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POSITION VECTORS OF CURVES IN THE GALILEAN SPACE G_3

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Abstract. In this paper, we study the position vector of an arbitrary curve in Galilean 3-space G_3. We first determine the position vector of an arbitrary curve with respect to the Frenet frame. Also, we deduce in terms of the curvature and torsion, the natural representation of the position vector of an arbitrary curve. Moreover, we define a plane curve, helix, general helix, Salkowski curves and anti-Salkowski curves in Galilean space G_3. Finally, the position vectors of some special curves are obtained and sketching.

1. Introduction

Helix is one of the most fascinating curves in science and nature. Scientist have long held a fascinating, sometimes bordering on mystical obsession, for helical structures in nature. Helices arise in nano-springs, carbon nano-tubes, α-helices, DNA double and collagen triple helix, lipid bilayers, bacterial flagella in salmonella and escherichia coli, aerial hyphae in actinomycetes, bacterial shape in spirochetes, horns, tendrils, vines, screws, springs, helical staircases and sea shells [7, 18, 27]. As well, helix curve or helical structures in fractal geometry, for instance hyperhelices [26]. In the field of computer aided design and computer graphics, helices can be used for the tool path description, the simulation of kinematic motion or the design of highways, etc. [28].

From the view of differential geometry in the Euclidean 3-space, a helix is a geometric curve with non-vanishing constant curvature κ and non-vanishing constant torsion τ [6]. The helix may be called a *circular helix* or *W-curve* [13]. Its known that straight lines ($\kappa(s) = 0$) and circles ($\tau(s) = 0$) are degenerate-helices examples [16]. In fact, circular helix is the simplest three-dimensional spirals. One of the most interesting spiral example is the k-Fibonacci spirals. These curves appear naturally in the study of the k-Fibonacci numbers $\{F_{k,n}\}_{n=0}^{\infty}$ and the related hyperbolic k-Fibonacci function. Fibonacci numbers and the related Golden Mean or Golden section appear very often in theoretical physics and physics of the high

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energy particles [9, 10]. Three-dimensional k-Fibonacci spirals was studied from a geometric point of view in [11].

Indeed a helix is a special case of the *general helix*. A curve of constant slope or general helix in Euclidean 3-space $\mathbb{E}^3$ is defined by the property that the tangent makes a constant angle with a fixed straight line called the axis of the general helix. A classical result stated by Lancret in 1802 and first proved by de Saint Venant in 1845 (see [25] for details) says that: *A necessary and sufficient condition that a curve be a general helix is that the ratio* $\dfrac{\kappa}{\tau}$ *is constant along the curve, where* κ *and* τ *denote the curvature and the torsion, respectively* [12, 19].

Izumiya and Takeuchi [14] have introduced the concept of slant helix by saying that the normal lines make a constant angle with a fixed straight line. They characterize a slant helix if and only if the geodesic curvature of the principal image of the principal normal indicatrix

$$\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'$$

is a constant function.

A family of curves with constant curvature but non-constant torsion is called Salkowski curves and a family of curves with constant torsion but non-constant curvature is called anti-Salkowski curves [24]. Monterde [20] studied some characterizations of these curves and he proved that the principal normal vector makes a constant angle with fixed straight line. As a direct consequence of Monterde results, Salkowski and anti-Salkowski curves are good examples of slant helices.

The problem of the determination of parametric representation of the position vector of an arbitrary space curve according to the intrinsic equations is still open in the Euclidean space $\mathbb{E}^3$ [8, 17]. This problem is not easy to solve in general case. However, this problem is solved in some special cases such as: the case of a plane curve ($\tau = 0$), the case of a helix (κ and τ are both non-vanishing constant). Recently, Ali [4, 5] adapted fundamental existence and uniqueness theorem for space curves in Euclidean space $\mathbb{E}^3$ and constructed a vector differential equation to solve this problem in the case of a general helix ($\frac{\tau}{\kappa}$ is constant) and in the case of a slant helix ($\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}(\frac{\tau}{\kappa})'$ is constant). However, this problem is neither solved in other cases of the space curve in Euclidean 3-space nor Minkowski 3-space [1–3].

