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GENERALIZED DISTANCE AND FIXED POINT THEOREMS IN PARTIALLY ORDERED PROBABILISTIC METRIC SPACES

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Abstract. Recently, Ćirić, Miheţ and Saadati [Topoplogy Appl. 156 (2009), 2838-2844] proved a common fixed point theorem in partially ordered probabilistic metric spaces. In this paper, we consider the generalized distance in probabilistic metric spaces introduced by Saadati, et. al., [Bull. Iranian Math. Soc. 35:2 (2009), 97–117] and prove some fixed point theorems in partially ordered probabilistic metric spaces.

1. Introduction

The Banach fixed point theorem for contraction mappings has been generalized and extended in many directions [1, 4, 6, 8, 9, 14, 15]. Recently Nieto and Rodriguez-Lopez [8] and Ran and Reurings [10] presented some new results for contractions in partially ordered metric spaces. The main idea in [8] and [10] involve combining the ideas of iterative technique in the contraction mapping principle with those in the monotone technique.

Recall that if $(X, \leq)$ is a partially ordered set and $F : X \to X$ is such that for $x, y \in X$, $x \leq y$ implies $F(x) \leq F(y)$, then a mapping F is said to be non-decreasing. The main result of Nieto and Rodriguez-Lopez [8] and Ran and Reurings [10] is the following fixed point theorem.

Theorem 1.1 *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose F is a non-decreasing mapping with*

$$d(F(x), F(y)) \leq k d(x, y) \tag{1.1}$$

for all $x, y \in X$, $x \leq y$, where $0 < k < 1$. Also suppose either

(a) *F is continuous or*

(b) *if $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to x$ in X,*

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then $x_n \leq x$ for all n hold. If there exists an $x_0 \in X$ with $x_0 \leq F(x_0)$ then F has a fixed point.

The works of Nieto and Rodriguez-Lopez [8] and Ran and Reurings [10] have motivated Agarwal et al. [1], Bhaskar and Lakshmikantham [2] and others authors [12] and [13] to undertake further investigation of fixed points in the area of ordered metric spaces.

2. Preliminaries

K. Menger introduced the notion of a probabilistic metric space in 1942 and since then the theory of probabilistic metric spaces has developed in many directions [16]. The idea of K. Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. The notion of a probabilistic metric space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. A probabilistic generalization of metric spaces appears to be of interest in the investigation of physical quantities and physiological thresholds. It is also of fundamental importance in probabilistic functional analysis.

Throughout this paper, the space of all probability distribution functions (briefly, d.f.'s) is denoted by $\Delta^+ = \{F : \mathbb{R} \cup \{-\infty, +\infty\} \longrightarrow [0,1] : F$ is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1\}$ and the subset $D^+ \subseteq \Delta^+$ is the set $D^+ = \{F \in \Delta^+ : l^-F(+\infty) = 1\}$. Here $l^-f(x)$ denotes the left limit of the function f at the point x, $l^-f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the d.f. given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 2.1. [16] A mapping $T : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t-norm if T satisfies the following conditions:

(a) T is commutative and associative;

(b) T is continuous;

(c) $T(a, 1) = a$ for all $a \in [0,1]$;

(d) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $c \leq d$, and $a, b, c, d \in [0,1]$.

Two typical examples of continuous t-norm are $T_P(a, b) = ab$ and $T_M(a, b) = \min(a, b)$.

Now t-norms are recursively defined by $T^1 = T$ and

$$T^n(x_1, \cdots, x_{n+1}) = T(T^{n-1}(x_1, \cdots, x_n), x_{n+1})$$

for $n \geq 2$ and $x_i \in [0,1]$, for all $i \in \{1, 2, \ldots, n+1\}$.

We say that a t-norm T is of Hadžić type if the family $\{T^n\}_{n\in\mathbb{N}}$ is equicontinuous at $x=1$, that is,

$$\forall\varepsilon\in(0,1)\ \exists\delta\in(0,1)\ a>1-\delta\ \Rightarrow\ T^n(a)>1-\varepsilon\quad(n\geq 1)\,.$$

T_M is a trivial example of a t-norm of Hadžić type, but T_P is not of Hadžić type.

