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Reg_G-STRONGLY SOLID VARIETIES OF COMMUTATIVE SEMIGROUPS

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Abstract. Generalized hypersubstitutions are mappings from the set of all fundamental operations into the set of all terms of the same language, which do not necessarily preserve the arities. Strong hyperidentities are identities which are closed under generalized hypersubstitutions and a strongly solid variety is a variety for which each of its identities is a strong hyperidentity. In this paper we determine the greatest Reg_G-strongly solid variety of commutative semigroups.

1. Introduction

Let $X := \{x_1, x_2, \dots\}$ be a countably infinite set of symbols called variables. We refer to these variables as letters, to X as an alphabet, and refer to the set $X_n =: \{x_1, x_2, \dots, x_n\}$ as an n-element alphabet. Let $(f_i)_{i\in I}$ be an indexed set which is disjoint from X. Each f_i is called an n_i-ary operation symbol, where $n_i \geq 1$ is a natural number. Let τ be a function which assigns to every f_i the number n_i as its arity. The function τ, on the values of τ written as $(n_i)_{i\in I}$ is called a type.

An n-ary term of type τ is defined inductively as follows :

(i) The variables $x_1, \dots, x_n$ are n-ary terms.

(ii) If $t_1, \dots, t_{n_i}$ are n-ary terms then $f_i(t_1, \dots, t_{n_i})$ is an n-ary term.

We denote by $W_\tau(X_n)$ the smallest set which contains $x_1, \dots, x_n$ and is closed under finite number of applications of (ii). Then the set $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ is the set of all terms of type τ. An *equation* of type τ is a pair (s, t) where s and t are from $W_\tau(X)$; such pairs are commonly written as $s \approx t$. An equation $s \approx t$ is an *identity* of an algebra $\underline{A}$, denoted by $\underline{A} \models s \approx t$ if $s^{\underline{A}} = t^{\underline{A}}$ where $s^{\underline{A}}$ and $t^{\underline{A}}$ are the corresponding term functions on $\underline{A}$. A generalized hypersubstitution

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of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \longrightarrow W_\tau(X)$ which does not necessarily preserve arities. We denote the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$. We define first the concept of a generalized superposition of terms $S^m : W_\tau(X)^{m+1} \longrightarrow W_\tau(X)$ by the following steps:

for any term $t \in W_\tau(X)$,

(i) if $t = x_j, 1 \leq j \leq m$, then $S^m(x_j, t_1, \dots, t_m) := t_j$,

(ii) if $t = x_j, m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \dots, t_m) := x_j$,

(iii) if $t = f_i(s_1, \dots, s_{n_i})$, then
$S^m(t, t_1, \dots, t_m) := f_i(S^m(s_1, t_1, \dots, t_m), \dots, S^m(s_{n_i}, t_1, \dots, t_m))$.

Then the generalized hypersubstitution σ can be extended to a mapping $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$ by the following steps:

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \dots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i where $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

We define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Let σ_{id} be the hypersubstitution mapping which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \dots, x_{n_i})$. It turns out that $(Hyp_G(\tau); \circ_G, \sigma_{id})$ is a monoid and the monoid $(Hyp(\tau); \circ_h, \sigma_{id})$ of all arity preserving hypersubstitutions of type τ forms a submonoid of $(Hyp_G(\tau); \circ_G, \sigma_{id})$.

If $\underline{M}$ is a submonoid of $\underline{Hyp_G(\tau)}$ and V is a variety, then an identity $s \approx t$ of V is called an *M-strong hyperidentity* of V if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity of V for every $\sigma \in M$. A variety V is called *M-strongly solid* if every identity satisfies an M-strong hyperidentity. In case of $M = Hyp_G(\tau)$ we will call *strong hyperidentity* and *strongly solid* respectively.

2. V-proper generalized hypersubstitutions and normal forms

Let V be a variety of algebras of type τ then to test whether an identity $s \approx t$ of V is a strong hyperidentity of V, our definition requires that we check, for each generalized hypersubstitution $\sigma \in Hyp_G(\tau)$ that $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity of V. In practice we restrict our testing to certain special generalized hypersubstitutions σ, those which correspond to V-normal form generalized hypersubstitutions.

Definition 2.1. [4] Let V be a variety of algebras of type τ. Two generalized hypersubstitutions σ_1 and σ_2 of type τ are called *V-generalized equivalent* if $\sigma_1(f_i) \approx \sigma_2(f_i)$ are identities in V for all $i \in I$. In this case we write $\sigma_1 \sim_{VG} \sigma_2$.

Theorem 2.2. [4] *Let V be a variety of algebras of type τ, and let $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Then the following statements are equivalent:*

(i) $\sigma_1 \sim_{VG} \sigma_2$.

(ii) For all $t \in W_\tau(X)$, the equations $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$ are identities in V.

(iii) For all $\underline{A} \in V$, $\sigma_1[\underline{A}] = \sigma_2[\underline{A}]$ where $\sigma_k[\underline{A}] = (A; (\sigma_k(f_i)^A)_{i \in I})$, for $k = 1, 2$.