Our main result in this work is to solve the above problem for all curves in the Galilean 3-space G_3. Firstly, we determine the position vector of an arbitrary curve with respect to the Frenet frame and deduce the natural representation of the position vector of an arbitrary curve in the Galilean 3-space G_3 in terms of the curvature and the torsion with respect to standard frame. Also, we shall give definitions of some special curves in Galilean 3-space G_3 such as: plane curve, helix, general helix, Salkowski curves and anti-Salkowski curves and plot such curves.

2. Preliminaries

The geometry of the Galilean space G_3 has been treated in detail in O. Roschl's habilitation in 1984 [23]. The Galilean space is a three dimensional complex projective space P_3 in which the absolute figure $\{w, f, I_1, I_2\}$ consists of a real plane w (the absolute plane), a real line $f \subset w$ (the absolute line) and two complex conjugate points $I_1, I_2 \in f$ (the absolute points) [15]. We shall take, as a real model of the space G_3, a real projective space P_3 with the absolute $\{w, f\}$ consisting of a real plane $w \subset \mathrm{G}_3$ and a real line $f \subset w$ on which an elliptic involution ε has been defined. In homogeneous coordinates

$$w \ldots x_0 = 0, \;\; f \ldots x_0 = x_1 = 0,$$
$$\varepsilon : (0 : 0 : x_2 : x_3) \rightarrow (0 : 0 : x_3 : -x_2),$$

while in the nonhomogeneous coordinates, the similarity group H_8 has the form

$$\begin{cases} x' = a_{11} + a_{12} x, \\ y' = a_{21} + a_{22} x + a_{23}(y \cos[\phi] + z \sin[\phi]), \\ z' = a_{31} + a_{32} x - a_{23}(y \sin[\phi] - z \cos[\phi]), \end{cases}$$

where a_{ij} and ϕ are real numbers. For $a_{12} = a_{23} = 1$, we have the subgroup B_6 which is the group of Galilean motions:

$$B_6 \ldots \begin{cases} x' = a + x, \\ y' = b + c\,x + y \cos[\phi] + z \sin[\phi], \\ z' = d + e\,x - y \sin[\phi] + z \cos[\phi]. \end{cases}$$

It is worth noting that [21]: in G_3 there are four classes of lines:

- **a:** (proper) nonisotropic lines: they do not meet the absolute line f.
- **b:** (proper) isotropic lines: lines that do not belong to the plane w but meet the absolute line f.
- **c:** (unproper) nonisotropic lines: all lines of w but f.
- **d:** the absolute line f.

In affine coordinates, the Galilean scalar product between two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ is defined by [22]:

$$(\mathbf{a}.\mathbf{b})_{\mathrm{G}} = \begin{cases} a_1\, b_1, & \text{if } a_1 \neq 0 \text{ or } b_1 \neq 0, \\ a_2\, b_2 + a_3\, b_3, & \text{if } a_1 = b_1 = 0. \end{cases}$$

We can define the Galilean cross product as:

$$(\mathbf{a} \wedge \mathbf{b})_{\mathrm{G}} = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, & \text{if } a_1 \neq 0 \text{ or } b_1 \neq 0, \\ \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, & \text{if } a_1 = b_1 = 0. \end{cases}$$

Let $\alpha : I \to \mathrm{G}_3$, $I \subset R$ be an unit speed curve in Galilean space G_3 given by $\alpha(x) = \big(x, y(x), z(x)\big)$, where x is a Galilean invariant parameter (the arc-length on α). The curvature and torsion of the curve α are defined by

$$\kappa(x) = \|\alpha''(x)\|, \quad \tau(x) = \frac{1}{\kappa^2(x)}\mathrm{Det}\big(\alpha'(x),\, \alpha''(x),\, \alpha'''(x)\big)$$

respectively. The orthonormal trihedron $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ of the curve α is defined by