Definition 2.2. A *Menger probabilistic metric space* (briefly, Menger PM-space) is a triple $(X,\mathcal{F},T)$, where X is a nonempty set, T is a continuous t-norm, and $\mathcal{F}$ is a mapping from $X\times X$ into D^+ such that, if $F_{x,y}$ denotes the value of $\mathcal{F}$ at the pair (x,y), the following conditions hold: for all x,y,z in X,

(PM1) $F_{x,y}(t)=\varepsilon_0(t)$ for all $t>0$ if and only if $x=y$;

(PM2) $F_{x,y}(t)=F_{y,x}(t)$;

(PM3) $F_{x,z}(t+s)\geq T(F_{x,y}(t),F_{y,z}(s))$ for all $x,y,z\in X$ and $t,s\geq 0$.

Definition 2.3. A *Menger probabilistic normed space* (briefly, Menger PN-space) is a triple (X,μ,T), where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that, the following conditions hold: for all x,y in X,

(PN1) $\mu_x(t)=\varepsilon_0(t)$ for all $t>0$ if and only if $x=0$;

(PN2) $\mu_{\alpha x}(t)=\mu_x(\frac{t}{|\alpha|})$ for $\alpha\neq 0$;

(PN3) $\mu_{x+y}(t+s)\geq T(\mu_x(t),\mu_y(s))$ for all $x,y,z\in X$ and $t,s\geq 0$.

Definition 2.4. Let $(X,\mathcal{F},T)$ be a Menger PM-space.

(1) A sequence $\{x_n\}_n$ in X is said to be *convergent* to x in X if, for every $\epsilon>0$ and $\lambda>0$, there exists positive integer N such that $F_{x_n,x}(\epsilon)>1-\lambda$ whenever $n\geq N$.

(2) A sequence $\{x_n\}_n$ in X is called *Cauchy sequence* if, for every $\epsilon>0$ and $\lambda>0$, there exists positive integer N such that $F_{x_n,x_m}(\epsilon)>1-\lambda$ whenever $n,m\geq N$.

(3) A Menger PM-space $(X,\mathcal{F},T)$ is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

Definition 2.5. Let $(X,\mathcal{F},T)$ be a Menger PM space. For each p in X and $\lambda>0$, the strong $\lambda-$*neighborhood* of p is the set

$$N_p(\lambda)=\{q\in X: F_{p,q}(\lambda)>1-\lambda\},$$

and the strong neighborhood system for X is the union $\bigcup_{p\in V}\mathcal{N}_p$ where $\mathcal{N}_p=\{N_p(\lambda):\lambda>0\}$.

The strong neighborhood system for X determines a Hausdorff topology for X.

Theorem 2.6. [16] *If $(X,\mathcal{F},T)$ is a PM-space and $\{p_n\}$ and $\{q_n\}$ are sequences such that $p_n\to p$ and $q_n\to q$, then $\lim_{n\to\infty}F_{p_n,q_n}(t)=F_{p,q}(t)$ for every continuity point t of $F_{p,q}$.*

3. r-distance

Kada, Suzuki and Takahashi [5] introduced the concept of w-distance on a metric space and proved some fixed point theorems. Using the concept of w-distance, Saadati et. al. [11] defined the concept of r-distance on a Menger PM-space. In this section, we review the r-distance and its properties, for more details, see [11].

Definition 3.1. Let $(X, \mathcal{F}, T)$ be a Menger PM-space. Then the function $f : X^2 \times [0, \infty] \longrightarrow [0, 1]$ is called a r-distance on X if the following are satisfied:

(r1) $f_{x,z}(t + s) \geq T(f_{x,y}(t), f_{y,z}(s))$ for all $x, y, z \in X$ and $t, s \geq 0$;

(r2) for any $x \in X$ and $t \geq 0$, $f_{x,.} : X \times [0, \infty] \longrightarrow [0, 1]$ is continuous;

(r3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $f_{z,x}(t) \geq 1-\delta$ and $f_{z,y}(s) \geq 1-\delta$ imply $F_{x,y}(t + s) \geq 1 - \varepsilon$.

Let us give some examples of r-distance.

Example 3.2. Let $(X, \mathcal{F}, T)$ be a Menger PM-space. Then $f = F$ is a r-distance on X.

Proof. Now (r1) and (r2) are obvious. We show (r3). Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that

$$T(1 - \delta, 1 - \delta) \geq 1 - \varepsilon.$$

Then, for $F_{z,x}(t) \geq 1 - \delta$ and $F_{z,y}(s) \geq 1 - \delta$ we have

$$F_{x,y}(t + s) \geq T(F_{z,x}(t), F_{z,y}(s)) \geq T(1 - \delta, 1 - \delta) \geq 1 - \varepsilon. \quad \blacksquare$$

Example 3.3. Let $(X, \mathcal{F}, T)$ be a Menger PM-space. Then the function $f : X^2 \times [0, \infty) \longrightarrow [0, 1]$ defined by $f_{x,y}(t) = 1 - c$ for every $x, y \in X$ and $t > 0$ is a r-distance on X, where $c \in]0, 1[$.

