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ON SANDWICH THEOREMS FOR SOME SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING EXTENDED MULTIPLIER TRANSFORMATIONS

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Abstract. In this paper we derive some subordination and superordination results for certain normalized analytic functions in the open unit disc, which are acted upon by a class of extended multiplier transformations. Relevant connections of the results, which are presented in this paper, with various known results are also considered.

1. Introduction

Let $H(U)$ be the class of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a, n]$ denote the subclass of the functions $f \in H(U)$ of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (a \in \mathbb{C}). \tag{1.1}$$

Also, let $A(n)$ be the subclass of the functions $f \in H(U)$ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \tag{1.2}$$

and set $A \equiv A(1)$.

For $f, g \in H(U)$, we say that the function $f(z)$ is subordinate to $g(z)$, written symbolically as follows:

$$f \prec g \quad \text{or} \quad f(z) \prec g(z),$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$, $(z \in U)$, such that $f(z) = g(w(z))$ for all $z \in U$. In particular, if the function $g(z)$ is univalent in U, then we have the following equivalence (cf., e.g., [10]; see also [11, p.4]):

$$f(z) \prec g\ (z) \Leftrightarrow f(0) \prec g(0) \quad \text{and} \quad f(U) \subset g(U).$$

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Supposing that p and h are two analytic functions in U, let

$$\varphi(r,s,t;z):\mathbb{C}^3\times U\to\mathbb{C}.$$

If p and $\varphi(p(z),zp'(z),z^2p''(z);z)$ are univalent functions in U and if p satisfies the second-order superordination

$$h(z)\prec\varphi(p(z),zp'(z),z^2p''(z);z), \tag{1.3}$$

then p is said to be a solution of the differential superordination (1.3). (If f is subordinate to F, then F is superordinate to f). An analytic function q is called a subordinant of (1.3), if $q(z)\prec p(z)$ for all the functions p satisfying (1.3). A univalent subordinant $\widetilde{q}$ that satisfies $q\prec\widetilde{q}$ for all of the subordinants q of (1.3), is called the best subordinant (cf., e.g.,[10], see also [11]).

Recently, Miller and Mocanu [12] obtained sufficient conditions on the functions h, q and φ for which the following implication holds:

$$h(z)\prec\varphi(p(z),zp'(z),z^2p''(z);z)\Rightarrow q(z)\prec p(z). \tag{1.4}$$

Using these results, Bulboaca [5] considered certain classes of first-order differential superordinations as well as superordination preserving integral operators [4]. Ali et al. [1], have used the results of Bulboaca [5] and obtained sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy

$$q_1(z)\prec\frac{zf'(z)}{f(z)}\prec q_2(z), \tag{1.5}$$

where q_1 and q_2 are given univalent functions in U with $q_1(0)=1$, Shanmugam et al. [17] obtained sufficient conditions for normalized analytic functions $f(z)$ to satisfy

$$q_1(z)\prec\frac{f(z)}{zf'(z)}\prec q_2(z),\quad\text{and}\quad q_1(z)\prec\frac{z^2f'(z)}{\{f(z)\}^2}\prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0)=1.$ and $q_2(0)=1.$ Liu [9] introduced and studied the class of functions $B(\beta,\alpha,\rho)$ defined by $f\in B(\beta,\alpha,\rho)$ if and only if

$$\operatorname{Re}\left\{(1-\beta)(\frac{f(z)}{z})^\alpha+\beta\frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^\alpha\right\}>\rho,$$

where $f(z)\in A,\beta\geq 0,\alpha>0$ and $\rho\geq 0$.

Many essentially equivalent definitions of multiplier transformation have been given in literature (see [7], [8], and [20]). In [6] Catas defined the operator $I^m(\lambda,\ell)$ as follows:

Definition 1 [6]. Let the function $f(z)\in A(n)$. For $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\lambda\geq 0$, $\ell\geq 0$, the extended multiplier transformation $I^m(\lambda,\ell)$ on $A(n)$ is defined by the following infinite series:

$$I^m(\lambda,\ell)f(z)=z+\sum_{k=n+1}^{\infty}\left[\frac{\ell+1+\lambda(k-1)}{\ell+1}\right]^m a_kz^k,\quad m\in\mathbb{N}_0,\ z\in U. \tag{1.6}$$