$$\begin{aligned} \mathbf{t}(x) &= \alpha'(x) = \big(1, y'(x), z'(x)\big), \\ \mathbf{n}(x) &= \frac{\alpha''(x)}{\|\alpha''(x)\|} = \frac{1}{\kappa(x)}\big(0, y''(x), z''(x)\big), \\ \mathbf{b}(x) &= \big(\mathbf{t}(x) \wedge \mathbf{n}(x)\big)_{\mathrm{G}} = \frac{1}{\kappa(x)}\big(0, -z''(x), y''(x)\big), \end{aligned}$$

where $\mathbf{t}(x)$, $\mathbf{n}(x)$ and $\mathbf{b}(x)$ are called the tangent vector, principal normal vector and binormal vector, respectively. The Frenet equations for $\alpha(x)$ are given by

$$\begin{bmatrix} \mathbf{t}'(x) \\ \mathbf{n}'(x) \\ \mathbf{b}'(x) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(x) & 0 \\ 0 & 0 & \tau(x) \\ 0 & -\tau(x) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}(x) \\ \mathbf{n}(x) \\ \mathbf{b}(x) \end{bmatrix} \tag{2.1}$$

3. Position vector of a curve with respect to the Frenet frame in G_3

THEOREM 3.1 *The position vector $\alpha(x)$ of an arbitrary curve with curvature $\kappa(x)$ and torsion $\tau(x) \neq 0$ with respect to the Frenet frame in the Galilean space G_3 is given by:*

$$\begin{aligned} \alpha(x) = (x + c_1)\mathbf{t} + \Big[c_2 - \int (x + c_1)\kappa(x) \sin\Big[\int \tau(x)\,dx\Big]\,dx\Big] \times \\ \times\Big(\sin\Big[\int \tau(x)\,dx\Big]\mathbf{n} + \cos\Big[\int \tau(x)\,dx\Big]\mathbf{b}\Big) + \Big[c_3 + \int (x + c_1)\kappa(x) \cos\Big[\int \tau(x)\,dx\Big]\,dx\Big] \times \\ \times \sin\Big[\int \tau(x)\,dx\Big]\Big(\cos\Big[\int \tau(x)\,dx\Big]\mathbf{n} - \sin\Big[\int \tau(x)\,dx\Big]\mathbf{b}\Big), \end{aligned} \tag{3.1}$$

where c_1, c_2 and c_3 are arbitrary constants.

Proof. Let $\alpha(x)$ be an arbitrary curve in Galilean space G_3, then, we may express its position vector as follows:

$$\alpha(x) = \lambda(x)\,\mathbf{t} + \mu(x)\,\mathbf{n} + \gamma(x)\,\mathbf{b}, \tag{3.2}$$

where $\lambda(x)$, $\mu(x)$ and $\gamma(x)$ are differentiable functions of $x \in I \subset R$. Differentiating the above equation with respect to x and using the Frenet equations, we can have a system following system of ordinary differential equations:

$$\begin{cases} \lambda'(x) - 1 = 0 \\ \mu'(x) + \kappa(x)\lambda(x) - \tau(x)\gamma(x) = 0 \\ \gamma'(x) + \tau(x)\mu(x) = 0 \end{cases} \tag{3.3}$$

The first equation of (3.3) leads to

$$\lambda(x) = x + c_1, \tag{3.4}$$

where c_1 is an arbitrary constant.

It is useful to change the variable x by the variable $t = \int \tau(x)\,dx$. So that all functions of x will transform to functions of t, for example $\lambda(t) = (\lambda \circ x)(t)$, $\kappa(t) = (\kappa \circ x)(t)$, and so on. Here, we will use dot to denote derivation with respect to t (prime denotes derivative with respect to x). The third equation of (3.3) can be written as

$$\mu(t) = -\dot{\gamma}(t). \tag{3.5}$$

Substituting the above equation to the second equation of (3.3) we have the following equation for $\gamma(t)$

$$\ddot{\gamma}(t) + \gamma(t) = \frac{\lambda(t)\kappa(t)}{\tau(t)}. \tag{3.6}$$