Proof. Now (r1) and (r2) are obvious. To show (r3), for any $\varepsilon > 0$, put $\delta = 1 - c/2$. Then we have that $f_{z,x}(t) \geq 1 - c/2$ and $f_{z,y}(s) \geq 1 - c/2$ imply $F_{x,y}(t + s) \geq 1 - \varepsilon$.

Example 3.4. Let (X, μ, T) be a Menger PN-space. Then the function $f : X^2 \times [0, \infty) \longrightarrow [0, 1]$ defined by $f_{x,y}(t + s) = T(\mu_x(t), \mu_y(s))$ for every $x, y \in X$ and $t, s > 0$ is a r-distance on X.

Proof. Let $x, y, z \in X$ and $t, s > 0$. Then we have

$$\begin{aligned} f_{x,z}(t + s) &= T(\mu_x(t), \mu_z(s)) \geq T(T(\mu_x(t/2), \mu_y(t/2)), T(\mu_y(s/2), \mu_z(s/2))) \\ &= T(f_{x,y}(t), f_{y,z}(s)). \end{aligned}$$

Hence (r1) holds. Also (r2) is obvious. Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that $T(1 - \delta, 1 - \delta) \geq 1 - \varepsilon$. Then, for $f_{z,x}(t) \geq 1 - \delta$ and $f_{z,y}(s) \geq 1 - \delta$ we have

$$\begin{aligned} F_{x,y}(t + s) &= \mu_{x-y}(t + s) \geq T(\mu_x(t), \mu_y(s)) \\ &\geq T(T(\mu_x(t/2), \mu_z(t/2)), T(\mu_y(s/2), \mu_z(s/2))) \\ &= T(f_{z,x}(t), f_{z,y}(s)) \geq T(1 - \delta, 1 - \delta) \geq 1 - \varepsilon. \end{aligned}$$

Hence (r3)also holds. $\blacksquare$

EXAMPLE 3.5. Let (X, μ, T) be a Menger PN-space. Then the function $f : X^2 \times [0, \infty] \longrightarrow [0, 1]$ defined by $f_{x,y}(t) = \mu_x(t)$ for every $x, y \in X$ and $t > 0$ is a r-distance on X.

Proof. Let $x, y, z \in X$ and $t, s > 0$. Then we have

$$f_{x,z}(t+s) = \mu_z(t+s) \geq T(\mu_y(t), \mu_z(s)) = T(f_{x,y}(t), f_{y,z}(s)).$$

Hence (r1) holds. Also (r2) is obvious. Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that $T(1-\delta, 1-\delta) \geq 1-\varepsilon$. Then, for $f_{z,x}(t) \geq 1-\delta$ and $f_{z,y}(s) \geq 1-\delta$ we have

$$\begin{aligned} F_{x,y}(t+s) &= \mu_{x-y}(t+s) \geq T(\mu_x(t), \mu_y(s)) = T(f_{z,x}(t), f_{z,y}(s)) \\ &\geq T(1-\delta, 1-\delta) \geq 1-\varepsilon. \end{aligned}$$

Hence (r3) holds. ∎

EXAMPLE 3.6. Let $(X, \mathcal{F}, T)$ be a Menger PM-space and let A be a continuous mapping from X into X. Then the function $f : X^2 \times [0, \infty] \longrightarrow [0, 1]$ defined by

$$f_{x,y}(t) = \min(F_{Ax,y}(t), F_{Ax,Ay}(s))$$

for every $x, y \in X$ and $t, s > 0$ is a r-distance on X.

Proof. Let $x, y, z \in X$ and $t, s > 0$. If $F_{Ax,z}(t) \leq F_{Ax,Ay}(t)$ then we have

$$\begin{aligned} f_{x,z}(t+s) &= F_{Ax,z}(t+s) \geq T(F_{Ax,Ay}(t), F_{Ay,z}(s)) \\ &\geq T(\min(F_{Ax,y}(t), F_{Ax,Ay}(t)), \min(F_{Ay,z}(s), F_{Ax,Ay}(s)) \\ &= T(f_{x,y}(t), f_{y,z}(s)). \end{aligned}$$