We can write (1.6) as follows:

$$I^m(\lambda,\ell)f(z) = (\Phi_{\lambda,\ell}^{m} * f)(z),$$

where

$$\Phi_{\lambda,\ell}^m(z) = z + \sum_{k=n+1}^{\infty} \left[\frac{\ell+1+\lambda(k-1)}{\ell+1}\right]^m z^k.$$

It is easily verified from (1.6), that

$$\lambda z(I^m(\lambda,\ell)f(z))' = (1+\ell)I^{m+1}(\lambda,\ell)f(z) - [1-\lambda+\ell]I^m(\lambda,\ell)f(z)\ (\lambda > 0). \quad (1.7)$$

We note that:

$$I^0(\lambda,\ell)f(z) = f(z) \quad \text{and} \quad I^1(1,0)f(z) = zf'(z).$$

Also by specializing the parameters λ, ℓ and m we obtain the following operators studied by various authors:

(i) $I^m(1,\ell) = I^m(\ell)f(z)$ (see Cho and Srivastava [8] and Cho and Kim [7]);

(ii) $I^m(\lambda,0)f(z) = D_\lambda^m f(z)$ (see Al-Oboudi [2]);

(iii) $I^m(1,0) = D^m f(z)$ (see Salagean [16]);

(iv) $I^m(1,1) = I^m f(z)$ (see Uralegaddi and Somanatha [20]);

2. Preliminaries

In order to prove our subordination and superordination results, we make use of the following known definition and lemmas.

Definition 2. [12] Denote by Q the set of all functions $f(z)$ that are analytic and injective on $\overline{U} \setminus E(f)$ where

$$E(f) = \{\zeta : \zeta \in \partial U \ \ \textit{and} \ \ \lim_{z\to\zeta} f(z) = \infty\}, \quad (2.1)$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Lemma 1. [11] *Let the function $q(z)$ be univalent in the unit disc U, and let θ and φ be analytic in a domain D containing $q(U)$, with $\varphi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\varphi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*

(i) Q is a starlike function in U,

(ii) $\mathrm{Re}\left(\dfrac{zh'(z)}{Q(z)}\right) > 0$ for $z \in U$.

If p is analytic in U with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \quad (2.2)$$

then $p(z) \prec q(z)$, and q is the best dominant.

Lemma 2. [17] *Let q be a convex function in U and let $\psi \in \mathbb{C}$ with $\delta \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ with*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -\operatorname{Re}\frac{\psi}{\delta}\right\}, \quad z \in U.$$

If $p(z)$ is analytic in U, and

$$\psi p(z) + \delta z p'(z) \prec \psi q(z) + \delta z q'(z), \tag{2.3}$$

then $p(z) \prec q(z)$, and q is the best dominant.

Lemma 3. [4] *Let $q(z)$ be a convex univalent function in the unit disc U and let θ and φ be analytic in a domain D containing $q(U)$. Suppose that*

(i) $\operatorname{Re}\left\{\frac{\theta'(q(z))}{\varphi(q(z))}\right\} > 0$ *for $z \in U$;*

(ii) $zq'(z)\varphi(q(z))$ is starlike in U.

If $p \in H[q(0), 1] \cap Q$ with $p(U) \subseteq D$, and $\theta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in U, and

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(p(z)) + zp'(z)\varphi(p(z)),$$

then $q(z) \prec p(z)$, and q is the best subordinant.

Lemma 4. [12] *Let q be convex univalent in U and let $\delta \in \mathbb{C}$, with $\operatorname{Re}(\delta) > 0$. If $p \in H[q(0), 1] \cap Q$ and $p(z) + \delta z p'(z)$ is univalent in U, then*

$$q(z) + \delta z q'(z) \prec p(z) + \delta z p'(z), \tag{2.4}$$

implies $q(z) \prec p(z)$ $(z \in U)$, and q is the best subordinant.

This last lemma gives us a necessary and sufficient condition for the univalence of a special function which will be used in some particular cases:

Lemma 5. [15] *The function $q(z) = (1-z)^{-2ab}$ is univalent in U if and only if $|2ab - 1| \le 1$ or $|2ab + 1| \le 1$.*

3. Subordination results for analytic functions

Unless otherwise mentioned we shall assume throughout the paper that $\beta \in \mathbb{C}^*$, $\alpha > 0$, $\lambda > 0$, $\ell \ge 0$, $n \in \mathbb{N}$, $m \in \mathbb{N}_0$ and the powers understood as principle values.