The general solution of this equation is

$$\gamma(t) = \Big[c_2 - \int \frac{\lambda(t)\kappa(t)}{\tau(t)} \sin[t]\,dt\Big] \cos[t] + \Big[c_3 + \int \frac{\lambda(t)\kappa(t)}{\tau(t)} \cos[t]\,dt\Big] \sin[t]. \tag{3.7}$$

where c_2 and c_3 are arbitrary constants. From (3.5), the function $\mu(t)$ is given by

$$\mu(t) = \Big[c_2 - \int \frac{\lambda(t)\kappa(t)}{\tau(t)} \sin[t]\,dt\Big] \sin[t] - \Big[c_3 + \int \frac{\lambda(t)\kappa(t)}{\tau(t)} \cos[t]\,dt\Big] \cos[t]. \tag{3.8}$$

Hence the equations (3.7) and (3.8) take the following form

$$\begin{aligned}\gamma(x) = &\Big[c_2 - \int (x+c_1)\kappa(x) \sin\Big[\int \tau(x)\,dx\Big]dx\Big] \cos\Big[\int \tau(x)\,dx\Big] \\ &+ \Big[c_3 + \int (x+c_1)\kappa(x) \cos\Big[\int \tau(x)\,dx\Big]\,dx\Big] \sin\Big[\int \tau(x)\,dx\Big].\end{aligned} \tag{3.9}$$

$$\begin{aligned}\mu(x) = &\Big[c_2 - \int (x+c_1)\kappa(x) \sin\Big[\int \tau(x)\,dx\Big]dx\Big] \sin\Big[\int \tau(x)\,dx\Big] \\ &- \Big[c_3 + \int (x+c_1)\kappa(x) \cos\Big[\int \tau(x)\,dx\Big]\,dx\Big] \cos\Big[\int \tau(x)\,dx\Big].\end{aligned} \tag{3.10}$$

Substituting equations (3.4), (3.9) and (3.10) to (3.2) we arrive to equation (3.1), so the proof of the theorem is complete. ■

4. Position vector of a curve with respect to standard frame of $\mathbf{G_3}$

THEOREM 4.1 *The position vector $\alpha(x)$ of an arbitrary curve with curvature $\kappa(x)$ and torsion $\tau(x)$ in the Galilean space G_3 is computed from the natural representation form*

$$\alpha(x) = \Big(x, \int \Big[\int \kappa(x) \cos\Big[\int \tau(x)\,dx\Big]\,dx\Big]\,dx, \int \Big[\int \kappa(x) \sin\Big[\int \tau(x)\,dx\Big]\,dx\Big]\,dx\Big). \tag{4.1}$$

Proof: If $\alpha(x)$ is an arbitrary curve in Galilean space G_3, then the Frenet equations (2.1) are hold. From the second equation in (2.1), we have

$$\mathbf{b}(x) = \frac{1}{\tau(x)}\mathbf{n}'(x).$$

Substituting the above equation to the third equation of (2.9) we have the following ordinary differential equation according to principal normal vector $\mathbf{n}$ as

$$\Big(\frac{1}{\tau(x)}\mathbf{n}'(x)\Big)' + \tau(x)\mathbf{n}(x) = 0.$$

The above equation can be written in the form

$$\frac{d^2\mathbf{n}}{dt^2} + \mathbf{n} = 0, \tag{4.2}$$

where t is the new variable equal to $t = \int \tau(x)\,dx$.

On other hand, we can write the principal normal vector in the following form

$$\mathbf{n} = \big(0, \cos[\theta(t)], \sin[\theta(t)]\big).$$

If we put the second and the third components from the vector $\mathbf{n}$ in the equation (4.2) we have the following two equations

$$\begin{aligned}\big(1-\dot{\theta}^2(t)\big)\cos[\theta(t)] - \ddot{\theta}(t)\sin[\theta(t)] = 0,\\ \big(1-\dot{\theta}^2(t)\big)\sin[\theta(t)] + \ddot{\theta}(t)\cos[\theta(t)] = 0.\end{aligned}$$

It is easy to prove that the above equations lead to the following two equations:

$$\dot{\theta}(t) = \pm 1, \qquad \ddot{\theta}(t) = 0,$$

which lead to $\theta(t) = \pm\, t = \pm \int \tau(x)dx$. Without loss of generality, we can take the positive sign for $\theta(t)$. Then the principal normal vector takes the form

$$\mathbf{n}(x) = \Big(0, \cos\Big[\int \tau(x)\,dx\Big], \sin\Big[\int \tau(x)\,dx\Big]\Big).$$