With this inequality, we have

$$\begin{aligned} f_{x,z}(t+s) &= F_{Ax,Az}(t+s) \geq T(F_{Ax,Ay}(t), F_{Ay,Az}(s)) \\ &\geq T(\min(F_{Ax,y}(t), F_{Ax,Ay}(t)), \min(F_{Ay,z}(s), F_{Ax,Ay}(s)) \\ &= T(f_{x,y}(t), f_{y,z}(s)). \end{aligned}$$

Hence (r1) holds. Since A is continuous, (r2) is obvious . Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that $T(1-\delta, 1-\delta) \geq 1-\varepsilon$. Then, from $f_{z,x}(t) \geq 1-\delta$ and $f_{z,y}(s) \geq 1-\delta$ we have $F_{Az,x}(t) \geq 1-\delta$ and $F_{Az,y}(s) \geq 1-\delta$. Therefore

$$F_{x,y}(t+s) \geq T(F_{Az,x}(t), F_{Az,y}(s)) \geq T(1-\delta, 1-\delta) \geq 1-\varepsilon.$$

Hence (r3) holds. ∎

Next, we discuss some properties of r-distance.

LEMMA 3.7. *Let $(X, \mathcal{F}, T)$ be a Menger PM-space and let f be a r-distance on it. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, \infty)$ converging to zero, and let $x, y, z \in X$ and $t, s > 0$. Then the following hold:*

(1) *if $f_{x_n,y}(t) \geq 1-\alpha_n$ and $f_{x_n,z}(s) \geq 1-\beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $f_{x,y}(t) = 1$ and $f_{x,z}(s) = 1$, then $y = z$;*

(2) *if $f_{x_n,y_n}(t) \geq 1-\alpha_n$ and $f_{x_n,z}(s) \geq 1-\beta_n$ for any $n \in \mathbb{N}$, then $F_{y_n,z}(t+s) \to 1$;*

(3) *if* $f_{x_n,x_m}(t) \geq 1 - \alpha_n$ *for any* $n, m \in \mathbb{N}$ *with* $m > n$, *then* $\{x_n\}$ *is a Cauchy sequence;*

(4) *if* $f_{y,x_n}(t) \geq 1 - \alpha_n$ *for any* $n \in \mathbb{N}$, *then* $\{x_n\}$ *is a Cauchy sequence.*

Proof. We first prove (2). Let $\varepsilon > 0$ be given. From the definition of r-distance, there exists $\delta > 0$ such that $f_{u,v}(t) \geq 1 - \delta$ and $f_{u,z}(s) \geq 1 - \delta$ imply $F_{v,z}(t + s) \geq 1 - \varepsilon$. Choose $n_0 \in \mathbb{N}$ such that $\alpha_n \leq \delta$ and $\beta_n \leq \delta$ for every $n \geq n_0$. Then we have, for any $n \geq n_0$ $f_{x_n,y_n}(t) \geq 1 - \alpha_n \geq 1 - \delta$ and $f_{x_n,z}(t) \geq 1 - \beta_n \geq 1 - \delta$ and hence $F_{y_n,z}(t + s) \geq 1 - \varepsilon$. This implies that $\{y_n\}$ converges to z. It follows from (2) that (1) holds. Let us prove (3). Let $\varepsilon > 0$ be given. As in the proof of (1), choose $\delta > 0$ and then $n_0 \in \mathbb{N}$. Then for any $n, m \geq n_0 + 1$,

$$f_{x_{n_0},x_n}(t) \geq 1 - \alpha_{n_0} \geq 1 - \delta \text{ and } f_{x_{n_0},x_m}(s) \geq 1 - \alpha_{n_0} \geq 1 - \delta$$

and hence $F_{x_n,x_m}(t + s) \geq 1 - \varepsilon$. This implies that $\{x_n\}$ is a Cauchy sequence. ∎

3. Main results

We introduce first the following concept.

Definition 4.1. Suppose $(X, \leq)$ is a partially ordered set and $f : X \to X$ be a self mapping on X. We say f is inverse increasing if for $x, y \in X$,

$$f(x) \leq f(y) \text{ implies } x \leq y. \tag{4.1}$$

In the proof of our first theorem we use the following two lemmas:

Lemma 4.2. *Let* (X, F, T) *be a PM space with* T *of Hadžić-type and* f *be a* r*-distance on* $(X, \mathcal{F}, T)$. *Let* $\{x_n\}$ *be a sequence in* X *such that, for some* $k \in (0, 1)$,

$$f_{x_n,x_{n+1}}(kt) \geq f_{x_{n-1},x_n}(t) \quad (n \geq 1, t > 0).$$

Then $\{x_n\}$ *is a Cauchy sequence.*

Proof. The proof is similar to Lemma 2.1 of [7], see also [3]. ∎

Theorem 4.3. *Let* $(X, \leq)$ *be a partially ordered set and* $(X, \mathcal{F}, T)$ *be a complete PM-space under a t-norm* T_M *of Hadžić-type. Let* f *be an* r*-distance on* $(X, \mathcal{F}, T)$. *Let* $A : X \to X$ *be a non-decreasing self-mapping on* X *and there exists* $k \in (0, 1)$ *such that*

$$f_{A(x),A^2(x)}(kt) \geq f_{x,A(x)}(t), \quad \textit{for all } x \leq Ax. \tag{4.2}$$

Suppose also that:

(i) *for every* $x \in X$ *with* $x \leq Ax$

$$\sup\{T(f_{x,y}(t), f_{x,Ax}(t))\} < 1, \textit{ for every } y \in X \textit{ with } y \neq Ay. \tag{4.3}$$

(ii) *there exists* $x_0 \in X$ *such that* $x_0 \leq Ax_0$.

Then A *has a fixed point in* X.

Proof. If $Ax_0 = x_0$, then the proof is finished. Suppose that $Ax_0 \neq x_0$. Since $x_0 \leq Ax_0$ and A is non-decreasing, we obtain

$$x_0 \leq Ax_0 \leq A^2 x_0 \leq \cdots \leq A^{n+1} x_0 \leq \cdots .$$

Hence, for each $n \in \mathbb{N}$ we have

$$f_{A^n x_0, A^{n+1} x_0}(t) \geq f_{x_0, Ax_0}\left(\frac{t}{k^n}\right). \tag{4.4}$$

Now, since $\frac{k^n}{1-k} \geq \sum_{i=n}^{m} k^i$, then for $m \geq n \in \mathbb{N}$, we successively have

$$\begin{aligned} f_{A^n x_0, A^m x_0}\left(\frac{k^n}{1-k} t\right) &\geq f_{A^n x_0, A^m x_0}\left(\sum_{i=n}^{m} k^i t\right) \\ &\geq T(f_{x_0, Ax_0}(k^n t), \ldots, f_{x_0, Ax_0}(k^m t) \geq f_{x_0, Ax_0}(t), \end{aligned}$$

and therefore,

$$f_{A^n x_0, A^m x_0}(t) \geq f_{x_0, Ax_0}\left(\frac{1-k}{k^n} t\right).$$

By Lemma 4.2, we conclude that $\{A^n x_0\}$ is Cauchy sequence in $(X, \mathcal{F}, T)$. Since $(X, \mathcal{F}, T)$ is a complete PM-space, there exists $z \in X$ such that $\lim_{n \to \infty} A^n x_0 = z$. Let $n \in \mathbb{N}$ be an arbitrary but fixed. Then since $\{A^n x_0\}$ converges to z in $(X, \mathcal{F}, T)$ and $f_{A^n x_0, \cdot}$ is continuous, we have

$$f_{A^n x_0, z}(t) \geq \lim_{m \to \infty} f_{A^n x_0, A^m x_0}(t) \geq f_{x_0, Ax_0}\left(\frac{1-k}{k^n} t\right).$$

Assume that $z \neq Az$. Since $A^n x_0 \leq A^{n+1} x_0$, by (4.3), we have

$$\begin{aligned} 1 &> \sup\{T(f_{A^n x_0, z}(t), f_{A^n x_0, A^{n+1} x_0}(t))\} \\ &\geq \sup\left\{T(f_{x_0, Ax_0}\left(\frac{1-k}{k^n} t\right), f_{x_0, Ax_0}\left(\frac{t}{k^n}\right))\right\} = 1. \end{aligned}$$

This is a contradiction. Therefore, we have $z = Az$. ∎

Another result of this type is the following

Theorem 4.4. *Let $(X, \leq)$ be a partially ordered set and $(X, \mathcal{F}, T)$ be a complete PM-space under a t-norm T_M of Hadžić-type. Let f be a r-distance on $(X, \mathcal{F}, T)$. Let $A : X \to X$ be a non-decreasing mapping and there exists $k \in (0, 1)$ such that* (4.2) *holds. Assume that one of the following assertions holds:*

(i) *for every $x \in X$ with $x \leq Ax$*

$$\sup\{T(f_{x,y}(t), f_{x,Ax}(t))\} < 1, \text{ for every } y \in X \text{ with } y \neq Ay \text{ and } t > 0. \tag{4.5}$$

(ii) *if both $\{x_n\}$ and $\{Ax_n\}$ converge to y, then $y = Ay$;*

(iii) *A is continuous.*

If there exists $x_0 \in X$ with $x_0 \leq Ax_0$, then A has a fixed point in X.

