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ON SEMI-INVARIANT SUBMANIFOLDS OF A NEARLY KENMOTSU MANIFOLD WITH THE CANONICAL SEMI-SYMMETRIC SEMI-METRIC CONNECTION

Mobin Ahmad

Abstract. We define the canonical semi-symmetric semi-metric connection in a nearly Kenmotsu manifold and we study semi-invariant submanifolds of a nearly Kenmotsu manifold endowed with the canonical semi-symmetric semi-metric connection. Moreover, we discuss the integrability of distributions on semi-invariant submanifolds of a nearly Kenmotsu manifold with the canonical semi-symmetric semi-metric connection.

1. Introduction

In [9], K. Kenmotsu introduced and studied a new class of almost contact manifolds called Kenmotsu manifolds. The notion of nearly Kenmotsu manifold was introduced by A. Shukla in [13]. Semi-invariant submanifolds in Kenmotsu manifolds were studied by N. Papaghuic [11] and M. Kobayashi [10]. Semi-invariant submanifolds of a nearly Kenmotsu manifolds were studied by M.M. Tripathi and S.S. Shukla in [14]. In this paper we study semi-invariant submanifolds of a nearly Kenmotsu manifold with the canonical semi-symmetric semi-metric connection.

Let ∇ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T and the curvature tensor R of ∇ are given respectively by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$
$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The connection ∇ is symmetric if the torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is a metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection.

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In [8, 12], A. Friedmann and J. A. Schouten introduced the idea of a semi-symmetric linear connection A linear connection ∇ is said to be a semi-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)X - \eta(X)Y,$$

where η is a 1-form.

Some properties of semi-invariant submanifolds, hypersurfaces and submanifolds with respect to semi-symmetric or quarter symmetric connections were studied in [1, 7], [2, 3] and [4] respectively.

This paper is organized as follows. In Section 2, we give a brief introduction of nearly Kenmotsu manifold. In Section 3, we show that the induced connection on semi-invariant submanifolds of a nearly Kenmotsu manifold with the canonical semi-symmetric semi-metric connection is also semi-symmetric semi-metric. In Section 4, we establish some lemmas on semi-invariant submanifolds and in Section 5, we discuss the integrability conditions of distributions on semi-invariant submanifolds of nearly Kenmotsu manifolds with the canonical semi-symmetric semi-metric connection.

2. Preliminaries

Let $\bar{M}$ be $(2m+1)$-dimensional almost contact metric manifold [6] with a metric tensor g, a tensor field ϕ of type (1,1), a vector field ξ, a 1-form η which satisfy

$$\phi^2 = -I + \eta\otimes\xi, \quad \phi\xi = 0, \quad \eta\phi = 0, \quad \eta(\xi) = 1 \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any vector fields X, Y on $\bar{M}$. If in addition to the above conditions we have $d\eta(X,Y) = g(X, \phi Y)$, the structure is said to be a contact metric structure.

The almost contact metric manifold $\bar{M}$ is called a nearly Kenmotsu manifold if it satisfies the condition [13]

$$(\bar{\bar{\nabla}}_X\phi)(Y) + (\bar{\bar{\nabla}}_Y\phi)(X) = -\eta(Y)\phi X - \eta(X)\phi Y, \tag{2.3}$$

where $\bar{\bar{\nabla}}$ denotes the Riemannian connection with respect to g. If, moreover, M satisfies

$$(\bar{\bar{\nabla}}_X\phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.4}$$

then it is called Kenmotsu manifold [9]. Obviously a Kenmotsu manifold is also a nearly Kenmotsu manifold.

Definition. An n-dimensional Riemannian submanifold M of a nearly Kenmotsu manifold $\bar{M}$ is called a semi-invariant submanifold if ξ is tangent to M and there exists on M a pair of distributions $(D, D^{\perp})$ such that [10]:

(i) TM orthogonally decomposes as $D\oplus D^{\perp}\oplus\langle\xi\rangle$,

(ii) the distribution D is invariant under ϕ, that is, $\phi D_x \subset D_x$ for all $x \in M$,

(iii) the distribution $D^\perp$ is anti-invariant under ϕ, that is, $\phi D_x^\perp \subset T_x^\perp M$ for all $x \in M$, where $T_x M$ and $T_x^\perp M$ are the tangent and normal spaces of M at x.