Proposition 2.3. [4] *Let V be a variety of algebras of type τ. Then the following statements hold:*

(i) *For all $\sigma_1, \sigma_2 \in Hyp_G(\tau)$, if $\sigma_1 \sim_{VG} \sigma_2$ then σ_1 is a V-proper generalized hypersubstitution iff σ_2 is a V- proper generalized hypersubstitution.*

(ii) *For all $s, t \in W_\tau(X)$ and for all $\sigma_1, \sigma_2 \in Hyp_G(\tau)$, if $\sigma_1 \sim_{VG} \sigma_2$ then $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t]$ is an identity in V iff $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t]$ is an identity in V.*

The relation $\sim_{VG}$ is an equivalence relation on $Hyp_G(\tau)$, but it is not necessarily a congruence relation. Since $\sim_{VG}$ is not always a congruence, the structure obtained by factoring $Hyp_G(\tau)$ by this relation is not necessarily going to be a monoid. Recall that the quotient set gives a monoid if and only if the equivalence relation used to factor it is a congruence. We factorize $Hyp_G(\tau)$ by $\sim_{VG}$ and consider the submonoid $\underline{P_G(V)}$ of $\underline{Hyp_G(\tau)}$ is the union of equivalence classes of the relation $\sim_{VG}$. This may also be done for a submonoid $\underline{M}$ of $\underline{Hyp_G(\tau)}$ and the relation $\sim_{VG_{|M}}$.

Lemma 2.4. [4] *Let $\underline{M}$ be a submonoid of $\underline{Hyp_G(\tau)}$ and let V be a variety of type τ. Then the monoid $P_G \cap M$ is the union of all equivalence classes of the restricted relation $\sim_{VG_{|M}}$.*

Definition 2.5. [4] Let $\underline{M}$ be a monoid of generalized hypersubstitutions of type τ, and let V be a variety of type τ. Let ϕ be a choice function which chooses from M one generalized hypersubstitution from each equivalence class of the relation $\sim_{VG_{|M}}$, and let $N_\phi^M(V)$ be the set of generalized hypersubstitutions which are chosen. Thus $N_\phi^M(V)$ is a set of distinguished generalized hypersubstitutions from M, which we might call *V-normal form generalized hypersubstitutions*. We will say that the variety V is $N_\phi^M(V)$*-strongly solid* if for every identity $s \approx t \in IdV$ and for every generalized hypersubstitution $\sigma \in N_\phi^M(V)$, $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$.

Theorem 2.6. [4] *Let $\underline{M}$ be a monoid of generalized hypersubstitutions of type τ and let V be a variety of type τ. For any choice function ϕ, V is M-strongly solid if and only if V is $N_\phi^M(V)$-strongly solid.*

3. Reg$_G$-strongly solid varieties of commutative semigroups

In this section we determine the greatest Reg_G-strongly solid varieties of commutative semigroups. We recall first the definition of a regular generalized hypersubstitution.

Definition 3.1. A generalized hypersubstitution $\sigma \in Hyp_G(\tau)$ is called a *regular generalized hypersubstitution* if for every $i \in I$, each of the variables $x_1, x_2, \ldots, x_{n_i}$ occur in $\hat{\sigma}[f_i(x_1, \ldots, x_{n_i})]$. (The other variables may also occur in $\hat{\sigma}[f_i(x_1, \ldots, x_{n_i})]$ too.)

Let $Reg_G(\tau)$ be the set of all regular generalized hypersubstitutions of type τ. $Reg_G(\tau)$ is also forms a submonoid of $(Hyp_G(\tau); \circ_G, \sigma_{id})$ [2].

For a class K of algebras of type τ and for a set Σ of identities of this type we fix the following notations:

IdK - the set of all identities of K,

$HIdK$ - the set of all hyperidenties of K,

$H_{Reg_G}IdK$ - the set of all regular-strong hyperidenties of K,

$Mod\Sigma = \{\underline{A} \in Alg(\tau) | \underline{A} \text{ satisfies} \Sigma\}$ - the variety defined by Σ,

$HMod\Sigma = \{\underline{A} \in Alg(\tau) | \underline{A} \text{ hypersatisfies} \Sigma\}$ - the hyperequational class defined by Σ,

$H_{Reg_G}Mod\Sigma = \{\underline{A} \in Alg(\tau) | \underline{A} \text{ regular-strong hypersatisfies} \Sigma\}$ - the regular-strong hyperequational class defined by Σ.

Definition 3.2. Let $\underline{A}$ be an algebra of type τ and let $\underline{M}$ be a submonoid of the monoid $\underline{Reg_G(\tau)}$. Then, we define

$$\chi_M^A : \mathcal{P}(Alg(\tau)) \longrightarrow \mathcal{P}(Alg(\tau)), \quad \chi_M^E : \mathcal{P}(W_\tau(X)^2) \longrightarrow \mathcal{P}(W_\tau(X)^2)$$

by

$$\chi_M^A(\underline{A}) := \{\sigma[\underline{A}] \mid \sigma \in M\}$$
$$\chi_M^E[s \approx t] := \{\hat{\sigma}[s] \approx \hat{\sigma}[t] \mid \sigma \in M\}.$$

For $K \subseteq Alg(\tau)$ and $\Sigma \subseteq W_\tau(X)^2$ we define $\chi_M^A(K) := \bigcup_{A \in K} \chi_M^A(\underline{A})$ and $\chi_M^E[\Sigma] := \bigcup_{s\approx t \in \Sigma} \chi_M^E[s \approx t]$.

Since we are henceforth considering only type (2) varieties of commutative semigroups, we can denote the binary operation of our variety simply by juxtaposition, and omit brackets where convenient due to associativity.

