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# COMMON FIXED POINT RESULTS FOR NON-LINEAR CONTRACTIONS IN $G$-METRIC SPACES

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**Abstract.** We establish common fixed point results for three self-mappings on a $G$-metric space satisfying non linear contractions. Also, we prove the uniqueness of such common fixed point, as well as studying the $G$-continuity at such point. Our results extend some known works. Also, an example is given to illustrate our obtained results.

## 1. Introduction and preliminaries

The notion of generalized metric spaces was introduced in 2004 by Z. Mustafa and B. Sims [3, 5, 6]. They generalized the concept of a metric space. Then, based on the notion of generalized metric spaces, many authors obtained some fixed point results for a self-mapping under some contractive conditions, see [1, 3–10]. In the present work, we study some common fixed point results for three self-mappings in a complete generalized metric space $X$ involving non linear contractions related to a function $\varphi \in \Phi$, where $\Phi$ is given by the following

Definition 1.1. Let $\Phi$ be the set of non-decreasing continuous functions $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying:

(a) $0 < \varphi(t) < t$ for all $t > 0$,

(b) the series $\sum_{n \geq 1} \varphi^n(t)$ converge for all $t > 0$.

From (b), we may have $\lim_{n \to +\infty} \varphi^n(t) = 0$ for all $t > 0$. Again from (a), we have $\varphi(0) = 0$. Now, we present some necessary definitions and results in generalized metric spaces, which will be needed in the sequel.

Definition 1.2. [5] Let $X$ be a nonempty set, and let $G : X \times X \times X \to R_+$ be a function satisfying the following properties

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$ for all $x, y \in X$, with $x \neq y$,

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(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, (symmetry in all three variables),

(G5) $G(x, y, z) \leq G(x, , a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality).

Then the function $G$ is called a generalized metric, or, more specially, a $G$-metric on $X$, and the pair $(X, G)$ is called a $G$-metric space.

Definition 1.3. [5] Let $(X, G)$ be a $G$-metric space and let $(x_n)$ be a sequence of points of $X$, a point $x \in X$ is said to be the limit of the sequence $(x_n)$, if $\lim_{n,m \to +\infty} G(x, x_n, x_m) = 0$, and we say that the sequence $(x_n)$ is $G$-convergent to $x$ or $(x_n)$ $G$-converges to $x$.

Thus, $x_n \to x$ in a $G$-metric space $(X, G)$ if for any $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$ for all $m, n \geq k$.

Proposition 1.4. [5] *Let $(X, G)$ be a $G$-metric space. Then the following are equivalent*

*(1) $\{x_n\}$ is is $G$-convergent to $x$;*

*(2) $G(x_n, x_n, x) \to 0$ as $n \to +\infty$*

*(3) $G(x_n, x, x) \to 0$ as $n \to +\infty$*

*(4) $G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.*

Definition 1.5. [5] Let $(X, G)$ be a $G$-metric space. A sequence $\{x_n\}$ is is called a $G$-Cauchy sequence if for any $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $m, n, l \geq k$, that is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 1.6. [6] *Let $(X, G)$ be a $G$-metric space. Then the following are equivalent:*

*(1) the sequence $\{x_n\}$ is $G$-Cauchy;*

*(2) for any $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $m, n \geq k$.*

Proposition 1.7. [5] *Let $(X, G)$ be a $G$-metric space. Then, $f : X \to X$ is $G$-continuous at $x \in X$ if and only if it is $G$-sequentially continuous at $x$, that is, whenever $(x_n)$ is $G$-convergent to $x$, $(f(x_n))$ is $G$-convergent to $f(x)$.*

Proposition 1.8. [5] *Let $(X, G)$ be a $G$-metric space. Then, the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Proposition 1.9. [5] A $G$-metric space $(X, G)$ is called $G$-complete if every $G$-Cauchy sequence is $G$-convergent in $(X, G)$.

Every $G$-metric on $X$ will define a metric $d_G$ on $X$ by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \text{ for all } x, y \in X. \tag{1.1}$$

In this paper, we address the question to find some common fixed point results on $G$-metric spaces. More precisely, taking three self-mappings on a complete $G$-metric space satisfying non-linear contractions, we establish a common fixed point result. Also, some corollaries and an example are given.

