МАТЕМАТИЧКИ ВЕСНИК
64, 1 (2012), 73–78
March 2012

оригинални научни рад
research paper

# SOME RESULTS CONCERNING THE ZEROS AND COEFFICIENTS OF POLYNOMIALS

**S. A. Baba, A. Liman and W. M. Shah**

**Abstract.** In this paper, we establish some relations between the zeros and coefficients of a polynomial and thereby prove a few results concerning stable polynomials.

## 1. Introduction and statement of results

Let $P(z)$ be a polynomial of degree $n$ with complex coefficients and $z_1$, $z_2$, $\dots$, $z_n$ be its zeros. The process of finding the relation between the zeros and coefficients of $P(z)$ is one of the classical problems in the theory of equations. A number of research papers (see for reference [1–4]) have been devoted to obtain such relations. Polynomials in various forms have recently come under extensive revision because of their applications in linear control systems, electrical networks, signal processing, coding theory and several areas of physical sciences, where among others root location and stability problems arise in a natural way. Here, we recall that $P(z)$ is Schur stable if all its zeros lie in the open unit disk $|z| < 1$, and Hurwitz stable if all its zeros lie in the open half plane $\operatorname{Re}(z) < 0$.

Recently Rubio-Massegu, Diaz-Barrero and Rubio-Diaz [6] obtained several inequalities involving zeros and coefficients of polynomials with real zeros. In fact they proved:

THEOREM A. *Let $A(x) = \sum_{k=0}^{n} a_k x^k$, $a_n \neq 0$, be a hyperbolic polynomial with all its zeros $x_1, x_2, \dots, x_n$ strictly positive. If $\alpha$, $p$ and $b$ are strictly positive real numbers such that $\alpha < p$, then*

$$\sum_{k=1}^{n} \frac{1}{(x_k^p + b)^{\frac{1}{\alpha}}} \leq \left[ \frac{\alpha^{\frac{1}{p}}}{p^{\frac{1}{\alpha}}} \left( \frac{p-\alpha}{b} \right)^{\frac{1}{\alpha} - \frac{1}{p}} \right] \left| \frac{a_1}{a_0} \right|.$$

*Equality holds when* $A(x) = a_n \left( x - \left( \dfrac{b\alpha}{p-\alpha} \right)^{\frac{1}{p}} \right)^n$.

*2010 AMS Subject Classification*: 12D10, 26C10, 26D15.

*Keywords and phrases*: Zeros and coefficients; stable polynomials; inequalities in the complex domain.

Influenced by their method of proof, in this paper, we consider polynomials $P(z)=\sum_{k=0}^{n} a_k z^k$ with real or complex zeros and prove the following:

THEOREM 1. *Let $P(z)=\sum_{k=0}^{n} a_k z^k$, $a_0 a_n \neq 0$, be a polynomial of degree $n$ and $z_1, z_2, \ldots, z_n$ be its zeros. If $p$, $q$ and $\alpha$ are strictly positive real numbers such that $q<p$, then*

$$\prod_{k=1}^{n} \frac{1}{(|z_k|^p+\alpha)^{\frac{1}{q}}} \leq \left[\frac{q^{\frac{1}{p}}}{p^{\frac{1}{q}}}\left(\frac{p-q}{\alpha}\right)^{\frac{1}{q}-\frac{1}{p}}\right]^n \left|\frac{a_n}{a_0}\right|. \tag{1}$$

*Equality holds if and only if $P(z)=a_n\left(z-\left(\frac{\alpha q}{p-q}\right)^{\frac{1}{p}}\right)^n$, $a_n \neq 0$.*

If $P(z)=\sum_{k=0}^{n} a_k z^k$ is an anti-Schur stable polynomial and $z_1, z_2, \ldots, z_n$ are its zeros, then $|z_k| \geq 1$, for all $k=1,2,\ldots,n$. Therefore

$$\left|\frac{a_0}{a_n}\right| = |z_1 z_2 \ldots z_n| \geq 1.$$

Thus from Theorem 1, we have the following:

COROLLARY 1. *Let $P(z)=\sum_{k=0}^{n} a_k z^k$, $a_n \neq 0$, be an anti-Schur stable complex polynomial with zeros $z_1, z_2, \ldots, z_n$. If $p$, $q$ and $\alpha$ are strictly positive real numbers such that $q<p$, then*

$$\prod_{k=1}^{n} \frac{1}{(|z_k|^p+\alpha)^{\frac{1}{q}}} \leq \left[\frac{q^{\frac{1}{p}}}{p^{\frac{1}{q}}}\left(\frac{p-q}{\alpha}\right)^{\frac{1}{q}-\frac{1}{p}}\right]^n.$$

THEOREM 2. *Let $P(z)=\sum_{k=0}^{n} a_k z^k$, $a_0 a_n \neq 0$, be an anti-Schur stable polynomial of degree $n$ with complex coefficients. If $p$, $q$ and $\alpha$ are strictly positive real numbers such that $q<p$, then*

$$\sum_{k=1}^{n} \frac{1}{(|z_k|^p+\alpha)^{\frac{1}{q}}} \leq n\left[\frac{q^{\frac{1}{p}}}{p^{\frac{1}{q}}}\left(\frac{p-q}{\alpha}\right)^{\frac{1}{q}-\frac{1}{p}}\right]. \tag{2}$$

*Equality holds for all polynomials $P(z)=a_n\left(z-\left(\frac{\alpha q}{p-q}\right)^{\frac{1}{p}}\right)^n$, $a_n \neq 0$ which satisfy the condition $\alpha=\frac{p-q}{q}$.*

Now, replacing $q$ by $\frac{1}{q}$ and $p$ by $\frac{1}{p}$ in (1) and (2), we get the following:

COROLLARY 2. *Let $P(z)=\sum_{k=0}^{n} a_k z^k$, $a_n \neq 0$, be an anti-Schur stable complex polynomial with zeros $z_1, z_2, \ldots, z_n$ and if $p$, $q$ and $\alpha$ are strictly positive real numbers such that $q>p$, then*

$$\prod_{k=1}^{n} \frac{1}{(|z_k|^{\frac{1}{p}}+\alpha)^q} \leq \left[\frac{p^p}{q^q}\left(\frac{q-p}{\alpha}\right)^{q-p}\right]^n$$

*and*

$$\sum_{k=1}^{n} \frac{1}{(|z_k|^{\frac{1}{p}} + \alpha)^q} \leq n\left[\frac{p^p}{q^q}\left(\frac{q-p}{\alpha}\right)^{q-p}\right].$$

The following result immediately follows from Theorems 1 and 2, if we choose $q = \frac{1}{2}, p = n$ and $\alpha = 2n - 1$ .

Corollary 3. *If $P(z) = \sum_{k=0}^{n} a_k z^k$, $a_0 a_n \neq 0$, is an anti-Schur stable polynomial of degree $n$ and $z_1, z_2, \ldots, z_n$ are its zeros, then*

$$\prod_{k=1}^{n} \frac{1}{(|z_k|^n + 2n - 1)^2} \leq \left[\frac{1}{4n^2}\right]^n$$

*and*

$$\sum_{k=1}^{n} \frac{1}{(|z_k|^n + 2n - 1)^2} \leq \frac{1}{4n}.$$

*The result is sharp and equality holds for the polynomial $P(z) = (z - 1)^n$.*

We also prove the following:

Theorem 3. *Let $P(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree $n$ with $z_1, z_2, \ldots, z_n$ as its zeros. If $p$, $q$ and $\alpha$ are strictly positive real numbers such that $q < p$, then*

$$\prod_{k=1}^{n} \left(\frac{|z_k|^p}{|z_k|^p + \alpha}\right)^{\frac{1}{q}} \leq \left[\frac{q^{\frac{1}{p}}}{p^{\frac{1}{q}} \alpha^{\frac{1}{p}}} (p - q)^{\frac{1}{q} - \frac{1}{p}}\right]^n \left|\frac{a_0}{a_n}\right|. \tag{3}$$

*Equality holds for polynomials of the form* $P(z) = a_n\left(z - \left(\frac{\alpha(p-q)}{q}\right)^{\frac{1}{p}}\right)^n$, $a_n \neq 0$.