Theorem 1. *Let $q(z)$ be convex univalent in U, with $q(0) = 1$. Suppose that*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -\operatorname{Re}\frac{\alpha}{\beta}\right\}. \tag{3.1}$$

If $f(z) \in A(n)$ satisfies the subordination:

$$\Phi(f, m, \lambda, \ell, \beta, \alpha) \prec q(z) + \frac{\beta}{\alpha} z q'(z), \tag{3.2}$$

where

$$\Phi(f,m,\lambda,\ell,\beta,\alpha)=\left[1-\beta(\frac{\ell+1}{\lambda})\right]\left(\frac{I^m(\lambda,\ell)f(z)}{z}\right)^{\alpha}$$
$$+\beta\left(\frac{\ell+1}{\lambda}\right)\frac{I^{m+1}(\lambda,\ell)f(z)}{I^m(\lambda,\ell)f(z)}\left(\frac{I^m(\lambda,\ell)f(z)}{z}\right)^{\alpha}, \quad (3.3)$$

then

$$\left(\frac{I^m(\lambda,\ell)f(z)}{z}\right)^{\alpha}\prec q(z), \quad (3.4)$$

and $q(z)$ *is the best dominant of* (3.2).

Proof. Define the function $p(z)$ by

$$p(z)=\left(\frac{I^m(\lambda,\ell)f(z)}{z}\right)^{\alpha} \quad (z\in U). \quad (3.5)$$

Then $p(z)$ is analytic in U and $p(0)=1$. Differentiating (3.5) logarithmically with respect to z, and using the identity (1.7) in the resulting equation, we have

$$\left[1-\beta\left(\frac{\ell+1}{\lambda}\right)\right]\left(\frac{I^m(\lambda,\ell)f(z)}{z}\right)^{\alpha}+\beta\left(\frac{\ell+1}{\lambda}\right)\frac{I^{m+1}(\lambda,\ell)f(z)}{I^m(\lambda,\ell)f(z)}\times$$
$$\times\left(\frac{I^m(\lambda,\ell)f(z)}{z}\right)^{\alpha}=p(z)+\frac{\beta}{\alpha}zp'(z). \quad (3.6)$$

Thus the subordination (3.2) is equivalent to

$$p(z)+\frac{\beta}{\alpha}zp'(z)\prec q(z)+\frac{\beta}{\alpha}zq'(z)\ . \quad (3.7)$$

Applying Lemma 2 with $\gamma=\frac{\beta}{\alpha}$ $(\alpha>0)$, the proof of Theorem 1 is completed. ■

REMARK 1. Putting $m=\ell=0$, $\lambda=n=1$ and $\beta\geq 0$ in Theorem 1, we obtain the result obtained by Shanmungam et al. [18, Theorem 3.1].

Putting $\lambda=1$ and $\ell=0$ in Theorem 1, we obtain the following corollary.

COROLLARY 1. *Let* $q(z)$ *be convex univalent in* U*, with* $q(0)=1$ *and suppose that* $q(z)$ *satisfies the condition* (3.1). *If* $f(z)\in A(n)$ *satisfies the subordination:*

$$\Phi(f,m,\beta,\alpha)\prec q(z)+\frac{\beta}{\alpha}zq'(z)\ ,$$

where

$$\Phi(f,m,\beta,\alpha)=(1-\beta)\left(\frac{D^mf(z)}{z}\right)^{\alpha}+\beta\frac{D^{m+1}f(z)}{D^mf(z)}\left(\frac{D^mf(z)}{z}\right)^{\alpha}, \quad (3.8)$$

then $\left(\frac{D^mf(z)}{z}\right)^{\alpha}\prec q(z)$ *and* $q(z)$ *is the best dominant.*

Putting $\ell = 0$ in Theorem 1, we obtain the following corollary.

Corollary 2. *Let $q(z)$ be convex univalent in U, with $q(0) = 1$ and suppose that $q(z)$ satisfy the condition (3.1). If $f(z) \in A(n)$ satisfies the subordination*

$$\Phi(f, m, \lambda, \beta, \alpha) \prec q(z) + \frac{\beta}{\alpha} zq'(z) \ ,$$

where

$$\Phi(f, m, \lambda, \beta, \alpha) = \left(1 - \frac{\beta}{\lambda}\right) \left(\frac{D_\lambda^m f(z)}{z}\right)^\alpha + \frac{\beta}{\lambda} \frac{D_\lambda^{m+1} f(z)}{D_\lambda^m f(z)} \left(\frac{D_\lambda^m f(z)}{z}\right)^\alpha , \quad (3.9)$$

then $\left(\frac{D_\lambda^m f(z)}{z}\right)^\alpha \prec q(z)$ *and* $q(z)$ *is the best dominant.*

Putting $\lambda = 1$ in Theorem 1, we obtain the following corollary.