Lemma 3.3. *Let $V \subseteq Mod\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1, x_1^2x_2 \approx x_1x_2^2\}$. Then*

(i) $x_1^3x_2 \approx x_1^4x_2 \in IdV$ (and thus $x_1^4 \approx x_1^5 \in IdV$),

(ii) $x_1^2x_2x_3 \approx x_1x_2^2x_3^2 \in IdV$ (and thus $x_1^2x_2x_3 \approx x_1^3x_2x_3 \in IdV$),

(iii) $x_1^4x_2^2x_3 \approx x_1^2x_2^2x_3 \in IdV$,

(iv) $x_1^9x_2^3x_3 \approx x_1^3x_2^3x_3 \in IdV$,

(v) $x_1^4x_2^2x_3x_4^3 \approx x_1^2x_2^2x_3x_4^2 \in IdV$,

(vi) $x_1^6x_2x_3^3 \approx x_1^2x_2^3x_3^2 \in IdV$,

(vii) $x_1^3x_2^6x_3 \approx x_1^2x_2^2x_3^3 \in IdV$,

(viii) $x_1^4x_2^9x_3^3x_4 \approx x_1^2x_2^3x_3^3x_4 \in IdV$,

(ix) $x_1^4x_2^{12}x_3 \approx x_1^2x_2^3x_3^4 \in IdV$.

Proof. (i) Using $x_1^2x_2 \approx x_1x_2^2 \in IdV$ we get

$$x_1^4x_2 \approx (x_1^2)^2x_2 \approx x_1^2x_2^2 \approx x_1x_1x_2^2 \approx x_1x_1^2x_2 \approx x_1^3x_2,$$

i.e. $x_1^3x_2 \approx x_1^4x_2 \in IdV$. Finally, $x_1^3x_2 \approx x_1^4x_2 \in IdV$ provides $x_1^4 \approx x_1^5 \in IdV$.

(ii) We substitute x_2 by x_2x_3 in $x_1^2x_2 \approx x_1x_2^2$. Using the commutative law we get $x_1^2x_2x_3 \approx x_1x_2^2x_3^2$ where $x_1x_2^2x_3^2 \approx x_1^3x_2x_3$ because $x_1^2x_2 \approx x_1x_2^2$.

(iii) Using $x_1^2x_2 \approx x_1x_2^2 \in IdV$, the associative law and the commutative law we get $x_1^4x_2^2x_3 \approx x_2^2(x_1^2x_2^2)x_3 \approx x_1^2(x_1x_2)^2x_3 \approx x_1^2(x_1x_2)x_3^2 \approx x_1^3x_2x_3^2 \approx x_1^3x_2^2x_3 \approx x_1(x_1^2x_2^2)x_3 \approx x_1(x_1x_2)^2x_3 \approx x_1(x_1x_2)x_3^2 \approx x_1^2x_2x_3^2 \approx x_1^2x_2^2x_3$.

(iv) Using $x_1^2x_2 \approx x_1x_2^2 \in IdV$, the associative law and the commutative law we get $x_1^9x_2^3x_3 \approx x_1^3(x_1^2x_2)(x_1^2x_2)(x_1^2x_2)x_3 \approx x_1^3(x_1x_2^2)(x_1x_2^2)(x_1x_2^2)x_3 \approx x_1^6x_2^6x_3 \approx (x_1^3x_2^3)^2x_3 \approx x_1^3x_2^3x_3^2 \approx (x_1x_2)^2x_1x_2x_3^2 \approx (x_1x_2)x_1^2x_2x_3^2 \approx x_1^3x_2^2x_3^2 \approx x_1^3x_2(x_2x_3^2) \approx x_1^3x_2(x_2^2x_3) \approx x_1^3x_2^3x_3$.

(v) Using $x_1^2x_2 \approx x_1x_2^2 \in IdV$, the associative law and the commutative law we get $x_1^2x_2^2x_3x_4^2 \approx x_1^2x_3x_2^2x_4^2 \approx x_1^2x_3x_2x_4^4 \approx x_3x_2x_1^2x_4^4 \approx x_3x_2x_1x_4^8 \approx (x_1x_4^2)(x_2x_4^2)x_3x_4^4 \approx (x_1^2x_4)(x_2^2x_4)x_3x_4^4 \approx x_1^2x_2^2x_3x_4^6 \approx x_1^2(x_4^3)^2x_2^2x_3 \approx x_1^4x_4^3x_2^2x_3 \approx x_1^4x_2^2x_3x_4^3$.

(vi) Using $x_1^2x_2 \approx x_1x_2^2 \in IdV$, the associative law and the commutative law we get $x_1^2x_2^3x_3^2 \approx x_1^2x_2x_2^2x_3^2 \approx x_1^2x_2^2x_3^4 \approx (x_1x_2)^2x_3^4 \approx x_1x_2x_3^8 \approx x_1x_3^2x_2x_3^2x_3^4 \approx x_1^2x_3x_2^2x_3x_3^4 \approx x_1^2x_2^2x_3^6 \approx x_2^2x_1^2(x_3^3)^2 \approx x_2^2x_1^4x_3^3 \approx x_1^2x_1^2x_2^2x_3^3 \approx x_1^2x_1^4x_2x_3^3 \approx x_1^6x_2x_3^3$.

