, $n \in Z$ [8]. Consider the long exact sequence [8]

$$\cdots \longrightarrow H_n(K';G) \longrightarrow H_n(f_*;G) \longrightarrow H_{n-1}(K;G) \longrightarrow H_{n-1}(K';G) \longrightarrow \cdots$$

induced by the chain map f_*. It is clear that for contiguous simplicial maps $f, g : K \to K'$, the long exact homological sequences induced by the chain maps f_* and g_* are isomorphic.

The association of the considered homological sequence to the simplicial map $f : K \to K'$ yields a functor $\mathrm{E} : \mathbf{K}(\mathbf{Mor_S}) \to \mathbf{LES}(\mathbf{Ab})$. Consequently, there exists a functor

$$\text{pro-E} : \mathbf{pro\text{-}K}(\mathbf{Mor_S}) \longrightarrow \mathbf{pro\text{-}LES}(\mathbf{Ab}),$$

whose composition with the functor $\mathrm{Mor}_{\check{C}}$ gives the functor

$$(\text{pro-E}) \circ \mathrm{Mor}_{\check{C}} : \mathbf{Mor_{Top}} \longrightarrow \mathbf{pro\text{-}LES}(\mathbf{Ab}).$$

Now for each map $f : X \to Y$ define a long exact sequence of pro-groups. S. Mardešić proved that there exists a functor

$$\gamma : \mathbf{pro\text{-}LES}(\mathbf{Ab}) \longrightarrow \mathbf{LES}(\mathbf{pro\text{-}Ab})$$

which to each object $\{s_\alpha\}_{\alpha\in A}$ of the category $\mathbf{pro\text{-}LES}(\mathbf{Ab})$ consisting of long exact sequences of the category $\mathbf{Ab}$

$$s_\alpha : \longrightarrow G_\alpha^{n+1} \longrightarrow G_\alpha^n \xrightarrow{h_\alpha^n} G_\alpha^{n-1} \longrightarrow \cdots, \quad n \in Z, \quad \alpha \in A,$$

assigns the sequence of the category $\mathbf{LES}(\mathbf{pro\text{-}Ab})$

$$\gamma(\{s_\alpha\}_{\alpha\in A}) : \cdots \longrightarrow \{G_\alpha^{n+1}\}_{\alpha\in A} \longrightarrow \{G_\alpha^n\}_{\alpha\in A} \xrightarrow{\mathbf{h}^n} \{G_\alpha^{n-1}\}_{\alpha\in A} \longrightarrow \cdots,$$

where $\mathbf{h}^n = (\{h_\alpha^n\}_{\alpha\in A})$ (see [7, 13]). Composing the natural functor γ with the functor $(\text{pro-E}) \circ \mathrm{Mor}_{\check{C}}$ we define the functor

$$\gamma \circ (\text{pro-E}) \circ \mathrm{Mor}_{\check{C}} : \mathbf{Mor_{Top}} \longrightarrow \mathbf{LES}(\mathbf{pro\text{-}Ab}).$$

Let $\text{pro-}H_n(f; g) = \{H_n(f_\lambda; G)\}_{\lambda=(\alpha,\beta,\nu)\in \check{\mathrm{C}}\mathrm{ov}_N(f)}$ and $\check{H}_n(f; G) = \varprojlim \text{pro-}H_n(f; G)$. Thus we obtain

Theorem 3.4. *Let $f : X \to Y$ be a map. Then there exists a long exact sequence*

$$\cdots \longrightarrow \text{pro-}H_n(X; G) \longrightarrow \text{pro-}H_n(Y; G) \longrightarrow \text{pro-}H_n(f; G) \longrightarrow \text{pro-}H_{n-1}(X; G) \longrightarrow \cdots.$$

Applying the limit functor to the pro-homology sequence of the map f, we obtain the sequence

$$\cdots \longrightarrow \check{H}_n(X; G) \longrightarrow \check{H}_n(Y; G) \longrightarrow \check{H}_n(f; G) \longrightarrow \check{H}_{n-1}(X; G) \longrightarrow \cdots$$

called the spectral Čech homology sequence of f. However, this sequence is not exact even for the map of compact metric spaces.

According to [7]. we say that a map $f : X \to Y$ is movable if and only if $\mathrm{Mor}_{\check{\mathrm{C}}}(f)$ is a movable object in $\mathbf{pro\text{-}K}(\mathbf{Mor_S})$ (see [13]).