Corollary 3. *Let $q(z)$ be convex univalent in U, with $q(0) = 1$ and suppose that $q(z)$ satisfy (3.1). If $f(z) \in A(n)$ satisfies the subordination*

$$\Phi(f, m, \ell, \beta, \alpha) \prec q(z) + \frac{\beta}{\alpha} zq'(z) \ ,$$

where

$$\Phi(f, m, \ell, \beta, \alpha) = [1 - \beta(\ell + 1)] \left(\frac{I^m(\ell) f(z)}{z}\right)^\alpha + \\ + \beta(\ell + 1) \frac{I^{m+1}(\ell) f(z)}{I^m(\ell) f(z)} \left(\frac{I^m(\ell) f(z)}{z}\right)^\alpha , \quad (3.10)$$

then $\left(\frac{I^m(\ell) f(z)}{z}\right)^\alpha \prec q(z)$ *and* $q(z)$ *is the best dominant.*

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem 1, we obtain the following corollary.

Corollary 4. *Let $-1 \le B < A \le 1$ and suppose that*

$$\mathrm{Re}\left\{\frac{1 - Bz}{1 + Bz}\right\} > \max\left\{0, -\mathrm{Re}\,\frac{\alpha}{\beta}\right\} \ .$$

If $f(z) \in A(n)$ satisfies the subordination

$$\Phi(f, m, \lambda, \ell, \beta, \alpha) \prec \frac{1 + Az}{1 + Bz} + \frac{\beta}{\alpha} \frac{(A - B)z}{(1 + Bz)^2} \ ,$$

where $\Phi(f, m, \lambda, \ell, \beta, \alpha)$ is given by (3.3), then

$$\left(\frac{I^m(\lambda, \ell) f(z)}{z}\right)^\alpha \prec \frac{1 + Az}{1 + Bz}$$

and $\frac{1 + Az}{1 + Bz}$ *is the best dominant.*

REMARK 2. Putting $m = \ell = 0$, $\lambda = n = 1$ and $\beta \geq 0$ in Corollary 4, we obtain the result obtained by Shanmungam et al. [18, Corollary 3.2].

THEOREM 2. *Let $q(z)$ be univalent in U, and $\alpha, \gamma \in \mathbb{C}$. Suppose that $q(z)$ satisfies*

$$\operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0. \tag{3.11}$$

If $f(z) \in A(n)$ satisfies the subordination

$$\psi(f, m, \lambda, \ell, \beta, \alpha) \prec 1 + \gamma \frac{zq'(z)}{q(z)}, \tag{3.12}$$

where

$$\psi(f, m, \lambda, \ell, \beta, \alpha) = 1 + \gamma\alpha \left(\frac{\ell + 1}{\lambda}\right) \left[\frac{I^{m+1}(\lambda, \ell) f(z)}{I^m(\lambda, \ell) f(z)} - 1\right], \tag{3.13}$$

then $\left(\frac{I^m(\lambda, \ell) f(z)}{z}\right)^\alpha \prec q(z)$ and $q(z)$ is the best dominant.

Proof. Let $p(z)$ be defined by (3.5). Then, simple computations show that

$$\frac{zp'(z)}{p(z)} = \alpha \left(\frac{\ell + 1}{\lambda}\right) \left[\frac{I^{m+1}(\lambda, \ell) f(z)}{I^m(\lambda, \ell) f(z)} - 1\right].$$

Putting $\theta(w) = 1$ and $\varphi(w) = \frac{\gamma}{w}$, we can observe that $\theta(w)$ is analytic in $\mathbb{C}$, $\varphi(w)$ is analytic in $\mathbb{C}^*$ and $\varphi(w) \neq 0$ $(w \in \mathbb{C}^*)$. If

$$\psi(z) = zq'(z) = \varphi(q(z)) = \gamma \frac{zq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + \psi(z) = 1 + \gamma \frac{zq'(z)}{q(z)},$$

then, from (3.11), we find that $\psi(z)$ is starlike univalent in U and

$$\operatorname{Re}\left(\frac{zh'(z)}{\psi(z)}\right) = \operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0.$$

Then applying Lemma 1, the proof is completed. ∎

REMARK 3. Taking $m = \ell = 0$ and $\lambda = n = 1$ in Theorem 2, we obtain the result obtained by Shanmugam et al. [18, Theorem 3.4].