(vii) Using $x_1^2x_2 \approx x_1x_2^2 \in IdV$, the associative law and the commutative law we get $x_1^2x_2^2x_3^3 \approx (x_1x_2)^2x_3^3 \approx x_1x_2x_3^6 \approx x_2x_1(x_3^3)^2 \approx x_2x_1^2x_3^3 \approx x_1^2x_2x_3^2x_3 \approx x_1^2x_3^2x_2x_3 \approx x_1^4x_2x_3^2 \approx x_1^4x_3^2x_2 \approx x_1^8x_2x_3 \approx x_1^4x_1^2x_2x_1^2x_3 \approx x_1^4x_1x_2^2x_1x_3^2 \approx x_1^6x_2^2x_3^2 \approx (x_1^3)^2x_2x_2x_3^2 \approx x_1^3x_2^2x_2x_3^2 \approx x_1^3x_2^3x_3^2 \approx x_1^3x_2^6x_3$.

(viii) Using $x_1^2x_2 \approx x_1x_2^2 \in IdV$, the associative law and the commutative law we get $x_1^2x_2^3x_3^3x_4 \approx x_1^2(x_2x_3)^2(x_2x_3)x_4 \approx x_1^4x_2^2x_3^2x_4 \approx x_1^4x_2^4x_3x_4 \approx x_1^4x_2^2x_2^2x_3x_4 \approx x_1^4x_2^2x_2x_3^2x_4 \approx x_1^4x_2^3x_3^2x_4 \approx x_1^4x_2^6x_3x_4 \approx x_1^4x_2^4x_2^2x_3x_4 \approx x_1^4x_2^4x_2x_3^2x_4 \approx x_1^4x_2^5x_3^2x_4 \approx x_1^4x_2^{10}x_3x_4 \approx x_1^4x_2^8x_2^2x_3x_4 \approx x_1^4x_2^8x_2x_3^2x_4 \approx x_1^4x_2^9x_3^2x_4 \approx x_1^4x_2^7x_2^2x_3^2x_4 \approx x_1^4x_2^8x_3^4x_4 \approx x_1^4x_2^7x_2x_3^2x_3^2x_4 \approx x_1^4x_2^9x_3^3x_4$.

(ix) Using $x_1^2x_2 \approx x_1x_2^2 \in IdV$, the associative law and the commutative law we get $x_1^2x_2^3x_3^4 \approx x_1^2(x_3^2)^2x_2^3 \approx x_1^4x_2^3x_3^2 \approx x_1^4x_2^6x_3 \approx x_1^4(x_2^3)^2x_3 \approx x_1^8x_2^3x_3 \approx (x_1^4)^2(x_2^3x_3) \approx x_1^4(x_2^3x_3)^2 \approx x_1^4x_2^6x_3^2 \approx x_1^4x_2^{12}x_3$. ■

Let V_{RC} be the variety of commutative semigroups defined by the identity $x_1^2x_2 \approx x_1x_2^2$, i.e. $V_{RC} = Mod\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1, x_1^2x_2 \approx x_1x_2^2\}$.

Theorem 3.4. *V_{RC} is the greatest Reg_G- solid variety of commutative semigroups.*

Proof. We have

$$H_{Reg_G}Mod\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1\} = Mod_{\chi^E_{Reg_G}}\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1\}.$$

The application of $\sigma_{x_1^2x_2}$ to the commutative law provides $x_1^2x_2 \approx x_1x_2^2$, i.e. $x_1^2x_2 \approx x_1x_2^2 \in \chi^E_{Reg_G}[(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1]$. This shows

$$Mod_{\chi^E_{Reg_G}}\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1\} \subseteq V_{RC}.$$

To prove the converse inclusion we have to check the associative law, the commutative law and $x_1^2x_2 \approx x_1x_2^2$ using all regular generalized hypersubstitutions.

From now on, the generalized hypersubstitution σ which maps f to the term t is denoted by σ_t.

By using Theorem 2.6 together with the identities of V_{RC}, we can restrict our checking to the following regular generalized hypersubstitutions σ_t where $t \in \{x_ix_j \mid i,j \in \mathbb{N}\} \cup \{x_ix_jx_k \mid i,j,k \in \mathbb{N}\} \cup \{x_ix_jx_kx_l \mid i,j,k,l \in \mathbb{N}\} \cup \{x_{i_1}x_{i_2}\ldots x_{i_k} \mid k, i_1,\ldots,i_k \in \mathbb{N}, k > 4$, and all of $i_1,\ldots,i_k$ are distinct$\}$.

If we apply $\sigma_{x_ix_j}$; $i,j \in \mathbb{N}$ on both sides of the associative law, the commutative law and $x_1^2x_2 \approx x_1x_2^2$ we have the following table.

