

## SOME STABILITY RESULTS FOR TWO HYBRID FIXED POINT ITERATIVE ALGORITHMS IN NORMED LINEAR SPACE

M. O. Olatinwo

**Abstract.** In this paper, we prove some stability results for two newly introduced hybrid fixed point iterative algorithms of Kirk-Ishikawa and Kirk-Mann Type in normed linear space using a certain contractive condition. Our results generalize, extend and improve some of the results of Harder and Hicks [11], Rhoades [29,30], Osilike [26], Berinde [2,3] as well as the recent results of the author [12,23,24,25].

### 1. Introduction

Let  $(E, d)$  be a complete metric space and  $T : E \rightarrow E$  a selfmap of  $E$ . Suppose that  $F_T = \{p \in E \mid Tp = p\}$  is the set of fixed points of  $T$ .

There are several iterative processes in the literature for which the fixed points of operators have been approximated over the years by various authors. In a complete metric space, the Picard iterative process  $\{x_n\}_{n=0}^{\infty}$  defined by

$$x_{n+1} = Tx_n, \quad n = 0, 1, \dots, \quad (1.1)$$

has been employed to approximate the fixed points of mappings satisfying the inequality relation

$$d(Tx, Ty) \leq \alpha d(x, y), \quad \forall x, y \in E \text{ and } \alpha \in [0, 1). \quad (1.2)$$

Condition (1.2) is called the *Banach's contraction condition*. Any operator satisfying (1.2) is called a *strict contraction*. Also, condition (1.2) is significant in the celebrated Banach's fixed point theorem [1].

In the Banach space setting, we shall state some of the iterative processes generalizing (1.1) as follows.

For  $x_0 \in E$ , the sequence  $\{x_n\}_{n=0}^{\infty}$  defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n = 0, 1, \dots, \quad (1.3)$$

where  $\{\alpha_n\}_{n=0}^{\infty} \subset [0, 1]$ , is called the Mann iterative process (see Mann [21]).

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For  $x_0 \in E$ , the sequence  $\{x_n\}_{n=0}^\infty$  defined by

$$\left. \begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Tz_n \\ z_n &= (1 - \beta_n)x_n + \beta_n Tx_n \end{aligned} \right\} \quad n = 0, 1, \dots, \quad (1.4)$$

where  $\{\alpha_n\}_{n=0}^\infty$  and  $\{\beta_n\}_{n=0}^\infty$  are sequences in  $[0, 1]$ , is called the Ishikawa iterative process (see Ishikawa [14]). See Berinde [3] for details on various iteration processes.

Kannan [16] established an extension of the Banach's fixed point theorem by using the following contractive definition. For a selfmap  $T$ , there exists  $\beta \in (0, \frac{1}{2})$  such that

$$d(Tx, Ty) \leq \beta [d(x, Tx) + d(y, Ty)], \quad \forall x, y \in E. \quad (1.5)$$

Chatterjea [6] used the following contractive condition. For a selfmap  $T$ , there exists  $\gamma \in (0, \frac{1}{2})$  such that

$$d(Tx, Ty) \leq \gamma [d(x, Ty) + d(y, Tx)], \quad \forall x, y \in E. \quad (1.6)$$

Zamfirescu [38] established a nice generalization of the Banach's fixed point theorem by combining (1.2), (1.5) and (1.6). That is, for a mapping  $T: E \rightarrow E$ , there exist real numbers  $\alpha, \beta, \gamma$  satisfying  $0 \leq \alpha < 1$ ,  $0 \leq \beta < \frac{1}{2}$ ,  $0 \leq \gamma < \frac{1}{2}$  such that for each  $x, y \in E$ , at least one of the following is true:

$$\left. \begin{aligned} (z_1) \quad & d(Tx, Ty) \leq \alpha d(x, y) \\ (z_2) \quad & d(Tx, Ty) \leq \beta [d(x, Tx) + d(y, Ty)] \\ (z_3) \quad & d(Tx, Ty) \leq \gamma [d(x, Ty) + d(y, Tx)]. \end{aligned} \right\} \quad (1.7)$$

The mapping  $T: E \rightarrow E$  satisfying (1.7) is called the *Zamfirescu contraction*. Any mapping satisfying condition  $(z_2)$  of (1.7) is called a *Kannan mapping*, while the mapping satisfying condition  $(z_3)$  is called a *Chatterjea operator*. The contractive condition (1.7) implies

$$d(Tx, Ty) \leq 2\delta d(x, Tx) + \delta d(x, y), \quad \forall x, y \in E, \quad (1.8)$$

where  $\delta = \max \left\{ \alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma} \right\}$ ,  $0 \leq \delta < 1$ .