Putting $\lambda = 1$ and $\ell = 0$ in Theorem 2, we obtain the following corollary.

COROLLARY 5. *Assume that (3.11) holds. If $f(z) \in A(n)$, and*

$$1 + \gamma\alpha \left[\frac{D^{m+1} f(z)}{D^m f(z)} - 1\right] \prec 1 + \gamma \frac{zq'(z)}{q(z)},$$

then $\left(\frac{D^m f(z)}{z}\right)^\alpha \prec q(z)$ and $q(z)$ is the best dominant.

Putting $\ell = 0$ in Theorem 2, we obtain the following corollary.

Corollary 6. *Assume that* (3.11) *holds. If* $f(z) \in A(n)$, *and*

$$1 + \frac{\gamma\alpha}{\lambda}\left[\frac{D_{\lambda}^{m+1}f(z)}{D_{\lambda}^{m}f(z)} - 1\right] \prec 1 + \gamma\frac{zq'(z)}{q(z)},$$

then $\left(\frac{D_{\lambda}^{m}f(z)}{z}\right)^{\alpha} \prec q(z)$ *and* $q(z)$ *is the best dominant.*

Putting $\lambda = 1$ in Theorem 2, we obtain the following corollary.

Corollary 7. *Assume that* (3.11) *holds. If* $f(z) \in A(n)$, *and*

$$1 + \gamma\alpha(\ell+1)\left[\frac{I^{m+1}(\ell)f(z)}{I^{m}(\ell)f(z)} - 1\right] \prec 1 + \gamma\frac{zq'(z)}{q(z)},$$

then $\left(\frac{I^{m}(\ell)f(z)}{z}\right)^{\alpha} \prec q(z)$ *and* $q(z)$ *is the best dominant.*

Taking $q(z) = \frac{1}{(1-z)^{2\alpha b}}$ $(\alpha, b \in \mathbb{C}^*)$, $\gamma = \frac{1}{\alpha b}$, $\lambda = n = 1$ and $m = \ell = 0$ in Theorem 2, we obtain the next result due to Obradović et al. [13, Theorem 1].

Corollary 8. [13] *Let* $\alpha, b \in \mathbb{C}^*$ *such that* $|2\alpha b - 1| \leq 1$ *or* $|2\alpha b + 1| \leq 1$. *Let* $f(z) \in A$ *and suppose that* $\frac{f(z)}{z} \neq 0$ *for all* $z \in U$. *If*

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1+z}{1-z},$$

then $\left(\frac{f(z)}{z}\right)^{\alpha} \prec (1-z)^{-2\alpha b}$ *and* $(1-z)^{-2\alpha b}$ *is the best dominant.*

Remark 4. For $\alpha = 1$, Corollary 8 reduces to the recent result of Srivastava and Lashin [19, Corollary 1].

Taking $q(z) = (1+Bz)^{\frac{\alpha(A-B)}{B}}$, $-1 \leq B < A \leq 1$, $B \neq 0$, $\alpha \in \mathbb{C}^*$, $\gamma = 1$, $m = \ell = 0$ and $\lambda = 1$ in Theorem 2, we obtain the following corollary.

Corollary 9. *Let* $-1 \leq B < A \leq 1$, *with* $B \neq 0$, *and suppose that*

$$\left|\frac{\alpha(A-B)}{B} - 1\right| \leq 1 \quad \text{or} \quad \left|\frac{\alpha(A-B)}{B} + 1\right| \leq 1.$$

If $f(z) \in A(n)$ *such that* $\frac{f(z)}{z} \neq 0$ *for all* $z \in U$, *and let* $\alpha \in \mathbb{C}^*$. *If*

$$1 + \alpha\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1 + [B + \alpha(A-B)]z}{1+Bz},$$

then

$$\left(\frac{f(z)}{z}\right)^{\alpha} \prec (1+Bz)^{\frac{\alpha(A-B)}{B}},$$

and $(1+Bz)^{\frac{\alpha(A-B)}{B}}$ *is the best dominant.*

Remark 5. For $\alpha = n = 1$, Corollary 9 reduces to the recent result of Obradović and Owa [14].