Proof. The case (i), was proved in Theorem 4.4.

Let us prove first that (ii)$\Longrightarrow$(i). Assume that there exists $y \in X$ with $y \neq Ay$ such that

$$\sup\{T(f_{x,y}(t), f_{x,Ax}(t))\} = 1.$$

Then there exists $\{z_n\} \in X$ such that $z_n \leq Az_n$ and

$$\lim_{n\to\infty} T(f_{z_n,y}(t), f_{z_n,Az_n}(t)) = 1.$$

Then $f_{z_n,y} \longrightarrow 1$ and $f_{z_n,Az_n} \longrightarrow 1$. By Lemma 3.7, we have that $Az_n \longrightarrow y$. We also have

$$\begin{aligned} f_{z_n,A^2z_n}(t) &\geq T\left(f_{z_n,Az_n}\left(\frac{t}{2}\right), f_{Az_n,A^2z_n}\left(\frac{t}{2}\right)\right) \\ &\geq f_{z_n,Az_n}\left(\frac{t}{2k}\right) \longrightarrow 1 \end{aligned}$$

Again by Lemma 3.7, we get $A^2z_n \longrightarrow y$. Put $x_n = Az_n$. Then both $\{x_n\}$ and $\{Ax_n\}$ converges to y. Thus, by (ii) we have $y = Ay$. Thus (ii)$\Longrightarrow$(i) holds.

Now, we show that (iii)$\Longrightarrow$(ii). Let A be continuous. Further assume that $\{x_n\}$ and $\{Ax_n\}$ converges to y. Then we have

$$Ay = A(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} x_n = y. \quad \blacksquare$$

5. Common fixed point theorem for commuting mappings

The next example shows that if the mapping $h : X \to X$ is continuous with respect to $(X, \mathcal{F}, T)$ and $g : X \to X$ satisfies the condition

$$f_{g(x),g(y)}(t) \geq f_{h(x),h(y)}\left(\frac{t}{k}\right), \text{ for all } x, y \in X,\ t > 0 \text{ and some } k \in (0,1),$$

then, in general, g may be not continuous in $(X, \mathcal{F}, T)$.

Example 5.1. Let $X := (\mathbb{R}, |\cdot|)$ be a normed linear space. Consider Example 3.5 with r-distance defined by

$$f_{x,y}(t) = \mu_y(t) = \frac{t}{t+|y|} \text{ for every } x, y \in \mathbb{R} \text{ and } t \geq 0.$$

Consider the functions h and g defined by $h(x) = 4$ and

$$g(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then

$$f_{g(x),g(y)}(t) = \frac{t}{t+|g(y)|} \geq \frac{t}{t+1} \geq \frac{t}{t+\frac{4}{3}} = \frac{t}{t+|h(y)|} = f_{h(x),h(y)}\left(\frac{t}{\frac{1}{3}}\right).$$

Definition 5.2. Let $(X, \leq)$ be a partially ordered set and $g, h : X \to X$. By definition, we say that g is h-non-decreasing if for $x, y \in X$,

$$h(x) \leq h(y) \text{ implies } g(x) \leq g(y). \tag{5.1}$$

Theorem 5.3. *Let $(X, \leq)$ be a partially ordered set and $(X, \mathcal{F}, T)$ be a complete PM-space under a t-norm T_M of Hadžić-type. Let f be a r-distance on $(X, \mathcal{F}, T)$. Let $h, g : X \longrightarrow X$ be mappings that satisfy the following conditions:*

(a) *$g(X) \subseteq h(X)$;*

(b) *g is h-non-decreasing and h is inverse increasing;*

(c) *g commutes with h and h, g are continuous in $(X, \mathcal{F}, T)$;*

(d) *$f_{g(x),g(y)}(t) \geq f_{h(x),h(y)}\left(\frac{t}{k}\right)$ for all $x, y \in X$ with $x \leq y$, $t > 0$ and some $0 < k < 1$.*

(e) *there exists $x_0 \in X$ such that:*

(i) *$h(x_0) \leq g(x_0)$ and* (ii) *$h(x_0) \leq h(g(x_0))$.*

Then h and g have a common fixed point $u \in X$. Moreover, if $g(v) = g^2(v)$ for all $v \in X$, then $f_{u,u} = 1$.