Let $f : X \to Y$ be a movable map of compact metric spaces. It is clear that the set $\check{\mathrm{C}}\mathrm{ov}_N(f)$ has a countable cofinal subset. Hence, each pro-group in the long exact sequence

$$\cdots \longrightarrow \text{pro-}H_n(X;G) \longrightarrow \text{pro-}H_n(Y;G) \longrightarrow \text{pro-}H_n(f;G) \longrightarrow \text{pro-}H_{n-1}(X;G) \longrightarrow \cdots$$

is indexed by a countable set. By the condition of the theorem the object $((\text{pro-E})\circ \mathrm{Mor}_{\check{\mathrm{C}}})(f)$ is movable in $\mathbf{pro\text{-}LES}(\mathbf{Ab})$. The functor γ takes a movable object in $\mathbf{pro\text{-}LES}(\mathbf{Ab})$ to a sequence of movable pro-groups. Consequently, each term of this sequence satisfies the Mittag-Leffler condition [7, 13]. By Theorem 3.4 and Theorem IV.1.2 of [7], the limit sequence is an exact sequence. Thus, we have the following

THEOREM 3.5. *If $f : X \to Y$ is a movable map of compact metric spaces, then there exists a long exact sequence*

$$\cdots \longrightarrow \check{H}_n(X;G) \longrightarrow \check{H}_n(Y;G) \longrightarrow \check{H}_n(f;G) \longrightarrow \check{H}_{n-1}(X;G) \longrightarrow \cdots .$$

The spectral Chogoshvili singular cohomology functor ${}_s\check{\mathrm{H}}(\text{-};G) : \mathbf{Top} \to \mathbf{Ab}$ with coefficients in abelian group G is the composition of functors $\mathrm{CH} : \mathbf{Top} \to \mathbf{inj\text{-}CW}$, $\text{pro-}\mathrm{H}^* : \mathbf{inj\text{-}CW} \to \mathbf{pro\text{-}Ab}$ and $\varprojlim : \mathbf{pro\text{-}Ab} \to \mathbf{Ab}$, where H^* is the cohomology functor of the category of CW-complexes [9]. An n-dimensional spectral Chogoshvili singular cohomology pro-group and group of space X are defined by the following formulas:

$$\begin{aligned} {}_s\,\text{pro-}H^n(X;G) &= ((\text{pro-}\mathrm{H}^n)\cdot \mathrm{CH})(X), X \in \mathrm{Ob}(\mathbf{Top}),\\ {}_s\check{H}^n(X;G) &= (\varprojlim \cdot(\text{pro-}\mathrm{H}^n)\cdot \mathrm{CH})(X), X \in \mathrm{Ob}(\mathbf{Top}). \end{aligned}$$

The spectral Chogoshvili singular homology functor ${}_s\check{\mathrm{H}}_*(\text{-};G) : \mathbf{Top} \to \mathbf{Ab}$ with coefficients in abelian group G is the composition of functors $\mathrm{CH} : \mathbf{Top} \to \mathbf{inj\text{-}CW}$, $\text{inj-}\mathrm{H}_n : \mathbf{inj\text{-}CW} \to \mathbf{inj\text{-}Ab}$ and $\varinjlim : \mathbf{pro\text{-}Ab} \to \mathbf{Ab}$, where H_* is the homology functor of the category of CW-complexes. An n-dimensional spectral Chogoshvili singular homology inj-group and group of space X are defined by the following formulas:

$$\begin{aligned} {}_s\,\text{inj-}H_n(X;G) &= ((\text{inj-}\mathrm{H}_n)\cdot \mathrm{CH})(X), X \in \mathrm{Ob}(\mathbf{Top}),\\ {}_s\check{H}_n(X;G) &= (\varinjlim \cdot(\text{inj-}\mathrm{H}_n)\cdot \mathrm{CH})(X), X \in \mathrm{Ob}(\mathbf{Top}). \end{aligned}$$

Now show that any map of topological spaces induces the long exact sequences of singular inj-groups and singular pro-groups. The assignment to each map $f : X \to Y$ of CW-spaces, the long exact sequences

$$\cdots \longrightarrow H_n(X;G) \longrightarrow H_n(Y;G) \longrightarrow H_n(f;G) \longrightarrow H_{n-1}(X;G) \longrightarrow \cdots ,$$
$$\cdots \longrightarrow H^{n-1}(X;G) \longrightarrow H^n(f;G) \longrightarrow H^n(Y;G) \longrightarrow H^n(X;G) \longrightarrow \cdots ,$$

where $H_n(f;G)$ and $H^n(f;G)$ are the homology and cohomology groups of f, define the functors

$$\mathrm{H}_* : \mathbf{Mor_{CW}} \longrightarrow \mathbf{LES(Ab)},$$
$$\mathrm{H}^* : \mathbf{Mor_{CW}} \longrightarrow \mathbf{LES(Ab)}.$$

The application of inj-functor yield the functors

$$\mathrm{inj}\text{-}\mathrm{H}_* : \mathbf{inj\text{-}Mor_{CW}} \to \mathbf{inj\text{-}LES(Ab)},$$
$$\mathrm{inj}\text{-}\mathrm{H}^* : \mathbf{inj\text{-}Mor_{CW}} \to \mathbf{pro\text{-}LES(Ab)}.$$