Rhoades [32,33] used condition (1.7) to obtain some convergence results for Mann and Ishikawa iterative processes in a uniformly convex Banach space, while Berinde [4] extended the results of [32,33] to arbitrary Banach space for the same iterative processes.

The following definition of stability of iterative process is due to Harder and Hicks [11].

DEFINITION 1.1. Let  $(E, d)$  be a complete metric space,  $T: E \rightarrow E$  a selfmap of  $E$ . Suppose that  $F_T = \{p \in E \mid Tp = p\}$  is the set of fixed points of  $T$ . Let  $\{x_n\}_{n=0}^\infty \subset E$  be the sequence generated by an iterative procedure involving  $T$  which is defined by

$$x_{n+1} = f(T, x_n), \quad n = 0, 1, \dots, \quad (1.9)$$

where  $x_0 \in E$  is the initial approximation and  $f$  is some function. Suppose  $\{x_n\}_{n=0}^{\infty}$  converges to a fixed point  $p$  of  $T$ . Let  $\{y_n\}_{n=0}^{\infty} \subset E$  and set  $\epsilon_n = d(y_{n+1}, f(T, y_n))$ ,  $n = 0, 1, \dots$ . Then, the iterative procedure (1.9) is said to be  $T$ -stable or stable with respect to  $T$  if and only if  $\lim_{n \rightarrow \infty} \epsilon_n = 0$  implies  $\lim_{n \rightarrow \infty} y_n = p$ .

Since the metric is induced by the norm, we have

$$\epsilon_n = \|y_{n+1} - f(T, y_n)\|, \quad n = 0, 1, \dots,$$

in place of

$$\epsilon_n = d(y_{n+1}, f(T, y_n)), \quad n = 0, 1, \dots,$$

in the definition of stability whenever we are working in normed linear space or Banach space.

If in (1.9),

$$f(T, x_n) = Tx_n, \quad n = 0, 1, \dots,$$

then we have the Picard iterative process defined in (1.1), while we obtain the Ishikawa iterative process (1.4) from (1.9) if

$$f(T, x_n) = (1 - \alpha_n)x_n + \alpha_n Tz_n, \quad z_n = (1 - \beta_n)x_n + \beta_n Tx_n, \\ n = 0, 1, \dots, \quad \alpha_n, \beta_n \in [0, 1].$$

Several stability results established in metric space and normed linear space are available in the literature. Some of the various authors whose contributions are of great value in the study of stability of the fixed point iterative procedures are Ostrowski [28], Harder and Hicks [11], Rhoades [29,31], Osilike [26], Osilike and Udomene [27], Jachymski [15], Berinde [2,3] and Singh et al. [37]. Harder and Hicks [11], Rhoades [29,31], Osilike [26] and Singh et al. [37] used the method of the summability theory of infinite matrices to prove various stability results for certain contractive definitions. The method has also been adopted to establish various stability results for certain contractive definitions in Olatinwo et al. [23,24]. Osilike and Udomene [27] introduced a shorter method of proof of stability results and this has also been employed by Berinde [2], Imoru and Olatinwo [12], Olatinwo et al. [25] and some others. In Harder and Hicks [11], the contractive definition stated in (1.2) was used to prove a stability result for the Kirk's iterative process. The first stability result on  $T$ -stable mappings was due to Ostrowski [28] where he established the stability of the Picard iteration by using condition (1.2). In addition to (1.2), the contractive condition in (1.9) was also employed by Harder and Hicks [11] to establish some stability results for both Picard and Mann iterative processes. Rhoades [29,31] extended the stability results of [11] to more general classes of contractive mappings. Rhoades [29] extended the results of [11] to the following independent contractive condition: there exists  $c \in [0, 1)$  such that

$$d(Tx, Ty) \leq c \max \{d(x, y), d(x, Ty), d(y, Tx)\}, \quad \forall x, y \in E. \quad (1.10)$$

Rhoades [31] used the following contractive definition: there exists  $c \in [0, 1)$  such that

$$d(Tx, Ty) \leq c \max \left\{ d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, d(x, Ty), d(y, Tx) \right\}, \quad (1.11)$$

$\forall x, y \in E$ .