Proof. We claim that for every $h(x) \leq g(x)$ and $t > 0$,

$$\sup\{T(f_{h(x),g(x)}(t), f_{h(x),z}(t), f_{g(x),z}(t), f_{g(x),g(g(x)}(t)))\} < 1$$

for every $z \in X$ with $g(z) \neq g(g(z))$. For the moment suppose the claim is true. Let $x_0 \in X$ with $h(x_0) \leq g(x_0)$. By (a) we can find $x_1 \in X$ such that $h(x_1) = g(x_0)$. By induction, we can define a sequence $\{x_n\}_n \in X$ such that

$$h(x_n) = g(x_{n-1}). \tag{5.2}$$

Since $h(x_0) \leq g(x_0)$ and $h(x_1) = g(x_0)$, we have

$$h(x_0) \leq h(x_1). \tag{5.3}$$

Then from (b), $g(x_0) \leq g(x_1)$, that means, by (5.2), that $h(x_1) \leq h(x_2)$. Again by (b) we get $g(x_1) \leq g(x_2)$, that is, $h(x_2) \leq h(x_3)$. By this procedure, we obtain

$$g(x_0) \leq g(x_1) \leq g(x_2) \leq g(x_3) \leq \cdots \leq g(x_n) \leq g(x_{n+1}) \leq \cdots. \tag{5.4}$$

Hence from (5.2) and (5.4) we have $h(x_{n-1}) \leq h(x_n)$ and by (4.1) we have $x_{n-1} \leq x_n$. By induction we get that

$$f_{h(x_n),h(x_{n+1})}(t) = f_{g(x_{n-1}),g(x_n)}(t) \geq f_{h(x_{n-1}),h(x_n)}\left(\frac{t}{k}\right) \geq \cdots \geq f_{h(x_0),h(x_1)}\left(\frac{t}{k^n}\right)$$

for $n = 1, 2, \cdots$. Also, since $\frac{k^n}{1-k} \geq \sum_{i=n}^{m} k^i$, then for $m \geq n \in \mathbb{N}$, with $m > n$,

$$\begin{aligned} f_{h(x_n),h(x_m)}\left(\frac{k^n}{1-k}t\right) &\geq f_{h(x_n),h(x_m)}\left(\sum_{i=n}^{m} k^i t\right) \\ &\geq T(f_{h(x_{m-1},h(x_m)}(k^m t), \cdots f_{h(x_n),h(x_{n+1})}(k^n t)) \\ &\geq f_{h(x_0),h(x_1)}(t), \end{aligned}$$

which implies that,

$$f_{h(x_n),h(x_m)}(t) \geq f_{h(x_0),h(x_1)}\left(\frac{1-k}{k^n}t\right).$$

Thus, by Lemma 3.7, we obtain that $\{h(x_n)\}$ is a Cauchy sequence in $(X, \mathcal{F}, T)$. Since $(X, \mathcal{F}, T)$ is complete, there exists $y \in X$ such that $\lim_{n\to\infty} h(x_n) = y$. As a result the sequence $g(x_{n-1}) = h(x_n)$ tends to y as $n \to +\infty$ and hence $\{g(h(x_n))\}_n$ converges to $g(y)$ as $n \to +\infty$. However, $g(h(x_n)) = h(g(x_n))$, by the commutativity of h and g, implies that $h(g(x_n))$ converges to $h(y)$ as $n \to +\infty$. Because limit is unique, we get $h(y) = g(y)$ and, thus, $h(h(y)) = h(g(y))$. On the other hand, by lower continuity of $f_{x,\cdot}$ we have, for each $n \in \mathbb{N}$, that

$$f_{h(x_n),y}(t) \geq \lim_{m\to\infty} f_{h(x_n),h(x_m)}(t) \geq f_{h(x_0),h(x_1)}\left(\frac{1-k}{k^n}t\right),$$
$$f_{g(x_n),y}(t) \geq \lim_{m\to\infty} f_{h(x_{n+1}),h(x_m)}(t) \geq f_{h(x_0),h(x_1)}\left(\frac{1-k}{k^{n+1}}t\right).$$