$i,j \in \mathbb{N}$	$\hat{\sigma}_{x_ix_j}[(x_1x_2)x_3] = S^2(x_ix_j, S^2(x_ix_j,x_1,x_2),x_3)$	$\hat{\sigma}_{x_ix_j}[x_1(x_2x_3)] = S^2(x_ix_j,x_1,S^2(x_ix_j,x_2,x_3))$
$i=1, j=2$	$x_1x_2x_3$	$x_1x_2x_3$

$i,j \in \mathbb{N}$	$\hat{\sigma}_{x_ix_j}[x_1x_2] = S^2(x_ix_j,x_1,x_2)$	$\hat{\sigma}_{x_ix_j}[x_2x_1] = S^2(x_ix_j,x_2,x_1)$
$i=1, j=2$	x_1x_2	x_2x_1

$i,j \in \mathbb{N}$	$\hat{\sigma}_{x_ix_j}[(x_1x_1)x_2] = S^2(x_ix_j, S^2(x_ix_j,x_1,x_1),x_2)$	$\hat{\sigma}_{x_ix_j}[x_1(x_2x_2)] = S^2(x_ix_j,x_1,S^2(x_ix_j,x_2,x_2))$
$i=1, j=2$	$x_1x_1x_2$	$x_1x_2x_2$

Using the associative law, the commutative law and the identity $x_1^2x_2 \approx x_1x_2^2$ we have both sides are equal.

If we apply $\sigma_{x_ix_jx_k}$; $i,j,k \in \mathbb{N}$ on both sides of the associative law, the commutative law and $x_1^2x_2 \approx x_1x_2^2$ we have the following table.

$i,j,k \in \mathbb{N}$	$\hat{\sigma}_{x_ix_jx_k}[(x_1x_2)x_3] = S^2(x_ix_jx_k, S^2(x_ix_jx_k,x_1,x_2),x_3)$	$\hat{\sigma}_{x_ix_jx_k}[x_1(x_2x_3)] = S^2(x_ix_jx_k,x_1,S^2(x_ix_jx_k,x_2,x_3))$
$i=j=1, k=2$	$x_1x_1x_2x_1x_1x_2x_3$	$x_1x_1x_2x_2x_3$
$i=1, j=2, k>2$	$x_1x_2x_kx_3x_k$	$x_1x_2x_3x_kx_k$

$i,j,k \in \mathbb{N}$	$\hat{\sigma}_{x_ix_jx_k}[x_1x_2] = S^2(x_ix_jx_k,x_1,x_2)$	$\hat{\sigma}_{x_ix_jx_k}[x_2x_1] = S^2(x_ix_jx_k,x_2,x_1)$
$i=j=1, k=2$	$x_1x_1x_2$	$x_2x_2x_1$
$i=1, j=2, k>2$	$x_1x_2x_k$	$x_2x_1x_k$

$i,j,k \in \mathbb{N}$	$\hat{\sigma}_{x_ix_jx_k}[(x_1x_1)x_2] = S^2(x_ix_jx_k, S^2(x_ix_jx_k,x_1,x_1),x_2)$	$\hat{\sigma}_{x_ix_jx_k}[x_1(x_2x_2)] = S^2(x_ix_jx_k,x_1,S^2(x_ix_jx_k,x_2,x_2))$
$i=j=1, k=2$	$x_1x_1x_1x_1x_1x_1x_2$	$x_1x_1x_2x_2x_2$
$i=1, j=2, k>2$	$x_1x_1x_kx_2x_k$	$x_1x_2x_2x_kx_k$

Using the associative law, the commutative law and the identity $x_1^2x_2 \approx x_1x_2^2$ we have both sides are equal.

If we apply $\sigma_{x_ix_jx_kx_l}$; $i,j,k,l \in \mathbb{N}$ on both sides of the associative law, the commutative law and $x_1^2x_2 \approx x_1x_2^2$ we have the following table.

$i,j,k \in \mathbb{N}$	$\hat{\sigma}_{x_ix_jx_kx_l}[(x_1x_2)x_3] = S^2(x_ix_jx_kx_l, S^2(x_ix_jx_kx_l, x_1, x_2), x_3)$	$\hat{\sigma}_{x_ix_jx_kx_l}[x_1(x_2x_3)] = S^2(x_ix_jx_kx_l, x_1, S^2(x_ix_jx_kx_l, x_2, x_3))$
$i=j=k=1, l=2$	$x_1x_1x_1x_2x_1x_1x_1x_2x_1x_1x_1x_2x_3$	$x_1x_1x_1x_2x_2x_2x_3$
$i=j=1, k=2, l>2$	$x_1x_1x_2x_lx_1x_1x_2x_lx_3x_l$	$x_1x_1x_2x_2x_3x_lx_l$
$i=1, j=2, k,l>2$	$x_1x_2x_kx_lx_3x_kx_l$	$x_1x_2x_3x_kx_lx_kx_l$

$i,j,k \in \mathbb{N}$	$\hat{\sigma}_{x_ix_jx_kx_l}[x_1x_2] = S^2(x_ix_jx_kx_l, x_1, x_2)$	$\hat{\sigma}_{x_ix_jx_kx_l}[x_2x_1] = S^2(x_ix_jx_kx_l, x_2, x_1)$
$i=j=k=1, l=2$	$x_1x_1x_1x_2$	$x_2x_2x_2x_1$
$i=j=1, k=2, l>2$	$x_1x_1x_2x_l$	$x_2x_2x_1x_l$
$i=1, j=2, k,l>2$	$x_1x_2x_kx_l$	$x_2x_1x_kx_l$

$i,j,k \in \mathbb{N}$	$\hat{\sigma}_{x_ix_jx_k}[(x_1x_1)x_2] = S^2(x_ix_jx_k, S^2(x_ix_jx_k, x_1, x_1), x_2)$	$\hat{\sigma}_{x_ix_jx_k}[x_1(x_2x_2)] = S^2(x_ix_jx_k, x_1, S^2(x_ix_jx_k, x_2, x_2))$
$i=j=k=1, l=2$	$x_1x_1x_1x_1x_1x_1x_1x_1x_1x_1x_1x_1x_2$	$x_1x_1x_1x_2x_2x_2$
$i=j=1, k=2, l>2$	$x_1x_1x_1x_lx_1x_1x_1x_lx_2x_l$	$x_1x_1x_2x_2x_2x_lx_l$
$i=1, j=2, k,l>2$	$x_1x_1x_kx_lx_2x_kx_l$	$x_1x_2x_2x_kx_lx_kx_l$

Using the associative law, the commutative law and the identity $x_1^2x_2 \approx x_1x_2^2$ we have both sides are equal.