If $P(z) = \sum_{k=0}^{n} a_k z^k$ is a Schur stable polynomial and $z_1, z_2, \ldots, z_n$ are its zeros, then $|z_k| < 1$, for all $k = 1, 2, \ldots, n$ and therefore

$$\left|\frac{a_0}{a_n}\right| = |z_1 z_2 \ldots z_n| < 1.$$

Thus from Theorem 3, we have the following:

Corollary 4. *Let $P(z) = \sum_{k=0}^{n} a_k z^k$, $a_n \neq 0$, be a Schur stable complex polynomial with zeros $z_1, z_2, \ldots, z_n$ and if $p$, $q$ and $\alpha$ are strictly positive real numbers such that $q < p$, then*

$$\prod_{k=1}^{n} \left(\frac{|z_k|^p}{|z_k|^p + \alpha}\right)^{\frac{1}{q}} < \left[\frac{q^{\frac{1}{p}}}{p^{\frac{1}{q}} \alpha^{\frac{1}{p}}} (p - q)^{\frac{1}{q} - \frac{1}{p}}\right]^n.$$

Theorem 4. *Let $P(z)=\sum_{k=0}^{n} a_k z^k$, $a_0 a_n \neq 0$, be a Schur stable polynomial of degree $n$ with zeros $z_k$, $k=1,2,\dots,n$. If $p$, $q$ and $\alpha$ are strictly positive real numbers such that $q<p$, then*

$$\sum_{k=1}^{n}\left(\frac{|z_k|^p}{|z_k|^p+\alpha}\right)^{\frac{1}{q}} < n\left[\frac{q^{\frac{1}{p}}}{p^{\frac{1}{q}}\alpha^{\frac{1}{p}}}(p-q)^{\frac{1}{q}-\frac{1}{p}}\right]. \tag{4}$$

## 2. A lemma

For the proofs of these theorems, we need the following lemma (for reference see [5]).

Lemma. *If $x_1, x_2, \dots, x_n$ are nonnegative real numbers and $\lambda_1, \lambda_2, \dots, \lambda_n$ are positive real numbers such that $\sum_{i=1}^{n}\lambda_i = 1$, then*

$$\prod_{i=1}^{n} x_i^{\lambda_i} \le \sum_{i=1}^{n}\lambda_i x_i.$$

*Equality holds if and only if $x_1 = x_2 = \dots = x_n$.*

## 3. Proofs of the theorems

*Proof of Theorem 1.* Let $\beta$ and $a$ be strictly positive real numbers such that $\beta = 1-\frac{q}{p} > 0$ and $a = \frac{q\alpha}{p\beta} > 0$. Using above Lemma and noting that $\frac{q}{p}+\beta = 1$, we have for all $k$, $1 \le k \le n$,

$$(|z_k|^p)^{\frac{q}{p}} a^{\beta} \le \frac{q}{p}|z_k|^p + \beta a \tag{5}$$

with equality in (5) if and only if $|z_k|^p = a$.