## 2. Main results

Our first main result is the following

THEOREM 2.1. *Let $(X,G)$ be a complete $G$-metric space. Suppose the maps $T_1,T_2,T_3 : X \to X$ satisfy for all $x,y,z \in X$*

$$G(T_1x,T_2y,T_3z) \leq \varphi(M(x,y,z)), \tag{2.1}$$

*where*

$$M(x,y,z) =: \max\{G(x,y,z),G(x,T_1x,T_1x),G(y,T_2y,T_2y),G(z,T_3z,T_3z)\},$$

*and $\varphi \in \Phi$. Then $T_1$, $T_2$ and $T_3$ have a unique common fixed point, say $u$. Moreover, each $T_i$, $i = 1,2,3$, is continuous at $u$.*

*Proof.* Let $x_0$ be an arbitrary point in $X$. Take $x_1 = T_1x_0$, $x_2 = T_2x_1$ and $x_3 = T_3x_2$. Then, we can construct a sequence $\{x_n\}$ in $X$ such that for any $n \in \mathbb{N}$

$$\begin{cases} x_{3n+1} = T_1x_{3n} \\ x_{3n+2} = T_2x_{3n+1} \\ x_{3n+3} = T_3x_{3n+2}. \end{cases} \tag{2.2}$$

• If there exists $p \in \mathbb{N}^*$ such that $x_{3p} = x_{3p+1} = x_{3p+2}$, then applying the contractive condition (2.1) with $x = x_{3p}$, $y = x_{3p+1}$ and $z = x_{3p+2}$, we get

$$\begin{aligned} G(x_{3p+1},x_{3p+2},x_{3p+3}) &=: G(T_1x_{3p},T_2x_{3p+1},T_3x_{3p+2}) \\ &\leq \varphi(\max\{G(x_{3p},x_{3p+1},x_{3p+2}),G(x_{3p},T_1x_{3p},T_1x_{3p}), \\ &\qquad G(x_{3p+1},T_2x_{3p+1},T_2x_{3p+1}),G(x_{3p+2},T_3x_{3p+2},T_3x_{3p+2})\}) \\ &= \varphi(\max\{G(x_{3p},x_{3p+1},x_{3p+2}),G(x_{3p},x_{3p+1},x_{3p+1}), \\ &\qquad G(x_{3p+1},x_{3p+2},x_{3p+2}),G(x_{3p+2},x_{3p+3},x_{3p+3})\}) \\ &= \varphi(G(x_{3p+2},x_{3p+3},x_{3p+3})). \end{aligned} \tag{2.3}$$

If $x_{3p+3} \neq x_{3p+1}$, then from the conditions (G3), (G4) and the property (a) of $\varphi$, we get

$$0 < G(x_{3p+1},x_{3p+2},x_{3p+3}) \leq \varphi(G(x_{3p+1},x_{3p+2},x_{3p+3})) < G(x_{3p+1},x_{3p+2},x_{3p+3}),$$

that is a contradiction. So we find $x_n = x_{3p}$ for any $n \geq 3p$. This implies that $(x_n)$ is a $G$-cauchy sequence. The same conclusion holds if $x_{3p+1} = x_{3p+2} = x_{3p+3}$, or $x_{3p+2} = x_{3p+3} = x_{3p+4}$ for some $p \in \mathbb{N}$.

• Assume for the rest that $x_n \neq x_m$ for any $n \neq m$. Applying again (2.1) with $x = x_{3n}$, $y = x_{3n+1}$ and $z = x_{3n+2}$ and using the condition (G3), we get that

$$G(x_{3n+1},x_{3n+2},x_{3n+3}) \leq \varphi(\max\{G(x_{3n},x_{3n+1},x_{3n+2}),G(x_{3n},x_{3n+1},x_{3n+1}),$$

$$G(x_{3n+1}, x_{3n+2}, x_{3n+2}), G(x_{3n+2}, x_{3n+3}, x_{3n+3})\})$$
$$= \varphi(\max\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n+1}, x_{3n+2}, x_{3n+3})\}). \tag{2.4}$$