If we apply σ_t where $t = x_{i_1}x_{i_2}\ldots x_{i_k}$ and $k, i_1, \ldots, i_k \in \mathbb{N}, k > 4$ on both sides of the associative law we have $\hat{\sigma}_t[(x_1x_2)x_3] = S^2(t, S^2(t, x_1, x_2), x_3)$ and $\hat{\sigma}_t[x_1(x_2x_3)] = S^2(t, x_1, S^2(t, x_2, x_3))$.

(i) If there exists a unique $n \in \{1, \ldots, k\}$ such that $i_n = 1$, there exists a unique $l \in \{1, \ldots, k\}$ such that $i_l = 2$ and $i_m > 2$ for all $m \neq n \neq l$ with $n < l$, then

$$\begin{aligned}\hat{\sigma}_t[(x_1x_2)x_3] = {} & x_{i_1}\ldots x_{i_{n-1}}(x_{i_1}\ldots x_{i_{n-1}}x_1x_{i_{n+1}}\ldots x_{i_{l-1}}x_2x_{i_{l+1}}\ldots x_{i_k})\\ & x_{i_{n+1}}\ldots x_{i_{l-1}}x_3x_{i_{l+1}}\ldots x_{i_k},\\ \hat{\sigma}_t[x_1(x_2x_3)] = {} & x_{i_1}\ldots x_{i_{n-1}}x_1x_{i_{n+1}}\ldots x_{i_{l-1}}(x_{i_1}\ldots x_{i_{n-1}}x_2\\ & x_{i_{n+1}}\ldots x_{i_{l-1}}x_3x_{i_{l+1}}\ldots x_{i_k})x_{i_{l+1}}\ldots x_{i_k}.\end{aligned}$$

(ii) If there exists $j, n \in \{1, \ldots, k\}$ such that $i_j = 1 = i_n$, there exist a unique $l \in \{1, \ldots, k\}$ such that $i_l = 2$ and $i_m > 2$ for all $j \neq n \neq l \neq m$ with $j < n < l$, then

$$\hat{\sigma}_t[(x_1x_2)x_3] = x_{i_1}\ldots x_{i_{j-1}}(x_{i_1}\ldots x_{i_{j-1}}x_1x_{i_{j+1}}\ldots x_{i_{n-1}}x_1x_{i_{n+1}}\ldots x_{i_{l-1}}x_2$$

$$x_{i_{l+1}}\dots x_{i_k})x_{i_{j+1}}\dots x_{i_{n-1}}(x_{i_1}\dots x_{i_{j-1}}x_1x_{i_{j+1}}\dots x_{i_{n-1}}x_1$$
$$x_{i_{n+1}}\dots x_{i_{l-1}}x_2x_{i_{l+1}}\dots x_{i_k})x_{i_{n+1}}\dots x_{i_{l-1}}x_3x_{i_{l+1}}\dots x_{i_k},$$
$$\hat{\sigma}_t[x_1(x_2x_3)] = x_{i_1}\dots x_{i_{j-1}}x_1x_{i_{j+1}}\dots x_{i_{n-1}}x_1x_{i_{n+1}}\dots x_{i_{l-1}}(x_{i_1}\dots x_{i_{j-1}}x_2$$
$$x_{i_{j+1}}\dots x_{i_{n-1}}x_2x_{i_{n+1}}\dots x_{i_{l-1}}x_3x_{i_{l+1}}\dots x_{i_k})x_{i_{l+1}}\dots x_{i_k}.$$

(iii) If there exist $h,j,n\in\{1,\dots,k\}$ such that $i_h=i_j=i_n=1$, there exists a unique $l\in\{1,\dots,k\}$ such that $i_l=2$ and $i_m>2$ for all $h\neq j\neq n\neq l\neq m$ with $h<j<n<l$, then