Multiplying the above equation by $\kappa(x)$ and integrating the result with respect to x, we have

$$\mathbf{t}(x) = \int \kappa(x)\Big(0, \cos\Big[\int \tau(x)\,dx\Big], \sin\Big[\int \tau(x)\,dx\Big]\Big)\,dx + \mathbf{c},$$

where $\mathbf{c}$ is a constant vector. Because the first component of tangent vector equal one, then with out loss of generality, we can take $\mathbf{c} = (1,0,0)$, and then

$$\mathbf{t}(x) = \Big(1, \int \kappa(x)\cos\Big[\int \tau(x)\,dx\Big]\,dx, \int \kappa(x)\sin\Big[\int \tau(x)\,dx\Big]\,dx\Big).$$

Integrating the above equation with respect to x, we obtain

$$\alpha(x) = \int \Big(1, \int \kappa(x)\cos\Big[\int \tau(x)\,dx\Big]\,dx, \int \kappa(x)\sin\Big[\int \tau(x)\,dx\Big]\,dx\Big)\,dx,$$

which leads to the equation (4.1) and the proof is complete. ■

5. Examples

EXAMPLE 1. When $\kappa(x) = 0$.

DEFINITION 5.1 Let α be a regular curve in Galilean space G_3, $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ the Frenet frame along the curve α and κ its curvature. If $\kappa = 0$, then α is called a straight line with respect to the Frenet frame.

If we take $\kappa = 0$ and put in the equation (4.1) we have the following lemma:

LEMMA 5.2. *The position vector $\alpha(x)$ of a straight line in the Galilean space G_3 is given by*

$$\alpha_1(x) = (x, c_1\, x + c_3, c_2\, x + c_4),$$

where c_i, $i = 1, 2, 3, 4$ are arbitrary constants.

EXAMPLE 2. When $\tau(x) = 0$.

DEFINITION 5.3. Let α be a regular curve in Galilean space G_3, $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ the Frenet frame along the curve α and τ its torsion. If $\tau = 0$, then α is called a plane curve with respect to the Frenet frame.

If we take $\tau = 0$ and put in equation (4.1) we have the following lemma:

LEMMA 5.4. *The position vector $\alpha(x)$ of a plane curve in the Galilean space G_3 is given by*

$$\alpha_2(x) = \Big(x, \cos[\varepsilon] \int \Big[\int \kappa(x)\, dx\Big]\, dx, \sin[\varepsilon] \int \Big[\int \kappa(x)\, dx\Big]\, dx\Big),$$

where ε is arbitrary constant.

The position vector of some plane curves with curvatures $\kappa(x) = x$, $\kappa(x) = \sin[x]$ and $\kappa(x) = \cosh[x]$ are

$$\begin{aligned} \alpha_3(x) &= \big(x, \cos[\varepsilon_1] x^3, \sin[\varepsilon_1] x^3\big), \\ \alpha_4(x) &= \big(x, -\cos[\varepsilon_2]\sin[x], -\sin[\varepsilon_2]\sin[x]\big), \\ \alpha_5(x) &= \big(x, \cos[\varepsilon_3]\cosh[x], \sin[\varepsilon_3]\cosh[x]\big), \end{aligned}$$

respectively, where ε_i, $i = 1, 2, 3$ are arbitrary constants. One can see the graphs of such curves on Figure 1.

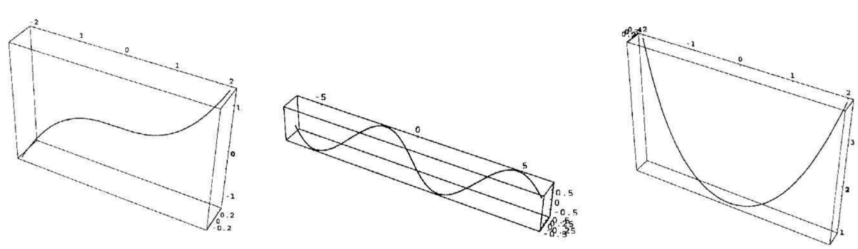

Fig. 1. Some plane curves with $\kappa(x) = x$, $\sin[x]$, $\cosh[x]$, respectively.