The case where

$$\max\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n+1}, x_{3n+2}, x_{3n+3})\} = G(x_{3n+1}, x_{3n+2}, x_{3n+3})$$

is excluded, because if it holds we have from (2.4)

$$0 < G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq \varphi(G(x_{3n+1}, x_{3n+2}, x_{3n+3}))$$
$$< G(x_{3n+1}, x_{3n+2}, x_{3n+3}),$$

which is a contradiction. Thus, we deduce

$$\max\{G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n+1}, x_{3n+2}, x_{3n+3})\} = G(x_{3n}, x_{3n+1}, x_{3n+2}).$$

Therefore, (2.4) gives us

$$G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq \varphi(G(x_{3n}, x_{3n+1}, x_{3n+2})) < G(x_{3n}, x_{3n+1}, x_{3n+2}). \tag{2.5}$$

Similarly

$$G(x_{3n+2}, x_{3n+3}, x_{3n+4}) < G(x_{3n+1}, x_{3n+2}, x_{3n+3}),$$
$$G(x_{3n+3}, x_{3n+4}, x_{3n+5}) < G(x_{3n+2}, x_{3n+3}, x_{3n+4}).$$

From the above three inequalities, one can assert that

$$G(x_n, x_{n+1}, x_{n+2}) < G(x_{n-1}, x_n, x_{n+1}) \quad \forall\, n \in \mathbb{N}^*. \tag{2.6}$$

If we take $t_n = G(x_n, x_{n+1}, x_{n+2})$, then $0 \leq t_n \leq t_{n-1}$, so the real sequence $(t_n)$ is decreasing, hence it converges to some $r \geq 0$. Assume that $r > 0$, then letting $n \to +\infty$ in (2.5),

$$r \leq \varphi(r) < r,$$

using the properties of $\varphi$. It is a contradiction, so we have $r = 0$. Thus

$$\lim_{n \to +\infty} G(x_n, x_{n+1}, x_{n+2}) = 0. \tag{2.7}$$

Next, we prove that $(x_n)$ is a $G$-Cauchy sequence. Following (2.5) and (2.6), one can write

$$G(x_n, x_{n+1}, x_{n+2}) \leq \varphi(G(x_{n-1}, x_n, x_{n+1})). \tag{2.8}$$

Consequently,

$$G(x_n, x_{n+1}, x_{n+2}) \leq \varphi^n(G(x_0, x_1, x_2)). \tag{2.9}$$

Therefore, using conditions (G3), (G4), (G5) and (2.9), we have for any $k \in \mathbb{N}$

$$G(x_n, x_{n+k}, x_{n+k})$$
$$\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3})$$
$$+ \cdots + G(x_{n+k-2}, x_{n+k-1}, x_{n+k-1}) + G(x_{n+k-1}, x_{n+k}, x_{n+k})$$

$$\begin{aligned}
&\le G(x_n,x_{n+1},x_{n+2})+G(x_{n+1},x_{n+2},x_{n+3})+G(x_{n+2},x_{n+3},x_{n+4})\\
&\quad +\cdots+G(x_{n+k-2},x_{n+k-1},x_{n+k})+G(x_{n+k-1},x_{n+k},x_{n+k+1})\\
&\le \varphi^n(G(x_0,x_1,x_2))+\varphi^{n+1}(G(x_0,x_1,x_2))+\varphi^{n+2}(G(x_0,x_1,x_2))\\
&\quad +\cdots+\varphi^{n+k}(G(x_0,x_1,x_2))\\
&=\sum_{i=n}^{n+k}\varphi^i(G(x_0,x_1,x_2))\le\sum_{i=n}^{+\infty}\varphi^i(G(x_0,x_1,x_2)).
\end{aligned}$$

The property (b) yields that $\sum_{i=n}^{+\infty}\varphi^i(G(x_0,x_1,x_2))$ tends to 0 as $n\to+\infty$. Therefore

$$\lim_{n\to+\infty}G(x_n,x_{n+k},x_{n+k})=0 \quad \forall\, k\in\mathbb{N}.$$

This means that $(x_n)$ is a $G$-Cauchy sequence and since $(X,G)$ is $G$-complete, $(x_n)$ is $G$-convergent to some $u\in X$, that is