Moreover, Osilike [26] generalized and extended some of the results of Rhoades [31] by using a more general contractive definition than those of Rhoades [29,31]. Indeed, he employed the following contractive definition: there exist  $a \in [0, 1]$ ,  $L \geq 0$  such that

$$d(Tx, Ty) \leq Ld(x, Tx) + ad(x, y), \quad \forall x, y \in E. \quad (1.12)$$

Osilike and Udomene [27] introduced a shorter method to prove stability results for the various iterative processes using the condition (1.12). Berinde [2] established the same stability results for the same iterative processes using the same set of contractive definitions as in Harder and Hicks [11], but the same method of shorter proof as in Osilike and Udomene [27].

More recently, Imoru and Olatinwo [12] established some stability results which are generalizations of some of the results of [2,11,26,27,29,31]. The following contractive definition was employed: there exist  $a \in [0, 1)$  and a monotone increasing function  $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $\varphi(0) = 0$ , such that

$$d(Tx, Ty) \leq \varphi(d(x, Tx)) + ad(x, y), \quad \forall x, y \in E. \quad (1.13)$$

Condition (1.13) was also employed in Olatinwo et al. [23] to establish some stability results in normed linear space setting with additional condition of continuity imposed on  $\varphi$ .

In the next section, we shall state our new iterative algorithms, contractive definition, some remarks and lemmas which are required in the sequel.

## 2. Preliminaries

We shall introduce and employ the following iterative processes. Let  $E$  be a Banach space,  $T: E \rightarrow E$  a selfmap of  $E$  and  $x_0 \in E$ . Then, define the sequence  $\{x_n\}_{n=0}^\infty$  by

$$\left. \begin{aligned} x_{n+1} &= \alpha_{n,0}x_n + \sum_{i=1}^k \alpha_{n,i}T^i z_n, \quad \sum_{i=0}^k \alpha_{n,i} = 1, \quad n = 0, 1, 2, \dots, \\ z_n &= \sum_{j=0}^s \beta_{n,j}T^j x_n, \quad \sum_{j=0}^s \beta_{n,j} = 1, \end{aligned} \right\} \quad (2.1)$$

$k \geq s$ ,  $\alpha_{n,i} \geq 0$ ,  $\alpha_{n,0} \neq 0$ ,  $\beta_{n,i} \geq 0$ ,  $\beta_{n,0} \neq 0$ ,  $\alpha_{n,i}, \beta_{n,j} \in [0, 1]$ , where  $k$  and  $s$  are fixed integers.

If  $s = 0$  in (2.1), we also obtain the following interesting iterative process in a Banach space:

$$x_{n+1} = \sum_{i=0}^k \alpha_{n,i}T^i x_n, \quad \sum_{i=0}^k \alpha_{n,i} = 1, \quad n = 0, 1, 2, \dots, \quad (2.2)$$

$\alpha_{n,i} \geq 0$ ,  $\alpha_{n,0} \neq 0$ ,  $\alpha_{n,i} \in [0, 1]$ , where  $k$  is a fixed integer.

The iterative process defined in (2.1) will be called the *Kirk-Ishikawa* iterative process, while that of (2.2) will be called the *Kirk-Mann* iterative process.

(i) If  $s = 0, k = 1$  in (2.1), then we have  $z_n = \beta_{n,0}x_n = x_n, \beta_{n,0} = 1$  and  $x_{n+1} = (1 - \alpha_{n,1})x_n + \alpha_{n,1}Tx_n$ , which is the usual Mann iterative process with  $\sum_{i=0}^1 \alpha_{n,i} = 1, \alpha_{n,1} = \alpha_n$ .

(ii) Also, if  $s = k = 1$ , in (2.1), we get

$$\left. \begin{aligned} x_{n+1} &= (1 - \alpha_{n,1})x_n + \alpha_{n,1}Tz_n \\ z_n &= (1 - \beta_{n,1})x_n + \beta_{n,1}Tx_n, \end{aligned} \right\}$$

which is the usual Ishikawa iterative process with  $\sum_{i=0}^1 \alpha_{n,i} = \sum_{i=0}^1 \beta_{n,i} = 1, \alpha_{n,1} = \alpha_n, \beta_{n,1} = \beta_n$ .

(iii) If  $s = 0$  and  $\alpha_{n,i} = \alpha_i$  in (2.1), then we obtain the usual Kirk's iterative process

$$x_{n+1} = \sum_{i=0}^k \alpha_i T^i x_n, \quad \sum_{i=0}^k \alpha_i = 1, \quad n = 0, 1, 2, \dots, \tag{2.3}$$

with  $z_n = \beta_{n,0}x_n = x_n$ , since  $\beta_{n,0} = 1$ .