Notice that, by (5.1), (5.2) and (5.3) we obtain $h(x_0) \leq h(h(x_1))$ and thus, by (5.1), we get $g(x_0) \leq g(h(x_1))$. Then

$$h(x_1) \leq g(h(x_1)) = h(g(x_1)) = h(h(x_2)).$$

By (5.1) we get that $g(x_1) \leq g(h(x_2))$ and thus $h(x_2) \leq h(g(x_2))$. Continuing this process we get

$$h(x_n) \leq h(g(x_n)),\ n = 0, 1, 2, 3, \dots,$$

and by (4.1) we get $x_n \leq g(x_n)$, $n = 0, 1, 2, \dots$

Using now the condition (d), we have

$$\begin{aligned} f_{g(x_n),g(g(x_n))}(t) &\geq f_{h(x_n),h(g(x_n)}\left(\frac{t}{k}\right) = f_{g(x_{n-1}),g(g(x_{n-1}))}\left(\frac{t}{k}\right) \\ &\geq f_{h(x_{n-1}),h(g(x_{n-1}))}\left(\frac{t}{k^2}\right) = f_{g(x_{n-2}),g(g(x_{n-2}))}\left(\frac{t}{k^2}\right) \\ &\geq \dots \geq f_{h(x_1),g(h(x_1))}\left(\frac{t}{k^n}\right). \end{aligned}$$

We will show that $g(y) = g(g(y))$. Suppose, by contradiction, that $g(y) \neq g(g(y)))$. Then, for every $t > 0$ we have:

$$\begin{aligned} 1 &> \sup\{T(f_{h(x),g(x)}(t), f_{h(x),y}(t), f_{g(x),y}(t), f_{g(x),g(g(x))}(t) : x \in X\} \\ &\geq \sup\{T(f_{h(x_n),g(x_n)}(t), f_{h(x_n),y}(t), f_{g(x_n),y}(t), f_{g(x_n),g(g(x_n))}(t)) : n \in \mathbb{N}\} \\ &= \sup\{T(f_{h(x_n),h(x_{n+1})}(t), f_{h(x_n),y}(t), f_{g(x_n),y}(t), f_{g(x_n),g(g(x_n))}(t) : n \in \mathbb{N}\} \\ &\geq \sup_n \Big\{T\Big(f_{h(x_0),h(x_1)}\left(\frac{t}{k^n}\right), f_{h(x_0),h(x_1)}\left(\frac{t}{k^n}\right), f_{h(x_0),h(x_1)}\left(\frac{t}{k^{n+1}}\right), \\ &\qquad f_{h(x_1),g(h(x_1))}\left(\frac{t}{k^n}\right)\Big) : n \in \mathbb{N}\Big\} = 1. \end{aligned}$$

This is a contradiction. Therefore $g(y) = g(g(y))$. Thus, $g(y) = g(g(y)) = h(g(y))$. Hence $u := g(y)$ is a common fixed point of h and g.

Furthermore, since $g(v) = g(g(v))$ for all $v \in X$, we have

$$f_{g(y),g(y)}(t) = f_{g(g(y)),g(g(y))}(t) \geq f_{h(g(y)),h(g(y))}\left(\frac{t}{k}\right) = f_{g(y),g(y)}\left(\frac{t}{k}\right),$$

which implies that, $f_{g(y),g(y)} = 1$.

Now it remains to prove the initial claim. Assume that there exists $y \in X$ with $g(y) \neq g(g(y))$ and

$$\sup\{T(f_{h(x),g(x)}(t), f_{h(x),y}(t), f_{g(x),y}(t), f_{g(x),g(g(x))}(t)) : x \in X\} = 1.$$

Then there exists $\{x_n\}$ such that

$$\lim_{n\to\infty}\{T(f_{h(x_n),g(x_n)}(t), f_{h(x_n),y}(t), f_{g(x_n),y}(t), f_{g(x_n),g(g(x_n))}(t))\} = 1.$$

Since $f_{h(x_n),g(x_n)}(t) \longrightarrow 1$ and $f_{h(x_n),y}(t) \longrightarrow 1$, by Lemma 3.7, we have

$$\lim_{n\to\infty} g(x_n) = y. \tag{5.5}$$

Also, since $f_{g(x_n),y}(t) \longrightarrow 1$ and $f_{g(x_n),g(g(x_n))}(t) \longrightarrow 1$, by Lemma 3.7, we have

$$\lim_{n\to\infty} g(g(x_n)) = y. \tag{5.6}$$

By (5.5), (5.6) and the continuity of g we have

$$g(y) = g(\lim_n g(x_n)) = \lim_n g(g(x_n)) = y.$$

Therefore, $g(y) = g(g(y))$, which is a contradiction. Hence, if $g(y) \neq g(g(y))$, then for $t > 0$ we have

$$\sup\{T(f_{h(x),g(x)}(t), f_{h(x),y}(t), f_{g(x),y}(t), f_{g(x),g(g(x))}(t)) : x \in X\} > 0. \quad \blacksquare$$

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