$$\lim_{n\to+\infty}G(x_n,x_n,u)=\lim_{n\to+\infty}G(x_n,u,u)=0. \tag{2.10}$$

Now, we show that $u$ is a common fixed point of the maps $T_i$, $i=1,2,3$. We start by proving the case $T_1u=u$. From (2.1), we get that

$$\begin{aligned}
&G(u,u,T_1u)\\
&\quad\le G(u,u,x_{3n+1})+G(x_{3n+1},x_{3n+1},T_1u)\\
&\quad= G(u,u,x_{3n+1})+G(T_1x_{3n},T_1x_{3n},T_1u)\\
&\quad\le G(u,u,x_{3n+1})+\varphi(M(x_{3n},x_{3n},u)),\quad (\text{here } T_3=T_2=T_1)\\
&\quad= G(u,u,x_{3n+1})+\varphi(\max\{G(x_{3n},x_{3n},u),G(x_{3n},T_1x_{3n},T_1x_{3n},G(u,u,T_1u)\})\\
&\quad= G(u,u,x_{3n+1})+\varphi(\max\{G(x_{3n},x_{3n},u),G(x_{3n},x_{3n+1},x_{3n+1}),G(u,u,T_1u)\}).
\end{aligned} \tag{2.11}$$

Using (2.10), the continuity of $\varphi$ and letting $n\to+\infty$ in (2.11), we get that $G(u,u,T_1u)\le\varphi(G(u,u,T_1u))$. Assume that $T_1u\neq u$; hence the condition (G2) implies that $G(u,u,T_1u)>0$, so

$$G(u,u,T_1u)\le\varphi(G(u,u,T_1u))<G(u,u,T_1u),$$

which is a contradiction, so $T_1u=u$. By symmetry, we can find that $T_2u=u=T_3u$, so $u$ is a common fixed point of the three maps $T_1$, $T_2$ and $T_3$. Let $v$ be another fixed point of each $T_i$, $i=1,2,3$. By (2.1)

$$\begin{aligned}
G(u,u,v)&=G(T_1u,T_2u,T_3v)\\
&\le\varphi(\max\{G(u,u,v),G(u,T_1u,T_1u),G(u,T_2u,T_2u),G(v,T_3v,T_3v)\})\\
&=\varphi(\max\{G(u,u,v),G(u,u,u),G(v,v,v)\})\\
&=\varphi(G(u,u,v)),
\end{aligned}$$

which is true unless $G(u,u,v)=0$. This yields that $u=v$. Let us show that each $T_i$, $i=1,2,3$, is $G$-continuous at $u$. By symmetry again, it suffices to prove the $G$-continuity of one of them, for example for $T_1$. For this, let $(u_n) \subseteq X$ be a sequence such that $(u_n)$ $G$-converges to $u$. First, we have

$$\begin{aligned} G(u,,T_1u_n,T_1u_n) &= G(T_1u,T_1u_n,T_1u_n) \\ &\leq \varphi(\max\{G(u,u_n,u_n),G(u,u,u),G(u_n,T_1u_n,T_1u_n)\}) \\ &= \varphi(\max\{G(u,u_n,u_n),G(u_n,T_1u_n,T_1u_n)\}) \\ &\leq \varphi(G(u,u_n,u_n)+G(u_n,T_1u_n,T_1u_n)) \\ &\leq \varphi(G(u,u_n,u_n)+G(u_n,u,u)+G(u,T_1u_n,T_1u_n)). \end{aligned}$$

Say $\lim_{n\to+\infty} G(u,T_1u_n,T_1u_n)=s$, then if $s>0$, using (2.10) and the continuity of $\varphi$ and letting $n\to+\infty$ in the above inequality we have

$$s \leq \varphi(s) < s;$$

it is a contradiction, hence $s=0$. On the other hand, we have

$$\begin{aligned} G(u,,u,T_1u_n) &= G(T_1u,T_1u,T_1u_n) \\ &\leq \varphi(\max\{G(u,u,u_n),G(u,u,u),G(u,u,u),G(u_n,T_1u_n,T_1u_n)\}) \\ &= \varphi(\max\{G(u,u,u_n),G(u_n,T_1u_n,T_1u_n)\}) \\ &\leq \varphi(\max\{G(u,u,u_n),G(u_n,u,u)+G(u,T_1u_n,T_1u_n)\}) \\ &= \varphi(G(u_n,u,u)+G(u,T_1u_n,T_1u_n)). \end{aligned}$$