The distribution D(resp. $D^\perp$) is called the horizontal (resp. vertical) distribution. A semi-invariant submanifold M is said to be an invariant (resp. anti-invariant) submanifold if we have $D_x^\perp = \{0\}$ (resp. $D_x = \{0\}$) for each $x \in M$. We also call M proper if neither D nor $D^\perp$ is null. It is easy to check that each hypersurface of $\bar{M}$ which is tangent to ξ inherits a structure of semi-invariant submanifold of $\bar{M}$.

Now, we remark that owing to the existence of the 1-form η, we can define the canonical semi-symmetric semi-metric connection $\bar{\nabla}$ in any almost contact metric manifold $(\bar{M}, \phi, \xi, \eta, g)$ by

$$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y - \eta(X)Y + g(X,Y)\xi \tag{2.5}$$

such that $(\bar{\nabla}_X g)(Y,Z) = 2\eta(X)g(Y,Z) - \eta(Y)g(X,Z) - \eta(Z)g(X,Y)$ for any $X, Y \in T\bar{M}$. In particular, if $\bar{M}$ is a nearly Kenmotsu manifold, then from (2.5) we have

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = -\eta(X)\phi Y - \eta(Y)\phi X. \tag{2.6}$$

Theorem 2.1. *Let $(\bar{M}, \phi, \xi, \eta, g)$ be an almost contact metric manifold and M be a submanifold tangent to ξ. Then, with respect to the orthogonal decomposition $TM \oplus T^\perp M$, the canonical semi-symmetric semi-metric connection $\bar{\nabla}$ induces on M a connection ∇ which is semi-symmetric and semi-metric.*

Proof. With respect to the orthogonal decomposition $TM \oplus T^\perp M$, we have

$$\bar{\nabla}_X Y = \nabla_X Y + m(X,Y), \tag{2.7}$$

where m is a $T^\perp M-$valued symmetric tensor field on M. If $\nabla^\star$ denotes the induced connection from the Riemannian connection $\bar{\bar{\nabla}}$, then

$$\bar{\bar{\nabla}}_X Y = \nabla^\star{}_X Y + h(X,Y), \tag{2.8}$$

where h is the second fundamental form. By the definition of semi-symmetric semi-metric connection

$$\bar{\nabla}_X Y = \bar{\bar{\nabla}}_X Y - \eta(X)Y + g(X,Y)\xi. \tag{2.9}$$

Now using above equations, we have

$$\nabla_X Y + m(X,Y) = \nabla^\star{}_X Y + h(X,Y) - \eta(X)Y + g(X,Y)\xi.$$

Equating tangential and normal components from both the sides, we get

$$h(X,Y) = m(X,Y)$$

and

$$\nabla_X Y = \nabla^\star{}_X Y - \eta(X)Y + g(X,Y)\xi.$$

Thus ∇ is also a semi-symmetric semi-metric connection. ∎

Now, Gauss equation for M in $(\bar{M}, \bar{\nabla})$ is

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.10}$$

and Weingarten formulas are given by

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N - \eta(X) N \tag{2.11}$$

for $X, Y \in TM$ and $N \in T^{\perp}M$. Moreover, we have

$$g(h(X, Y), N) = g(A_N X, Y). \tag{2.12}$$

From now on, we consider a nearly Kenmotsu manifold $\bar{M}$ and a semi-invariant submanifold M. Any vector X tangent to M can be written as

$$X = PX + QX + \eta(X)\xi, \tag{2.13}$$

where PX and QX belong to the distribution D and $D^{\perp}$ respectively. For any vector field N normal to M, we put

$$\phi N = BN + CN, \tag{2.14}$$

where BN (resp. CN) denotes the tangential (resp. normal) component of ϕN.

Definition. A semi-invariant submanifold is said to be mixed totally geodesic if $h(X, Z) = 0$ for all $X \in D$ and $Z \in D^{\perp}$.

Using the canonical semi-symmetric semi-metric connection, the Nijenhuis tensor of ϕ is expressed by

$$N(X, Y) = (\bar{\nabla}_{\phi X}\phi)(Y) - (\bar{\nabla}_{\phi Y}\phi)(X) - \phi(\bar{\nabla}_X\phi)(Y) + \phi(\bar{\nabla}_Y\phi)(X) \tag{2.15}$$

for any $X, Y \in T\bar{M}$.