EXAMPLE 3. When $\kappa(x) = \text{const}$ and $\tau(x) = \text{const}$.

DEFINITION 5.5. Let α be a regular curve in Galilean space G_3, $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ the Frenet frame along the curve α and κ, τ its curvature and torsion, respectively. If κ and τ are positive constants, then α is called a circular helix or W-curve with respect to the Frenet frame.

If we take κ and τ are constants and put in equation (4.1) we have the following lemma:

LEMMA 5.6. *The position vector $\alpha(x)$ of a circular helix in the Galilean space G_3 is given by*

$$\alpha_6(x) = \Big(x, -\frac{\kappa}{\tau^2}\cos[\tau\, x], \frac{\kappa}{\tau^2}\sin[\tau\, x]\Big).$$

One can see the graph of such curve ($\kappa = 5$ and $\tau = 1$) in the left side of Figure 2.

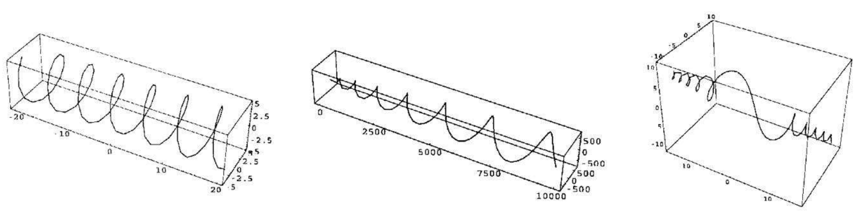

Fig. 2. Some general helices with $\kappa(x) = 5$, $\frac{1}{2\sqrt{x}}$, x, respectively.

EXAMPLE 4. When $\tau(x) = m\,\kappa(x)$, where m is arbitrary constant.

DEFINITION 5.7. Let α be a regular curve in Galilean space G_3, $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ the Frenet frame along the curve α and κ, τ its curvature and torsion, respectively. If the ratio $\frac{\tau}{\kappa}$ is constant, then α is called a general helix with respect to the Frenet frame.

If we put $\tau(x) = m\,\kappa(x)$ in equation (4.1) we have the following lemma:

LEMMA 5.8. *The position vector $\alpha(x)$ of a general helix in the Galilean space G_3 is given by*

$$\alpha_7(x) = \Big(x, \frac{1}{m}\int \sin\big[m\int \kappa(x)\,dx\big]\,dx, -\frac{1}{m}\int \cos\big[m\int \kappa(x)\,dx\big]\,dx\Big).$$

The position vector of some general helices with curvatures $\kappa(x) = \frac{1}{2\sqrt{x}}$ and $\kappa(x) = x$ are:

$$\begin{aligned}\alpha_8(x) = \Big(x, &\frac{2}{m_1^3}\big[\sin[m_1\sqrt{x}] - m_1\sqrt{x}\cos[m_1\sqrt{x}]\big],\\ &-\frac{2}{m^3}\big[\cos[m_1\sqrt{x}] + m_1\sqrt{x}\sin[m_1\sqrt{x}]\big]\Big),\\ \alpha_9(x) = \Big(x, &\frac{\sqrt{\pi}}{m_2\sqrt{m_2}}\text{FresnelS}\Big[\sqrt{\frac{m_2}{\pi}}\,x\Big], -\frac{\sqrt{\pi}}{m_2\sqrt{m_2}}\text{FresnelC}\Big[\sqrt{\frac{m_2}{\pi}}\,x\Big]\Big),\end{aligned}$$

respectively, where m_i, $i=1,2$ are arbitrary constants. It is worth noting that

$$\text{FresnelS}[z]=\int \sin\Big[\frac{\pi\, z^2}{2}\Big]\,dz,\quad \text{FresnelC}[z]=\int \cos\Big[\frac{\pi\, z^2}{2}\Big]\,dz.$$

One can see the graphs of such curves in the middle ($m_1=\frac{1}{2}$) and in the right ($m_2=\frac{1}{4}$) of Figure 2.