$$\hat{\sigma}_t[(x_1x_2)x_3] = x_{i_1}\dots x_{i_{h-1}}(x_{i_1}\dots x_{i_{h-1}}x_1x_{i_{h+1}}\dots x_{i_{j-1}}x_1x_{i_{j+1}}\dots x_{i_{n-1}}x_1$$
$$x_{i_{n+1}}\dots x_{i_{l-1}}x_2x_{i_{l+1}}\dots x_{i_k})x_{i_{h+1}}\dots x_{i_{j-1}}(x_{i_1}\dots x_{i_{h-1}}x_1$$
$$x_{i_{h+1}}\dots x_{i_{j-1}}x_1x_{i_{j+1}}\dots x_{i_{n-1}}x_1x_{i_{n+1}}\dots x_{i_{l-1}}x_2x_{i_{l+1}}\dots x_{i_k})$$
$$x_{i_{j+1}}\dots x_{i_{n-1}}(x_{i_1}\dots x_{i_{h-1}}x_1x_{i_{h+1}}\dots x_{i_{j-1}}x_1x_{i_{j+1}}\dots x_{i_{n-1}}x_1x_{i_{n+1}}$$
$$\dots x_{i_{l-1}}x_2x_{i_{l+1}}\dots x_{i_k})x_{i_{n+1}}\dots x_{i_{l-1}}x_3x_{i_{l+1}}\dots x_{i_k},$$
$$\hat{\sigma}_t[x_1(x_2x_3)] = x_{i_1}\dots x_{i_{h-1}}x_1x_{i_{h+1}}\dots x_{i_{j-1}}x_1x_{i_{j+1}}\dots x_{i_{n-1}}x_1x_{i_{n+1}}\dots x_{i_{l-1}}$$
$$(x_{i_1}\dots x_{i_{h-1}}x_2x_{i_{h+1}}\dots x_{i_{j-1}}x_2x_{i_{j+1}}\dots x_{i_{n-1}}x_2$$
$$x_{i_{n+1}}\dots x_{i_{l-1}}x_3x_{i_{l+1}}\dots x_{i_k})x_{i_{l+1}}\dots x_{i_k}.$$

Using the associative law, the commutative law and the identity $x_1^2x_2\approx x_1x_2^2$ we have both sides are equal.

If we apply σ_t where $t=x_{i_1}x_{i_2}\dots x_{i_k}$ and $k,i_1,\dots,i_k\in\mathbb{N},k>4$ on both sides of the commutative law we have $\hat{\sigma}_t[x_1x_2]=S^2(t,x_1,x_2)$ and $\hat{\sigma}_t[x_2x_1]=S^2(t,x_2,x_1)$.

(i) If there exists a unique $n\in\{1,\dots,k\}$ such that $i_n=1$, there exists a unique $l\in\{1,\dots,k\}$ such that $i_l=2$ and $i_m>2$ for all $m\neq n\neq l$ with $n<l$, then

$$\hat{\sigma}_t[x_1x_2] = x_{i_1}\dots x_{i_{n-1}}x_1x_{i_{i+1}}\dots x_{i_{l-1}}x_2x_{i_{l+2}}\dots x_{i_k},$$
$$\hat{\sigma}_t[x_2x_1] = x_{i_1}\dots x_{i_{n-1}}x_2x_{i_{i+1}}\dots x_{i_{l-1}}x_1x_{i_{l+2}}\dots x_{i_k}.$$

(ii) If there exist $j,n\in\{1,\dots,k\}$ such that $i_j=1=i_n$, there exists a unique $l\in\{1,\dots,k\}$ such that $i_l=2$ and $i_m>2$ for all $j\neq n\neq l\neq m$ with $j<n<l$, then

$$\hat{\sigma}_t[x_1x_2] = x_{i_1}\dots x_{i_{j-1}}x_1x_{i_{j+1}}\dots x_{i_{n-1}}x_1x_{i_{i+1}}\dots x_{i_{l-1}}x_2x_{i_{l+2}}\dots x_{i_k},$$
$$\hat{\sigma}_t[x_2x_1] = x_{i_1}\dots x_{i_{j-1}}x_2x_{i_{j+1}}\dots x_{i_{n-1}}x_2x_{i_{i+1}}\dots x_{i_{l-1}}x_1x_{i_{l+2}}\dots x_{i_k}.$$

(iii) If there exist $h,j,n\in\{1,\dots,k\}$ such that $i_h=i_j=i_n=1$, there exists a unique $l\in\{1,\dots,k\}$ such that $i_l=2$ and $i_m>2$ for all $h\neq j\neq n\neq l\neq m$ with $h<j<n<l$, then

$$\hat{\sigma}_t[x_1x_2] = x_{i_1}\dots x_{i_{h-1}}x_1x_{i_{h+1}}\dots x_{i_{j-1}}x_1x_{i_{j+1}}\dots x_{i_{n-1}}x_1$$
$$x_{i_{i+1}}\dots x_{i_{l-1}}x_2x_{i_{l+2}}\dots x_{i_k},$$
$$\hat{\sigma}_t[x_2x_1] = x_{i_1}\dots x_{i_{h-1}}x_2x_{i_{h+1}}\dots x_{i_{j-1}}x_2x_{i_{j+1}}\dots x_{i_{n-1}}x_2$$
$$x_{i_{i+1}}\dots x_{i_{l-1}}x_1x_{i_{l+2}}\dots x_{i_k}.$$

Using the associative law, the commutative law and the identity $x_1^2x_2 \approx x_1x_2^2$ we have both sides are equal.

If we apply σ_t where $t = x_{i_1}x_{i_2}\dots x_{i_k}$ and $k, i_1, \dots, i_k \in \mathbb{N}, k > 4$ on both sides of the identity $x_1^2x_2 \approx x_1x_2^2$ we have $\hat{\sigma}_t[(x_1x_1)x_2] = S^2(t, S^2(t, x_1, x_1), x_2)$ and $\hat{\sigma}_t[x_1(x_2x_2)] = S^2(t, x_1, S^2(t, x_2, x_2))$.