Putting $q(z) = (1-z)^{-2\alpha b \cos\lambda e^{-i\lambda}}$ ($\alpha, b \in \mathbb{C}^*$; $|\lambda| < \frac{\pi}{2}$), $\gamma = \frac{e^{i\lambda}}{\alpha b \cos\lambda}$, $n = \lambda = 1$ and $m = \ell = 0$ in Theorem 2, we obtain the next result due to Aouf et al. [3, Theorem 1].

Corollary 10. [3] *Let $\alpha, b \in \mathbb{C}^*$ and $|\lambda| < \frac{\pi}{2}$, and suppose that $|2\alpha b \cos\lambda e^{-i\lambda} - 1| \le 1$ or $|2\alpha b \cos\lambda e^{-i\lambda} + 1| \le 1$. Let $f(z) \in A$ such that $\frac{f(z)}{z} \neq 0$ for all $z \in U$. If*

$$1 + \frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1+z}{1-z} \ ,$$

then

$$\left(\frac{f(z)}{z}\right)^{\alpha} \prec (1-z)^{-2\alpha b \cos\lambda e^{-i\lambda}},$$

and $(1-z)^{-2\alpha b \cos\lambda e^{-i\lambda}}$ is the best dominant.

4. Superordination and Sandwich results

Theorem 3. *Let $q(z)$ be convex in U with $q(0) = 1$, and $\beta \in \mathbb{C}$, $\operatorname{Re}\beta > 0$. If $f(z) \in A(n)$ such that $(\frac{I^m(\lambda,\ell)f(z)}{f(z)})^{\alpha} \in H[q(0), 1] \cap Q$ and $\Phi(f, m, \lambda, \ell, \beta, \alpha)$ is univalent in U and satisfies the superordination:*

$$q(z) + \frac{\beta}{\alpha} zq'(z) \prec \Phi(f, m, \lambda, \ell, \beta, \alpha), \tag{4.1}$$

where $\Phi(f, m, \lambda, \ell, \beta, \alpha)$ is given by (3.3), then

$$q(z) \prec \left(\frac{I^m(\lambda,\ell)f(z)}{z}\right)^{\alpha}$$

and $q(z)$ is the best subordinant.

Proof. Let $p(z)$ be given by (3.5) and proceeding as in the proof of Theorem 1, the subordination (4.1) becomes

$$q(z) + \frac{\beta}{\alpha} zq'(z) \prec p(z) + \frac{\beta}{\alpha} zp'(z).$$

The proof follows by an application of Lemma 4. ■

Theorem 4. *Let $q(z)$ be convex univalent in U, $\beta \in \mathbb{C}$, $\operatorname{Re}(\beta) > 0$, and $(\frac{I^m(\lambda,\ell)f(z)}{z})^{\alpha} \in H[q(0), 1] \cap Q$. If $f(z) \in A(n)$ and*

$$1 + \gamma \frac{zq'(z)}{q(z)} \prec 1 + \gamma\alpha \left(\frac{\ell+1}{\lambda}\right)\left[\frac{I^{m+1}(\lambda,\ell)f(z)}{I^m(\lambda,\ell)f(z)} - 1\right], \tag{4.2}$$

then

$$q(z) \prec \left(\frac{I^m(\lambda,\ell)f(z)}{z}\right)^{\alpha}$$

and $q(z)$ is the best subordinant.

Remark 6. Putting $m = \ell = 0$, $\lambda = n = 1$ in Theorem 4, we obtain the result obtained by Shanmugam et al. [18, Theorem 4.3].

Combining Theorem 1 with Theorem 3 and Theorem 2 with Theorem 4, we state the following "Sandwich results".

Theorem 5. *Let q_1, q_2 be convex in U with $q_1(0) = q_2(0) = 1$, $\beta \in \mathbb{C}$, $\operatorname{Re}\beta > 0$ and satisfies* (3.1). *If $f(z) \in A(n)$, $(\frac{I^m(\lambda,\ell)f(z)}{z})^\alpha \in H[q(0), 1] \cap Q$, $\Phi(f, m, \lambda, \ell, \beta, \alpha)$ is univalent in the unit disc U, where $\Phi(f, m, \lambda, \ell, \beta, \alpha)$ is defined by* (3.3) *and*

$$q_1(z) + \frac{\beta}{\alpha} z q_1'(z) \prec \Phi(f, m, \lambda, \ell, \beta, \alpha) \prec q_2(z) + \frac{\beta}{\alpha} z q_2'(z), \tag{4.3}$$

then

$$q_1(z) \prec \left(\frac{I^m(\lambda,\ell)f(z)}{z}\right)^\alpha \prec q_2(z)$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and best dominant.