From (2.6), we have

$$(\bar{\nabla}_{\phi X}\phi)(Y) = \eta(Y)X - \eta(X)\eta(Y)\xi - (\bar{\nabla}_Y\phi)\phi X. \tag{2.16}$$

Also,

$$(\bar{\nabla}_Y\phi)\phi X = ((\bar{\nabla}_Y\eta)(X))\xi + \eta(X)\bar{\nabla}_Y\xi - \phi(\bar{\nabla}_Y\phi)X. \tag{2.17}$$

By virtue of (2.15), (2.16) and (2.17), we get

$$\begin{aligned} N(X, Y) = -\eta(Y)X - 3\eta(X)Y + 4\eta(X)\eta(Y)\xi + \eta(Y)\bar{\nabla}_X\xi \\ - \eta(X)\bar{\nabla}_Y\xi + 2d\eta(X, Y)\xi + 4\phi(\bar{\nabla}_Y\phi)X \end{aligned} \tag{2.18}$$

for any $X, Y \in T\bar{M}$.

3. Basic lemmas

LEMMA 3.1. *Let M be a semi-invariant submanifold of a nearly Kenmotsu manifold $\bar{M}$ with the canonical semi-symmetric semi-metric connection. Then*

$$2(\bar{\nabla}_X\phi)Y = \nabla_X\phi Y - \nabla_Y\phi X + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y]$$

for any $X, Y \in D$.

Proof. By Gauss formula we have

$$\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = \nabla_X\phi Y - \nabla_Y\phi X + h(X,\phi Y) - h(Y,\phi X). \tag{3.1}$$

Also by use of (2.10) covariant differentiation yields

$$\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = (\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X + \phi[X,Y]. \tag{3.2}$$

From (3.1) and (3.2), we get

$$(\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X = \nabla_X\phi Y - \nabla_Y\phi X + h(X,\phi Y) - h(Y,\phi X) - \phi[X,Y]. \tag{3.3}$$

Using $\eta(X) = 0$ for each $X \in D$ in (2.6), we get

$$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = 0. \tag{3.4}$$

Adding (3.3) and (3.4) we get the result. ■

Similar computations also yield

LEMMA 3.2. *Let M be a semi-invariant submanifold of a nearly Kenmotsu manifold with the canonical semi-symmetric semi-metric connection. Then*

$$2(\bar{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y]$$

for any $X{\in}D$ and $Y{\in}D^{\perp}$.

LEMMA 3.3. *Let M be a semi-invariant submanifold of a nearly Kenmotsu manifold $\bar{M}$ with the canonical semi-symmetric semi-metric connection. Then*

$$\begin{aligned} P\nabla_X\phi PY + P\nabla_Y\phi PX - PA_{\phi QY}X - PA_{\phi QX}Y \\ = -2\eta(Y)\phi PX - \eta(X)\phi PY + \phi P\nabla_XY + \phi P\nabla_YX \end{aligned} \tag{3.5}$$

$$\begin{aligned} Q\nabla_X\phi PY + Q\nabla_Y\phi PX - QA_{\phi QY}X - QA_{\phi QX}Y \\ = -\eta(Y)\phi QX - 2\eta(X)\phi QY + 2Bh(X,Y) \end{aligned} \tag{3.6}$$

$$\begin{aligned} h(X,\phi PY) + h(Y,\phi PX) + \nabla^{\perp}_X\phi QY + \nabla^{\perp}_Y\phi QX \\ = 2Ch(X,Y) + \phi Q\nabla_XY + \phi Q\nabla_YX \end{aligned} \tag{3.7}$$

$$\eta(\nabla_X\phi PY + \nabla_Y\phi PX - A_{\phi QY}X - A_{\phi QX}Y) = 0 \tag{3.8}$$

for all $X, Y{\in}TM$.