This gives

$$\frac{1}{\frac{q}{p}|z_k|^p+\beta a} \le \frac{1}{|z_k|^q a^{\beta}},\ 1 \le k \le n,$$

or, equivalently

$$\frac{1}{|z_k|^p+\frac{p}{q}\beta a} \le \frac{q}{p}\frac{1}{|z_k|^q a^{\beta}}.$$

Noting that $\frac{p}{q}\beta a = \alpha$ and $\beta = \frac{p-q}{p}$, we have

$$\begin{aligned}\frac{1}{|z_k|^p+\alpha} &\le \frac{q}{p}\frac{p^{\beta}\beta^{\beta}}{\alpha^{\beta}q^{\beta}}\frac{1}{|z_k|^q} = \frac{q}{p}\frac{p^{1-\frac{q}{p}}\left(\frac{p-q}{p}\right)^{1-\frac{q}{p}}}{\alpha^{1-\frac{q}{p}}q^{1-\frac{q}{p}}}\frac{1}{|z_k|^q}\\ &= \frac{q^{\frac{q}{p}}}{p}\left(\frac{p-q}{\alpha}\right)^{1-\frac{q}{p}}\frac{1}{|z_k|^q}.\end{aligned}$$

That is

$$\frac{1}{(|z_k|^p+\alpha)^{\frac{1}{q}}} \le \frac{q^{\frac{1}{p}}}{p^{\frac{1}{q}}}\left(\frac{p-q}{\alpha}\right)^{\frac{1}{q}-\frac{1}{p}}\frac{1}{|z_k|},\ 1 \le k \le n, \tag{6}$$

with equality in (6) if and only if $|z_k|^p = \left(\dfrac{\alpha q}{p-q}\right)$.

On taking the product of the inequality (6) for $1 \le k \le n$, we get

$$\prod_{k=1}^{n} \frac{1}{(|z_k|^p+\alpha)^{\frac{1}{q}}} \le \left[\frac{q^{\frac{1}{p}}}{p^{\frac{1}{q}}}\left(\frac{p-q}{\alpha}\right)^{\frac{1}{q}-\frac{1}{p}}\right]^n \prod_{k=1}^{n} \frac{1}{|z_k|},$$

with equality if and only if $|z_1| = |z_2| = \cdots = |z_n| = \left(\dfrac{\alpha q}{p-q}\right)^{\frac{1}{p}}$.

Thus in particular, we get

$$\prod_{k=1}^{n} \frac{1}{(|z_k|^p+\alpha)^{\frac{1}{q}}} \le \left[\frac{q^{\frac{1}{p}}}{p^{\frac{1}{q}}}\left(\frac{p-q}{\alpha}\right)^{\frac{1}{q}-\frac{1}{p}}\right]^n \left|\frac{a_n}{a_0}\right|.$$

The equality holds for all polynomials whose all zeros lie in the circle

$$|z| = \left(\frac{q}{p-q}\right)^{\frac{1}{p}}. \quad ■$$

*Proof of Theorem 2.* By taking the sum of the inequalities (6) for $1 \le k \le n$, we get

$$\sum_{k=1}^{n} \frac{1}{(|z_k|^p+\alpha)^{\frac{1}{q}}} \le \left[\frac{q^{\frac{1}{p}}}{p^{\frac{1}{q}}}\left(\frac{p-q}{\alpha}\right)^{\frac{1}{q}-\frac{1}{p}}\right] \sum_{k=1}^{n} \frac{1}{|z_k|}. \tag{7}$$

Since $P(z)$ is an anti-Schur stable polynomial, therefore $|z_k| \ge 1$, for $k = 1, 2, \dots, n$. This gives $\frac{1}{|z_k|} \le 1$ and hence $\sum_{k=1}^{n} \frac{1}{|z_k|} \le n$. Using this fact in (7), we get the desired result. ■

*Proof of Theorem 3.* Let $P^*(z) = z^n\overline{P(1/\bar{z})} = \sum_{j=0}^{n} \bar{a}_{n-j} z^j$. Since $z_1$, $z_2$, $\dots$, $z_n$ are zeros of $P(z)$, therefore $\frac{1}{\bar{z}_1}, \dots, \frac{1}{\bar{z}_n}$, are zeros of $P^*(z)$ and hence using inequality (1), we get

$$\prod_{k=1}^{n} \frac{1}{\left[\left(\frac{1}{|z_k|}\right)^p+\alpha\right]^{\frac{1}{q}}} \le \left[\frac{q^{\frac{1}{p}}}{p^{\frac{1}{q}}}\left(\frac{p-q}{\alpha}\right)^{\frac{1}{q}-\frac{1}{p}}\right]^n \left|\frac{a_0}{a_n}\right|.$$