The composition of functors CH, $\mathrm{inj}\text{-}\mathrm{H}_*$ and δ gives the functor

$$\delta \cdot (\mathrm{inj}\text{-}\mathrm{H}) \cdot \mathrm{CH} : \mathbf{Mor_{Top}} \to \mathbf{LES(inj\text{-}Ab)}.$$

Let $\mathrm{CH}(f) = (|f_\lambda|, |\pi_{\lambda\lambda'}|, \Lambda)$. By ${}_s\,\mathrm{inj}\text{-}H_n(f;G)$ denote the homology inj-group $\{H_n(|f_\lambda|;G)\}_{\lambda\in\Lambda}$ of map f consisting of homology groups of $|f_\lambda|$. The resulting long exact sequence of inj-groups looks as follows.

Theorem 3.6. *For any map $f : X \to Y$ there is a long exact sequence*

$$\cdots \longrightarrow_s \mathrm{inj}\text{-}H_n(X;G) \longrightarrow_s \mathrm{inj}\text{-}H_n(Y;G) \longrightarrow_s \mathrm{inj}\text{-}H_n(f;G) \longrightarrow_s \mathrm{inj}\text{-}H_{n-1}(X;G) \longrightarrow \cdots .$$

Applying the direct limit functor $\varinjlim$ and using Lemma 1 of [13, Ch. II, §3.2] we obtain the following corollaries.

Corollary 3.7. *For each map $f : X \to Y$ there is an exact sequence*

$$\cdots \longrightarrow_s \check{H}_n(X;G) \longrightarrow_s \check{H}_n(Y;G) \longrightarrow_s \check{H}_n(f;G) \longrightarrow_s \check{H}_{n-1}(X;G) \longrightarrow \cdots ,$$

where ${}_s\check{H}_n(f;G) = \varinjlim :_s \mathrm{inj}\text{-}H_n(f;G)$.

By Theorem 10 of [13, Ch. II, §2.3] there exists a functor

$$\gamma : \mathbf{pro\text{-}LES(Ab)} \to \mathbf{LES(pro\text{-}Ab)},$$

which to each family $\{s_\alpha\}_{\alpha\in A} \in \mathbf{pro\text{-}LES(Gr)}$ of exact sequences

$$s_\alpha : \cdots \longrightarrow G_\alpha^{n+1} \longrightarrow G_\alpha^n \xrightarrow{h_\alpha^n} G_\alpha^{n-1} \longrightarrow \cdots , \quad n \in Z, \quad \alpha \in A,$$

assigns the exact sequence

$$\gamma(\{s_\alpha\}_{\alpha\in A}) : \cdots \{G_\alpha^{n+1}\}_{\alpha\in A} \longrightarrow \{G_\alpha^n\}_{\alpha\in A} \xrightarrow{\mathbf{h}^n} \{G_\alpha^{n-1}\}_{\alpha\in A} \longrightarrow \cdots ,$$

where $\mathbf{h}^n = (\{h_\alpha^n\}_{\alpha\in A}, 1_A)$ (cf. [6]). The composition of functors γ, $\mathrm{inj}\text{-}H$ and CH yields the functor

$$\gamma \cdot (\mathrm{inj}\text{-}\mathrm{H}^*) \cdot \mathrm{CH} : \mathrm{Mor}_{\mathbf{Top}} \to \mathbf{LES(pro\text{-}Gr)}.$$

By ${}_s\,\mathrm{pro}\text{-}H^n(f;G)$ denote the cohomology pro-group $\{H^n(|f_\lambda|;G)\}_{\lambda\in\Lambda}$ of map f consisting of cohomology groups of $|f_\lambda|$.

Thus we have the following

THEOREM 3.8. *For any map $f : X \to Y$ there is an exact sequence*

$$\cdots \longrightarrow_s \mathrm{pro}\text{-}H^{n-1}(X;G) \longrightarrow_s \mathrm{pro}\text{-}H^n(f;G) \longrightarrow_s \mathrm{pro}\text{-}H^n(Y;G) \longrightarrow_s \mathrm{pro}\text{-}H^n(X;G) \longrightarrow \cdots .$$

Also note that the limit sequence

$$\cdots \longrightarrow_s \check{H}^{n-1}(X;G) \longrightarrow_s \check{H}^n(f;G) \longrightarrow_s \check{H}^n(Y;G) \longrightarrow_s \check{H}^n(X;G) \longrightarrow \cdots ,$$

where ${}_s\check{H}^n(f;G) = \varprojlim : {}_s\,\mathrm{pro}\text{-}H^n(f;G)$, generally speaking, is not exact.

REMARK 3.9. The approximation theorems of maps proved in [1] and the methods and ideas developed in this paper could be used for the construction of long exact sequences of maps of pairs for various (co)homology and (co)homotopy groups [5, 6, 8, 22, 26], too.

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