Equation (2.2) is also a generalization of Picard, Schaefer, Mann and Kirk's iterative processes. See Berinde [3,5] for detail on the various already existing fixed point iterative processes.

We shall employ the following contractive definition. For a selfmap  $T: E \rightarrow E$ , there exist a real number  $a \in [0, 1)$ , and a monotone increasing function  $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with  $\varphi(0) = 0$ , such that  $\forall x, y \in E$ ,

$$\|Tx - Ty\| \leq \varphi(\|x - Tx\|) + a\|x - y\|. \tag{2.4}$$

However, we shall employ the following lemmas in the sequel.

LEMMA 2.1. [2,3] *If  $\delta$  is a real number such that  $0 \leq \delta < 1$ , and  $\{\epsilon_n\}_{n=0}^\infty$  is a sequence of positive numbers such that  $\lim_{n \rightarrow \infty} \epsilon_n = 0$ , then for any sequence of positive numbers  $\{u_n\}_{n=0}^\infty$  satisfying*

$$u_{n+1} \leq \delta u_n + \epsilon_n, \quad n = 0, 1, \dots,$$

*we have  $\lim_{n \rightarrow \infty} u_n = 0$ .*

LEMMA 2.2. [25] *Let  $(E, \|\cdot\|)$  be a normed linear space and let  $T: E \rightarrow E$  be a selfmap of  $E$  satisfying (2.5). Let  $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a subadditive, monotone increasing function such that  $\varphi(0) = 0, \varphi(Lu) \leq L\varphi(u), L \geq 0, u \in \mathbb{R}_+$ . Then,  $\forall i \in \mathbb{N}$  and  $\forall x, y \in E$ ,*

$$\|T^i x - T^i y\| \leq \sum_{j=1}^i \binom{i}{j} a^{i-j} \varphi^j(\|x - Tx\|) + a^i \|x - y\|.$$

It is our purpose in this paper to prove some stability results for the iterative processes defined in (2.1) and (2.2). Our results are improvements, generalizations and/or extensions of some of the results of Harder and Hicks [11], Rhoades [29,30], Osilike [26], Osilike and Udomene [27], Berinde [2,3] as well as the recent results of the author [23,24,25].

### 3. Main Results

**THEOREM 3.1.** *Let  $(E, \|\cdot\|)$  be a normed linear space and  $T : E \rightarrow E$  a selfmap of  $E$  satisfying the contractive condition (2.4) and  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , a subadditive monotone increasing function such that  $\varphi(0) = 0$ . Let  $x_0 \in E$ , and  $\{x_n\}_{n=0}^\infty$  be the Kirk-Ishikawa iterative process defined by (2.1). Suppose that  $T$  has a fixed point  $p$ . Then, the Kirk-Ishikawa iterative process is  $T$ -stable.*

*Proof.* Let  $\{y_n\}_{n=0}^\infty \subset E$ ,  $\epsilon_n = \|y_{n+1} - \alpha_{n,0}y_n - \sum_{i=1}^k \alpha_{n,i}T^i b_n\|$  and  $b_n = \sum_{r=0}^s \beta_{n,r}T^r y_n$ . Suppose that  $\lim_{n \rightarrow \infty} \epsilon_n = 0$ . Then, we shall employ Lemma 2.2 and the triangle inequality to establish that  $\lim_{n \rightarrow \infty} y_n = p$ . Let  $(\sum_{i=1}^k \alpha_{n,i}a^i)(\sum_{r=0}^s \beta_{n,r}a^r) + \alpha_{n,0} \leq \delta$  with  $0 \leq \delta < 1$ . Then,