Take $\lim_{n\to+\infty} G(u,u,T_1u_n)=t$; then letting $n\to+\infty$ and using $s=0$ and the continuity of $\varphi$, we get that

$$t \leq \varphi(0)=0,$$

that is $t=0$. We rewrite this as

$$\lim_{n\to+\infty} G(u,u,T_1u_n) =: \lim_{n\to+\infty} G(T_1u,T_1u,T_1u_n)=0.$$

This means that the sequence $(T_1u_n)$ $G$-converges to $u=T_1u$, so $T_1$ is $G$-continuous at $u$. By symmetry, we deduce that each $T_i$, $i=1,2,3$, is $G$-continuous at $u$. ∎

Now, we give some corollaries of Theorem 2.1. The first corresponds to $\varphi(t)=kt$ where $0\leq k<1$.

Corollary 2.2. *Let $X$ be a complete $G$-metric space. Suppose the maps $T_1,T_2,T_3 : X\to X$ satisfy*

$$G(T_1x,T_2y,T_3z) \leq k\max\{G(x,y,z),G(x,T_1x,T_1x),G(y,T_2y,T_2y),G(z,T_3z,T_3z)\}, \tag{2.12}$$

*for all $x,y,z\in X$, where $0\leq k<1$. Then, the mappings $T_i$, $i=1,2,3$ have a unique common fixed point, say $u$, and each $T_i$ is $G$-continuous at $u$.*

Corollary 2.3. *Let $X$ be a complete $G$-metric space. Suppose the maps $T_1,T_2,T_3 : X\to X$ satisfy*

$$G(T_1^m x,T_2^m y,T_3^m z) \leq \varphi(M(x,y,z)),$$

*for all* $x, y, z \in X$ *and* $m \in \mathbb{N}$, *where*

$$M(x,y,z) =: \max\{G(x,y,z), G(x,T_1^m x, T_1^m x), G(y,T_2^m y, T_2^m y), G(z,T_3^m z, T_3^m z)\},$$

*and* $\varphi \in \Phi$. *Then* $T_1^m$, $T_2^m$ *and* $T_3^m$ *have a unique common fixed point, say* $u$, *and are* $G$*-continuous at* $u$.

*Proof.* From Theorem 2.1, we conclude that the maps $T_1^m$, $T_2^m$ and $T_3^m$ have a unique common fixed point say $u$. For any $i = 1, 2, 3$

$$T_i u = T_i(T_i^m u) = T_i^{m+1} u = T_i^m(T_i u),$$

meaning that $T_i u$ is also a fixed point of $T_i^m$. By uniqueness of $u$, we get $T_i u = u$. ∎

We have again a common fixed point result for Hardy and Rogers's contraction type [2]. It is a consequence of Corollary 2.2 with $k = a + b + c + d$.

Corollary 2.4. *Let* $X$ *be a complete* $G$*-metric space. Suppose the maps* $T_1, T_2, T_3 : X \to X$ *satisfy*

$$G(T_1x, T_2y, T_3z) \le aG(x,y,z) + bG(x,T_1x,T_1x) + cG(y,T_2y,T_2y) + dG(z,T_3z,T_3z),$$

*for all* $x, y, z \in X$, *where* $a, b, c, d$ *are non-negative reals such that* $a+b+c+d < 1$. *Then* $T_1$, $T_2$ *and* $T_3$ *have a unique common fixed point, say* $u$, *and are* $G$*-continuous at* $u$.

Our Theorem 2.1 is again an extension of some recent new results by taking particular cases of $\varphi$ or $T = T_1 = T_2 = T_3$ in 2.1 or in the above corollaries. We cite them in the following corollaries.

Corollary 2.5. [4] *Let* $X$ *be a complete* $G$*-metric space. Suppose the map* $T : X \longrightarrow X$ *satisfies*

$$G(Tx,Ty,Tz) \le k \max\{G(x,y,z), G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz)\},$$

*for all* $x, y, z \in X$, *where* $0 \le k < 1$. *Then,* $T$ *has a unique fixed point, say* $u$, *and* $T$ *is* $G$*-continuous at* $u$.