Example 5. When $\kappa(x)=a$, where a is arbitrary constant.

Definition 5.9. A regular curve in Galilean space G_3 with constant curvature and non-constant torsion is called Salkowski curve.

If we put $\kappa(x)=a$ in equation (4.1) we have the following lemma:

Lemma 5.10. *The position vector $\alpha(x)$ of a family of Salkowski curves in the Galilean space G_3 is given by*

$$\alpha_{10}(x)=\Big(x, a\int\Big[\int\cos\big[\int\tau(x)\,dx\big]\,dx\Big]\,dx, a\int\Big[\int\sin\big[\int\tau(x)\,dx\big]\,dx\Big]\,dx\Big).$$

Example 6. When $\tau(x)=b$, where b is arbitrary constant.

Definition 5.11. A regular curve in Galilean space G_3 with constant torsion and non-constant curvature is called Anti-Salkowski curve.

If we put $\tau(x)=b$ in equation (4.1) we have the following lemma:

Lemma 5.12. *The position vector $\alpha(x)$ of a family of anti-Salkowski curves in the Galilean space G_3 is given by:*

$$\alpha_{11}(x)=\Big(x, \int\Big[\int\kappa(x)\cos[b\,x]\,dx\Big]\,dx, \int\Big[\int\kappa(x)\sin[b\,x]\,dx\Big]\,dx\Big).$$

The position vector of Salkowski curve with $\tau(x)=\frac{1}{\sqrt{x}}$, anti-Salkowski curve with $\kappa(x)=\cosh\left[\frac{x}{4}\right]$ and general curve with $\kappa(x)=x^2$, $\tau(x)=x$ are:

$$\begin{aligned}
\alpha_{12}(x)&=\Big(x,\frac{1}{4}\Big[(3-4x)\cos[2\sqrt{x}]+6\sqrt{x}\sin[2\sqrt{x}]\Big],\\
&\qquad\frac{1}{4}\Big[(3-4x)\sin[2\sqrt{x}]-6\sqrt{x}\cos[2\sqrt{x}]\Big]\Big),\\
\alpha_{13}(x)&=\Big(x,\frac{16}{289}\Big[8\sin[t]\sinh\big[\frac{t}{4}\big]-15\cos[t]\cosh\big[\frac{t}{4}\big]\Big],\\
&\qquad-\frac{16}{289}\Big[8\cos[t]\sinh\big[\frac{x}{4}\big]+15\sin[t]\cosh\big[\frac{x}{4}\big]\Big]\Big),\\
\alpha_{14}(x)&=\Big(x,-\sqrt{\pi}\,x\,\text{FresnelS}\big[\frac{x}{\sqrt{\pi}}\big]-2\cos\big[\frac{x^2}{2}\big],\\
&\qquad\sqrt{\pi}\,x\,\text{FresnelC}\big[\frac{x}{\sqrt{\pi}}\big]-2\sin\big[\frac{x^2}{2}\big]\Big),
\end{aligned}$$

respectively. One can see the graphs of such curves in the left, middle and right hand side on Figure 3, respectively.

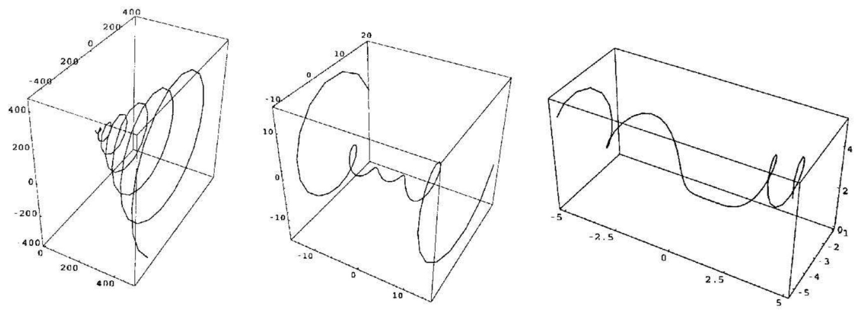

Fig. 3. Some general curves

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