Putting $\lambda = 1$ and $\ell = 0$ in Theorem 5, we obtain the following corollary.

Corollary 11. *Let $q_1(z), q_2(z)$ be convex in U with $q_1(0) = q_2(0) = 1$, $\beta \in \mathbb{C}$, $\operatorname{Re}\beta > 0$ and satisfies* (3.1). *If $f(z) \in A(n)$, $(\frac{D^m f(z)}{z})^\alpha \in H[q(0), 1] \cap Q$, $\Phi(f, m, \beta, \alpha)$ is univalent in the unit disc U, where $\Phi(f, m, \beta, \alpha)$ is defined by* (3.8) *and*

$$q_1(z) + \frac{\beta}{\alpha} z q_1'(z) \prec \Phi(f, m, \beta, \alpha) \prec q_2(z) + \frac{\beta}{\alpha} z q_2'(z),$$

then

$$q_1(z) \prec \left(\frac{D^m f(z)}{z}\right)^\alpha \prec q_2(z)$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and best dominant.

Putting $\ell = 0$ in Theorem 5, we obtain the following corollary.

Corollary 12. *Let $q_1(z), q_2(z)$ be convex in U with $q_1(0) = q_2(0) = 1$, $\beta \in \mathbb{C}$, $\operatorname{Re}\beta > 0$ and satisfies* (3.1). *If $f(z) \in A(n)$, $(\frac{D_\lambda^m f(z)}{z})^\alpha \in H[q(0), 1] \cap Q$, $\Phi(f, m, \lambda, \beta, \alpha)$ is univalent in the unit disc U, where $\Phi(f, m, \lambda, \beta, \alpha)$ is defined by* (3.9) *and*

$$q_1(z) + \frac{\beta}{\alpha} z q_1'(z) \prec \Phi(f, m, \lambda, \beta, \alpha) \prec q_2(z) + \frac{\beta}{\alpha} z q_2'(z),$$

then

$$q_1(z) \prec \left(\frac{D_\lambda^m f(z)}{z}\right)^\alpha \prec q_2(z)$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and best dominant.

Putting $\lambda = 1$ in Theorem 5, we obtain the following corollary.

Corollary 13. *Let $q_1(z), q_2(z)$ be convex in U with $q_1(0) = q_2(0) = 1$, $\beta \in \mathbb{C}$, $\operatorname{Re}\beta > 0$ and satisfies* (3.1). *If $f(z) \in A(n)$, $(\frac{I^m(\ell)f(z)}{z})^\alpha \in H[q(0), 1] \cap Q$,*

$\Phi(f,m,\ell,\beta,\alpha)$ is univalent in the unit disc U, where $\Phi(f,m,\ell,\beta,\alpha)$ is defined by (3.10) *and*

$$q_1(z)+\frac{\beta}{\alpha}zq_1'(z)\prec\Phi(f,m,\ell,\beta,\alpha)\prec q_2(z)+\frac{\beta}{\alpha}zq_2'(z),$$

then

$$q_1(z)\prec\left(\frac{I^m(\ell)f(z)}{z}\right)^{\alpha}\prec q_2(z)$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and best dominant.

Theorem 6. *Let $q_1(z),q_2(z)$ be convex in U with $q_1(0)=q_2(0)=1$, $\beta\in\mathbb{C}$, $\operatorname{Re}\beta>0$ and satisfies* (3.1). *If $f(z)\in A(n)$, $(\frac{I^m(\lambda,\ell)f(z)}{z})^{\alpha}\in H[q(0),1]\cap Q$, $\psi(f,m,\lambda,\ell,\beta,\alpha)$ is univalent in the unit disc U, where $\psi(f,m,\lambda,\ell,\beta,\alpha)$ is defined by* (3.13) *and*

$$1+\gamma\frac{zq_1'(z)}{q_1(z)}\prec\psi(f,m,\lambda,\ell,\beta,\alpha)\prec1+\gamma\frac{zq_2'(z)}{q_2(z)},\tag{4.4}$$

then

$$q_1(z)\prec\left(\frac{I^m(\lambda,\ell)f(z)}{z}\right)^{\alpha}\prec q_2(z)$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

Putting $\lambda=1$ and $\ell=0$ in Theorem 6, we obtain the following corollary.