(i) If there exists a unique $n \in \{1, \dots, k\}$ such that $i_n = 1$, there exists a unique $l \in \{1, \dots, k\}$ such that $i_l = 2$ and $i_m > 2$ for all $m \neq n \neq l$ with $n < l$, then

$$\begin{aligned}\hat{\sigma}_t[(x_1x_1)x_2] = {} & x_{i_1}\dots x_{i_{n-1}}(x_{i_1}\dots x_{i_{n-1}}x_1x_{i_{n+1}}\dots x_{i_{l-1}}x_1x_{i_{l+1}}\dots x_{i_k})\\ & x_{i_{n+1}}\dots x_{i_{l-1}}x_2x_{i_{l+1}}\dots x_{i_k},\\ \hat{\sigma}_t[x_1(x_2x_2)] = {} & x_{i_1}\dots x_{i_{n-1}}x_1x_{i_{n+1}}\dots x_{i_{l-1}}(x_{i_1}\dots x_{i_{n-1}}x_2x_{i_{n+1}}\dots x_{i_{l-1}}\\ & x_2x_{i_{l+1}}\dots x_{i_k})x_{i_{l+1}}\dots x_{i_k}.\end{aligned}$$

(ii) If there exist $j, n \in \{1, \dots, k\}$ such that $i_j = 1 = i_n$, there exists a unique $l \in \{1, \dots, k\}$ such that $i_l = 2$ and $i_m > 2$ for all $j \neq n \neq l \neq m$ with $j < n < l$, then

$$\begin{aligned}\hat{\sigma}_t[(x_1x_1)x_2] = {} & x_{i_1}\dots x_{i_{j-1}}(x_{i_1}\dots x_{i_{j-1}}x_1x_{i_{j+1}}\dots x_{i_{n-1}}x_1x_{i_{n+1}}\dots x_{i_{l-1}}\\ & x_1x_{i_{l+1}}\dots x_{i_k})x_{i_{j+1}}\dots x_{i_{n-1}}(x_{i_1}\dots x_{i_{j-1}}x_1x_{i_{j+1}}\dots x_{i_{n-1}}x_1\\ & x_{i_{n+1}}\dots x_{i_{l-1}}x_1x_{i_{l+1}}\dots x_{i_k})x_{i_{n+1}}\dots x_{i_{l-1}}x_2x_{i_{l+1}}\dots x_{i_k},\\ \hat{\sigma}_t[x_1(x_2x_2)] = {} & x_{i_1}\dots x_{i_{j-1}}x_1x_{i_{j+1}}\dots x_{i_{n-1}}x_1x_{i_{n+1}}\dots x_{i_{l-1}}(x_{i_1}\dots x_{i_{j-1}}x_2\\ & x_{i_{j+1}}\dots x_{i_{n-1}}x_2x_{i_{n+1}}\dots x_{i_{l-1}}x_2x_{i_{l+1}}\dots x_{i_k})x_{i_{l+1}}\dots x_{i_k}.\end{aligned}$$

(iii) If there exist $h, j, n \in \{1, \dots, k\}$ such that $i_h = i_j = i_n = 1$, there exists a unique $l \in \{1, \dots, k\}$ such that $i_l = 2$ and $i_m > 2$ for all $h \neq j \neq n \neq l \neq m$ with $h < j < n < l$, then

$$\begin{aligned}\hat{\sigma}_t[(x_1x_1)x_2] = {} & x_{i_1}\dots x_{i_{h-1}}(x_{i_1}\dots x_{i_{h-1}}x_1x_{i_{h+1}}\dots x_{i_{j-1}}x_1x_{i_{j+1}}\dots x_{i_{n-1}}x_1\\ & x_{i_{n+1}}\dots x_{i_{l-1}}x_1x_{i_{l+1}}\dots x_{i_k})x_{i_{h+1}}\dots x_{i_{j-1}}(x_{i_1}\dots x_{i_{h-1}}x_1\\ & x_{i_{h+1}}\dots x_{i_{j-1}}x_1x_{i_{j+1}}\dots x_{i_{n-1}}x_1x_{i_{n+1}}\dots x_{i_{l-1}}x_1x_{i_{l+1}}\dots x_{i_k})\\ & x_{i_{j+1}}\dots x_{i_{n-1}}(x_{i_1}\dots x_{i_{h-1}}x_1x_{i_{h+1}}\dots x_{i_{j-1}}x_1x_{i_{j+1}}\dots x_{i_{n-1}}x_1\\ & x_{i_{n+1}}\dots x_{i_{l-1}}x_1x_{i_{l+1}}\dots x_{i_k})x_{i_{n+1}}\dots x_{i_{l-1}}x_2x_{i_{l+1}}\dots x_{i_k},\\ \hat{\sigma}_t[x_1(x_2x_2)] = {} & x_{i_1}\dots x_{i_{h-1}}x_1x_{i_{h+1}}\dots x_{i_{j-1}}x_1x_{i_{j+1}}\dots x_{i_{n-1}}x_1x_{i_{n+1}}\dots x_{i_{l-1}}\\ & (x_{i_1}\dots x_{i_{h-1}}x_2x_{i_{h+1}}\dots x_{i_{j-1}}x_2x_{i_{j+1}}\dots x_{i_{n-1}}x_2x_{i_{n+1}}\dots x_{i_{l-1}}x_2\\ & x_{i_{l+1}}\dots x_{i_k})x_{i_{l+1}}\dots x_{i_k}.\end{aligned}$$

Using the associative law, the commutative law and the identity $x_1^2x_2 \approx x_1x_2^2$ we have both sides are equal. ■

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