Corollary 2.6. [4] *Let* $X$ *be a complete* $G$*-metric space. Suppose the map* $T : X \to X$ *satisfies for all* $x, y, z \in X$

$$G(Tx,Ty,Tz) \le aG(x,y,z) + bG(x,Tx,Tx) + cG(y,Ty,Ty) + dG(z,Tz,Tz),$$

*where* $a, b, c, d$ *are non-negative reals and* $a+b+c+d < 1$. *Then* $T$ *has a unique fixed point, say* $u$, *and* $T$ *is* $G$*-continuous at* $u$.

Corollary 2.7. [4] *Let* $X$ *be a complete* $G$*-metric space. Suppose the map* $T : X \to X$ *satisfies for* $m \in \mathbb{N}$ *and* $x, y, z \in X$

$$\begin{aligned} G(T^m x, T^m y, T^m z) \le\ & aG(x,y,z) + bG(x,T^m x,T^m x) \\ & + cG(y,T^m y,T^m y) + dG(z,T^m z,T^m z), \end{aligned}$$

*where $a, b, c, d$ are non-negative reals and $a+b+c+d<1$. Then $T^m$ has a unique fixed point, say $u$, and is G-continuous at $u$.*

We give an example illustrating our obtained results.

EXAMPLE 2.8 Let $X=[0,+\infty)$ be endowed with the complete $G$-metric given as follows:

$$G(x,y,z)=\max\{|x-y|,|x-z|,|y-z|\},$$

for all $x, y, z \in X$. Define $T_1, T_2, T_3 : X \to X$ by

$$T_1 t=\frac{t}{2}, \quad T_2 t=T_3 t=\frac{t}{4} \quad \forall\, t \geq 0.$$

Take $k=\frac{1}{4}$. Without loss of generality, we assume that $x \leq y \leq z$, so

$$G(x,y,z)=\max\{|x-y|,|x-z|,|y-z|\}=z-x,$$
$$G(x,T_1x,T_1x)=\frac{x}{2}, \quad G(y,T_2y,T_2y)=3\frac{y}{4} \quad \text{and} \quad G(z,T_3z,T_3z)=3\frac{z}{4}.$$

From these identities, the right-hand side of (2.12), denoted $R_{x,y,z}$, is equal to

$$R_{x,y,z}=\frac{1}{4}\max\{z-x,\frac{x}{2},\frac{3y}{4},\frac{3z}{4}\}=\frac{1}{4}\max\{z-x,\frac{3z}{4}\}. \tag{2.13}$$

While, the left-hand side of (2.12) is

$$G(T_1x,T_2y,T_3z)=\max\{|\frac{x}{2}-\frac{y}{4}|,|\frac{x}{2}-\frac{z}{4}|,|\frac{y}{4}-\frac{z}{4}|\}. \tag{2.14}$$

We distinguish the following cases:

- If $\frac{x}{2} \leq \frac{y}{4}$. From (2.14), we have $G(T_1x,T_2y,T_3z)=\frac{z}{4}-\frac{x}{2}$.

Case 1. If $\frac{z}{4} \geq x$. Here, we have from (2.13), $R_{x,y,z}=\frac{1}{4}(z-x)$. Then,

$$G(T_1x,T_2y,T_3z)=\frac{z}{4}-\frac{x}{2} \leq \frac{1}{4}(z-x)=R_{x,y,z}.$$

Case 2. If $\frac{z}{4} \leq x$. Here, we have from (2.13), $R_{x,y,z}=\frac{3}{16}z$. Then,

$$G(T_1x,T_2y,T_3z)=\frac{z}{4}-\frac{x}{2} \leq \frac{3}{16}z=R_{x,y,z}.$$

- If $\frac{x}{2} \geq \frac{y}{4}$. By (2.14), we have $G(T_1x,T_2y,T_3z)=\frac{z}{4}-\frac{y}{4}$.

Case 1. If $\frac{z}{4} \geq x$. By (2.13), we have $R_{x,y,z}=\frac{1}{4}(z-x)$, so

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Note that in all cases, the inequality (2.12) holds for all $x, y, z \in X$. The hypotheses of Corollary 2.2 satisfied, and 0 is the unique common fixed point of the mappings $T_1$, $T_2$ and $T_3$.

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