Corollary 14. *Let $q_1(z),q_2(z)$ be convex in U with $q_1(0)=q_2(0)=1$, $\beta\in\mathbb{C}$, $\operatorname{Re}\beta>0$ and satisfies* (3.1). *If $f(z)\in A(n)$, $(\frac{D^mf(z)}{z})^{\alpha}\in H[q(0),1]\cap Q$,*

$$1+\gamma\alpha\left[\frac{D^{m+1}f(z)}{D^mf(z)}-1\right]$$

is univalent in U and

$$1+\gamma\frac{zq_1'(z)}{q_1(z)}\prec1+\gamma\alpha\left[\frac{D^{m+1}f(z)}{D^mf(z)}-1\right]\prec1+\gamma\frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z)\prec\left(\frac{D^mf(z)}{z}\right)^{\alpha}\prec q_2(z)$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

Putting $\ell=0$ in Theorem 6, we obtain the following corollary.

Corollary 15. *Let $q_1(z),q_2(z)$ be convex in U with $q_1(0)=q_2(0)=1$, $\beta\in\mathbb{C}$, $\operatorname{Re}\beta>0$ and satisfies* (3.1). *If $f(z)\in A(n)$, $(\frac{D_\lambda^mf(z)}{z})^{\alpha}\in H[q(0),1]\cap Q$,*

$$1+\frac{\gamma\alpha}{\lambda}\left[\frac{D_\lambda^{m+1}f(z)}{D_\lambda^mf(z)}-1\right]$$

is univalent in U *and*

$$1+\gamma\frac{zq_1'(z)}{q_1(z)} \prec 1+\frac{\gamma\alpha}{\lambda}\left[\frac{D_\lambda^{m+1}f(z)}{D_\lambda^m f(z)}-1\right] \prec 1+\gamma\frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{D_\lambda^m f(z)}{z}\right)^\alpha \prec q_2(z)$$

and $q_1(z)$ *and* $q_2(z)$ *are, respectively, the best subordinant and the best dominant.*

Putting $\lambda = 1$ in Theorem 6, we obtain the following corollary.

Corollary 16. *Let* $q_1(z), q_2(z)$ *be convex in* U *with* $q_1(0) = q_2(0) = 1$, $\beta \in \mathbb{C}$, $\operatorname{Re}\beta > 0$ *and satisfies* (3.1). *If* $f(z) \in A(n)$, $(\frac{I^m(\ell)f(z)}{z})^\alpha \in H[q(0), 1] \cap Q$,

$$1+\gamma\alpha(\ell+1)\left[\frac{I^{m+1}(\ell)f(z)}{I^m(\ell)f(z)}-1\right]$$

is univalent in U *and*

$$1+\gamma\frac{zq_1'(z)}{q_1(z)} \prec 1+\gamma\alpha(\ell+1)\left[\frac{I^{m+1}(\ell)f(z)}{I^m(\ell)f(z)}-1\right] \prec 1+\gamma\frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{I^m(\ell)f(z)}{z}\right)^\alpha \prec q_2(z)$$

and $q_1(z)$ *and* $q_2(z)$ *are, respectively, the best subordinant and the best dominant.*

Remark 7. Putting $m = \ell = 0$, $\lambda = n = 1$ and $\beta \geq 0$ in Theorem 6, we obtain the following result which improves the result of Shanmugam et al. [18, Theorem 5.2].

Corollary 17. *Let* $q_1(z), q_2(z)$ *be convex in* U *with* $q_1(0) = q_2(0) = 1$, $\beta \in \mathbb{C}$, $\operatorname{Re}\beta > 0$ *and satisfies* (3.1). *If* $f(z) \in A(n)$, $(\frac{f(z)}{z})^\alpha \in H[q(0), 1] \cap Q$, $1+\gamma\alpha(\frac{zf'(z)}{f(z)}-1)$ *is univalent in* U *and*

$$1+\gamma\frac{zq_1'(z)}{q_1(z)} \prec 1+\gamma\alpha\left(\frac{zf'(z)}{f(z)}-1\right) \prec 1+\gamma\frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{f(z)}{z}\right)^\alpha \prec q_2(z)$$

and $q_1(z)$ *and* $q_2(z)$ *are, respectively, the best subordinant and the best dominant.*

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