Proof. Differentiating (2.13) covariantly and using (2.10) and (2.11), we have

$$\begin{aligned}(\bar{\nabla}_X\phi)Y + \phi(\nabla_X Y) + \phi h(X,Y) &= P\nabla_X(\phi PY) + Q\nabla_X(\phi PY)\\ &- \eta(A_{\phi QY}X)\xi + \eta(\nabla_X\phi PY)\xi - PA_{\phi QY}X - QA_{\phi QY}X\\ &+ \nabla^{\perp}_X\phi QY + h(X,\phi PY) + \eta(X)\phi PY. \qquad (3.9)\end{aligned}$$

Similarly,

$$\begin{aligned}(\bar{\nabla}_Y\phi)X + \phi(\nabla_Y X) + \phi h(Y,X) &= P\nabla_Y(\phi PX) + Q\nabla_Y(\phi PX)\\ &- \eta(A_{\phi QX}Y)\xi + \eta(\nabla_Y\phi PX)\xi - PA_{\phi QX}Y - QA_{\phi QX}Y\\ &+ \nabla^{\perp}_Y\phi QX + h(Y,\phi PX) + \eta(Y)\phi PX. \qquad (3.10)\end{aligned}$$

Adding (3.9) and (3.10) and using (2.6) and (2.14), we have

$$\begin{aligned}&- 2\eta(Y)\phi PX - 2\eta(Y)\phi QX - 2\eta(X)\phi PY - 2\eta(X)\phi QY + \phi P\nabla_X Y\\ &\quad + \phi Q\nabla_X Y + \phi P\nabla_Y X + \phi Q\nabla_Y X + 2Bh(Y,X) + 2Ch(Y,X)\\ &\quad = P\nabla_X(\phi PY) + P\nabla_Y(\phi PX) + Q\nabla_Y(\phi PX) - PA_{\phi QY}X\\ &\quad + Q\nabla_X(\phi PY) + \nabla^{\perp}_X\phi QY - PA_{\phi QX}Y - QA_{\phi QY}X\\ &\quad - QA_{\phi QX}Y + \nabla^{\perp}_Y\phi QX + h(Y,\phi PX) + h(X,\phi PY)\\ &\quad + \eta(\nabla_X\phi PY)\xi + \eta(\nabla_Y\phi PX)\xi - \eta(A_{\phi QX}Y)\xi - \eta(A_{\phi QY}X)\xi. \qquad (3.11)\end{aligned}$$

Equations (3.5)–(3.8) follow by comparison of tangential, normal and vertical components of (3.11). ∎

Definition. The horizontal distribution D is said to be parallel with respect to the connection ∇ on M if $\nabla_X Y \in D$ for all vector fields $X, Y \in D$.

Proposition 3.4. *Let M be a semi-invariant submanifold of a nearly Kenmotsu manifold $\bar{M}$ with the canonical semi-symmetric semi-metric connection. If the horizontal distribution D is parallel then $h(X,\phi Y) = h(Y,\phi X)$ for all $X, Y \in D$.*

Proof. Since D is parallel, therefore, $\nabla_X\phi Y \in D$ and $\nabla_Y\phi X \in D$ for each $X, Y \in D$. Now from (3.6) and (3.7), we get

$$h(X,\phi Y) + h(Y,\phi X) = 2\phi h(X,Y). \qquad (3.12)$$

Replacing X by ϕX in above equation, we have

$$h(\phi X,\phi Y) - h(Y,X) = 2\phi h(\phi X,Y). \qquad (3.13)$$

Replacing Y by ϕY in (3.12), we have

$$-h(X,Y) + h(\phi X,\phi Y) = 2\phi h(X,\phi Y). \qquad (3.14)$$

Comparing (3.13) and (3.14), we have $h(X,\phi Y) = h(\phi X,Y)$ for all $X, Y \in D$. ∎

LEMMA 3.5. *Let M be a semi-invariant submanifold of a nearly Kenmotsu manifold $\bar{M}$ with the canonical semi-symmetric semi-metric connection. Then M is mixed totally geodesic if and only if $A_N X \in D$ for all $X \in D$ and $N \in T^{\perp}M$.*

Proof. If $A_N X \in D$, then $g(h(X,Y),N) = g(A_N X, Y) = 0$, which gives $h(X,Y) = 0$ for $Y \in D^{\perp}$. Hence M is mixed totally geodesic. ∎

4. Integrability conditions for distributions

THEOREM 4.1. *Let M be a semi-invariant submanifold of a nearly Kenmotsu manifold $\bar{M}$ with the canonical semi-symmetric semi-metric connection. Then the following conditions are equivalent:*

(i) the distribution $D \oplus \langle \xi \rangle$ is integrable,

(ii) $N(X,Y) \in D \oplus \langle \xi \rangle$ and $h(X, \phi Y) = h(\phi X, Y)$ for any $X, Y \in D \oplus \langle \xi \rangle$.