This gives

$$\left(\frac{1}{\alpha^{\frac{1}{q}}}\right)^n \prod_{k=1}^{n} \left(\frac{|z_k|^p}{\frac{1}{\alpha}+|z_k|^p}\right)^{\frac{1}{q}} \le \left[\frac{q^{\frac{1}{p}}}{p^{\frac{1}{q}}}\left(\frac{p-q}{\alpha}\right)^{\frac{1}{q}-\frac{1}{p}}\right]^n \left|\frac{a_0}{a_n}\right|.$$

That is

$$\prod_{k=1}^{n} \left(\frac{|z_k|^p}{\frac{1}{\alpha}+|z_k|^p}\right)^{\frac{1}{q}} \le \left[\alpha^{\frac{1}{p}}\frac{q^{\frac{1}{p}}}{p^{\frac{1}{q}}}(p-q)^{\frac{1}{q}-\frac{1}{p}}\right]^n \left|\frac{a_0}{a_n}\right|. \tag{8}$$

Now, replacing $\alpha$ by $\frac{1}{\alpha}$ in (8), we get inequality (3). ■

*Proof of Theorem 4.* Let $z_1, z_2, \ldots, z_n$ be the zeros of $P(z)$. Since $P(z)$ is Schur stable polynomial, therefore $|z_k| < 1$, for $k = 1, 2, \ldots, n$. If $P^*(z) = z^n \overline{P(1/\bar{z})}$, then all the zeros of $P^*(z)$ lie in $|z| > 1$. Now using the same arguments as used in the proofs of Theorem 2 and Theorem 3, we get

$$\sum_{k=1}^{n} \left( \frac{|z_k|^p}{|z_k|^p + \alpha} \right)^{\frac{1}{q}} < n \left[ \frac{q^{\frac{1}{p}}}{p^{\frac{1}{q}} \alpha^{\frac{1}{p}}} (p-q)^{\frac{1}{q} - \frac{1}{p}} \right]. \quad \blacksquare$$

ACKNOWLEDGEMENTS. We are grateful to the referee for his useful comments and valuable suggestions.

REFERENCES

[**1**] M. Marden, *Geometry of Polynomials*, American Mathematical Society, Providence, Rhode Island, 1989.

[**2**] G.V. Milovanović, D.S. Mitrinović, Th. Rassias, *Topics in Polynomials, Extremal properties, Inequalities, Zeros*, World Scientific Publishing Co., Singapore, 1994.

[**3**] Q.I. Rahman, G. Schmeisser, *Analytic Theory of Polynomials,* Oxford University Press, New York, 2002.

[**4**] T.M. Rassias, *Inequalities for polynomial zeros*, in: Rassias, T.M.(ed.) *Topics in Polynomials*, Kluwer Academic Publisher, Dordrecht, 2000, pp. 165–202.

[**5**] J. Rooin, *Some new proofs for the AGM inequality*, Math. Inequalities Appl. **7** (2004), 517–521.

[**6**] J. Rubio-Massegu, J.L. Diaz-Barrero, P. Rubio-Diaz, *Zero and coefficient inequalities for hyperbolic polynomials*, J. Inequalities Pure Appl. Math. **7**, 1, (2006), 1–6.

(received 10.10.2010; in revised form 11.02.2011; available online 20.04.2011)

S. A. Baba, Department of Mathematics, National Institute of Technology, Kashmir, India - 190006

*E-mail*: `sajad2baba@yahoo.com`

A. Liman, Department of Mathematics, National Institute of Technology, Kashmir, India - 190006

*E-mail*: `abliman22@yahoo.com`

W. M. Shah, Department of Mathematics, Kashmir University, Srinagar, India - 190006

*E-mail*: `wmshah@rediffmail.com`