Proof. The condition $N(X,Y) \in D \oplus \langle \xi \rangle$ for any $X, Y \in D \oplus \langle \xi \rangle$ is equivalent to the following two

(I) $N(X,\xi) \in D \oplus \langle \xi \rangle$ for any $X \in D$,

(II) $N(X,Y) \in D \oplus \langle \xi \rangle$ for any $X, Y \in D$.

In the first case, using Gauss formula and (2.6) in (2.18), we get

$$N(X,\xi) = 3X - 3\nabla_X \xi + 2d\eta(X,\xi)\xi - 3h(X,\xi) + 4\eta(\nabla_X \xi)\xi$$

and

$$N(X,\xi) \in D \oplus \langle \xi \rangle \Leftrightarrow Q(\nabla_X \xi) = 0, h(X,\xi) = 0.$$

Using again (2.6) and computing its normal component we get

$$h(\xi, \phi X) - \phi Q(\nabla_\xi X) - 2C(h(\xi, X)) - \phi Q(\nabla_X \xi) = 0.$$

Hence for any $X \in D$

$$N(X,\xi) \in D \oplus \langle \xi \rangle \Rightarrow Q([X,\xi]) = 0, h(X,\xi) = 0. \tag{4.1}$$

In case (II), using Gauss formula in (2.18), we get

$$N(X,Y) = 2d\eta(X,Y)\xi + 4\phi(\nabla_Y \phi X) + 4\phi h(Y, \phi X) + 4h(Y,X) + 4\nabla_Y X - 4\eta(\nabla_Y X)\xi \tag{4.2}$$

for all $X, Y \in D$. From (4.2) we have that $N(X,Y) \in (D \oplus \langle \xi \rangle)$ implies

$$\phi Q(\nabla_Y \phi X) + Ch(Y, \phi X) + h(Y,X) = 0$$

for all $X, Y \in D$. Replacing Y by ϕZ, where $Z \in D$, we get

$$\phi Q(\nabla_{\phi Z} \phi X) + Ch(\phi Z, \phi X) + h(\phi Z, X) = 0.$$

Interchanging X and Z, we have

$$\phi Q(\nabla_{\phi X} \phi Z) + Ch(\phi X, \phi Z) + h(\phi X, Z) = 0.$$

Subtracting above two equations, we have

$$\phi Q[\phi X, \phi Z] + h(Z, \phi X) - h(X, \phi Z) = 0.$$

Thus, we get, for any $X, Y \in D$

$$N(X,Y) \in D \oplus \langle\xi\rangle \Rightarrow \phi Q([X,Y]) + h(\phi X, Y) - h(X, \phi Y) = 0. \tag{4.3}$$

Now, suppose that $D\oplus\langle\xi\rangle$ is integrable so for any $X, Y \in D\oplus\langle\xi\rangle$ we have $N(X,Y) \in D\oplus\langle\xi\rangle$, since $\phi(D\oplus\langle\xi\rangle) \subset D$. Moreover, $h(X,\xi) = 0, h(X,\phi Y) = h(\phi X, Y)$ for any $X, Y \in D$ and ii) is proven. Vice versa, if ii) holds, then from (4.1) and (4.3) we get the integrability of $D\oplus\langle\xi\rangle$. ■

Lemma 4.2. *Let M be a semi-invariant submanifold of a nearly Kenmotsu manifold $\bar{M}$ with the canonical semi-symmetric semi-metric connection. Then*

$$2(\bar{\nabla}_Y \phi)Z = A_{\phi Y} Z - A_{\phi Z} Y + \nabla^{\perp}_Y \phi Z - \nabla^{\perp}_Z \phi Y - \phi[Y, Z]$$

for $Y, Z \in D^{\perp}$.

Proof. From Weingarten equation, we have

$$\bar{\nabla}_Y \phi Z - \bar{\nabla}_Z \phi Y = -A_{\phi Z} Y + A_{\phi Y} Z + \nabla^{\perp}_Y \phi Z - \nabla^{\perp}_Z \phi Y. \tag{4.4}$$

Also by covariant differentiation, we get

$$\bar{\nabla}_Y \phi Z - \bar{\nabla}_Z \phi Y = (\bar{\nabla}_Y \phi)Z - (\bar{\nabla}_Z \phi)Y + \phi[Y, Z]. \tag{4.5}$$

From (4.4) and (4.5) we have

$$(\bar{\nabla}_Y \phi)Z - (\bar{\nabla}_Z \phi)Y = A_{\phi Y} Z - A_{\phi Z} Y + \nabla^{\perp}_Y \phi Z - \nabla^{\perp}_Z \phi Y - \phi[Y, Z]. \tag{4.6}$$

From (2.6) we obtain

$$(\bar{\nabla}_Y \phi)Z + (\bar{\nabla}_Z \phi)Y = 0 \tag{4.7}$$

for any $Y, Z \in D^{\perp}$. Adding (4.6) and (4.7), we get

$$2(\bar{\nabla}_Y \phi)Z = A_{\phi Y} Z - A_{\phi Z} Y + \nabla^{\perp}_Y \phi Z - \nabla^{\perp}_Z \phi Y - \phi[Y, Z]. \quad ■$$

Proposition 4.3. *Let M be a semi-invariant submanifold of a nearly Kenmotsu manifold $\bar{M}$ with the canonical semi-symmetric semi-metric connection. Then*

$$A_{\phi Y} Z - A_{\phi Z} Y = \frac{1}{3}\phi P[Y, Z]$$

for any $Y, Z \in D^{\perp}$.

Proof. Let $Y, Z \in D^{\perp}$ and $X \in TM$ then from (2.10) and (2.12), we have

$$2g(A_{\phi Z} Y, X) = -g(\bar{\nabla}_Y \phi X, Z) - g(\bar{\nabla}_X \phi Y, Z) + g((\bar{\nabla}_Y \phi)X + (\bar{\nabla}_X \phi)Y, Z).$$

By use of (2.6) and $\eta(Y) = 0$ for $Y \in D^{\perp}$, we have

$$2g(A_{\phi Z} Y, X) = -g(\phi \bar{\nabla}_Y Z, X) + g(A_{\phi Y} Z, X).$$

Interchanging Y and Z and subtracting we get

$$g(3A_{\phi Y}Z - 3A_{\phi Z}Y - \phi P[Y,Z], X) = 0 \tag{4.8}$$

from which, for any $Y, Z \in D^{\perp}$,

$$A_{\phi Y}Z - A_{\phi Z}Y = \frac{1}{3}\phi P[Y,Z]$$

follows. ■

THEOREM 4.4. *Let M be a semi-invariant submanifold of a nearly Kenmotsu manifold $\bar{M}$ with the canonical semi-symmetric semi-metric connection. Then the distribution $D^{\perp}$ is integrable if and only if*

$$A_{\phi Y}Z - A_{\phi Z}Y = 0$$

for all $Y, Z \in D^{\perp}$.

Proof. Suppose that the distribution $D^{\perp}$ is integrable. Then $[Y,Z] \in D^{\perp}$ for any $Y, Z \in D^{\perp}$. Therefore, $P[Y,Z] = 0$ and from (4.8), we get

$$A_{\phi Y}Z - A_{\phi Z}Y = 0. \tag{4.9}$$

Conversely, let (4.9) hold. Then by virtue of (4.8) we have $\phi P[Y,Z] = 0$ for all $Y, Z \in D^{\perp}$. Since rank $\phi = 2m$, we have $\phi P[Y,Z] = 0$ and $P[Y,Z] \in D \cap \langle \xi \rangle$. Hence $P[Y,Z] = 0$, which is equivalent to $[Y,Z] \in D^{\perp}$ for all $Y, Z \in D^{\perp}$ and $D^{\perp}$ is integrable. ■

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Department of Mathematics, Integral University, Kursi Road, Lucknow-226026, INDIA.

E-mail: mobinahmad@rediffmail.com