$$\begin{aligned}
\|y_{n+1} - p\| &\leq \|y_{n+1} - \alpha_{n,0}y_n - \sum_{i=1}^k \alpha_{n,i}T^i b_n\| + \|\alpha_{n,0}y_n + \sum_{i=1}^k \alpha_{n,i}T^i b_n - p\| \\
&= \|\alpha_{n,0}y_n + \sum_{i=1}^k \alpha_{n,i}T^i b_n - \sum_{i=0}^k \alpha_{n,i}T^i p\| + \epsilon_n \\
&= \left\| \sum_{i=1}^k \alpha_{n,i}(T^i b_n - T^i p) + \alpha_{n,0}(y_n - p) \right\| + \epsilon_n \\
&\leq \sum_{i=1}^k \alpha_{n,i} \|T^i p - T^i b_n\| + \alpha_{n,0} \|y_n - p\| + \epsilon_n \\
&\leq \sum_{i=1}^k \alpha_{n,i} \left\{ \sum_{j=1}^i \binom{i}{j} a^{i-j} \varphi^j(\|p - Tp\|) + a^i \|p - b_n\| \right\} + \alpha_{n,0} \|y_n - p\| + \epsilon_n \\
&= \sum_{i=1}^k \alpha_{n,i} \left\{ \sum_{j=1}^i \binom{i}{j} a^{i-j} \varphi^j(0) + a^i \|p - b_n\| \right\} + \alpha_{n,0} \|y_n - p\| + \epsilon_n \\
&= \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \|p - b_n\| + \alpha_{n,0} \|y_n - p\| + \epsilon_n \\
&= \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \left\| \sum_{r=0}^s \beta_{n,r} T^r p - \sum_{r=0}^s \beta_{n,r} T^r y_n \right\| + \alpha_{n,0} \|y_n - p\| + \epsilon_n \\
&= \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \left\| \sum_{r=0}^s \beta_{n,r} (T^r p - T^r y_n) \right\| + \alpha_{n,0} \|y_n - p\| + \epsilon_n \\
&= \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \left\{ \left\| \sum_{r=1}^s \beta_{n,r} (T^r p - T^r y_n) + \beta_{n,0} (p - y_n) \right\| \right\} + \alpha_{n,0} \|y_n - p\| + \epsilon_n \\
&\leq \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \left\{ \sum_{r=1}^s \beta_{n,r} \|T^r p - T^r y_n\| + \beta_{n,0} \|y_n - p\| \right\} + \alpha_{n,0} \|y_n - p\| + \epsilon_n \\
&= \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \sum_{r=1}^s \beta_{n,r} \|T^r p - T^r y_n\| + \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \beta_{n,0} \|y_n - p\| \\
&\quad + \alpha_{n,0} \|y_n - p\| + \epsilon_n \\
&\leq \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \sum_{r=1}^s \beta_{n,r} \left[ \sum_{j=1}^r \binom{r}{j} a^{r-j} \varphi^j(\|p - Tp\|) + a^r \|p - y_n\| \right]
\end{aligned}$$

$$\begin{aligned}
 & + \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \beta_{n,0} \|y_n - p\| + \alpha_{n,0} \|y_n - p\| + \epsilon_n \\
 = & \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \sum_{r=1}^s \beta_{n,r} \left[ \sum_{j=1}^r \binom{r}{j} a^{r-j} \varphi^j(0) + a^r \|y_n - p\| \right] \\
 & + \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \beta_{n,0} \|y_n - p\| + \alpha_{n,0} \|y_n - p\| + \epsilon_n \\
 = & \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \left( \sum_{r=1}^s \beta_{n,r} a^r \right) \|y_n - p\| + \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \beta_{n,0} \|y_n - p\| \\
 & + \alpha_{n,0} \|y_n - p\| + \epsilon_n \\
 = & \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \left\{ \sum_{r=1}^s \beta_{n,r} a^r + \beta_{n,0} \right\} \|y_n - p\| + \alpha_{n,0} \|y_n - p\| + \epsilon_n \\
 = & \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \left( \sum_{r=0}^s \beta_{n,r} a^r \right) \|y_n - p\| + \alpha_{n,0} \|y_n - p\| + \epsilon_n \\
 = & \left[ \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \left( \sum_{r=0}^s \beta_{n,r} a^r \right) + \alpha_{n,0} \right] \|y_n - p\| + \epsilon_n \\
 \leq & \delta \|y_n - p\| + \epsilon_n. \tag{3.1}
 \end{aligned}$$

Since  $0 \leq \delta < 1$ , using Lemma 2.1 in (3.1) yields  $\lim_{n \rightarrow \infty} y_n = p$ .

Conversely, let  $\lim_{n \rightarrow \infty} y_n = p$ . Then, we shall show that  $\lim_{n \rightarrow \infty} \epsilon_n = 0$ , using Lemma 2.2 and the triangle inequality as follows:

$$\begin{aligned}
 \epsilon_n & = \|y_{n+1} - \alpha_{n,0} y_n - \sum_{i=1}^k \alpha_{n,i} T^i b_n\| \\
 & \leq \|y_{n+1} - p\| + \|p - \alpha_{n,0} y_n - \sum_{i=1}^k \alpha_{n,i} T^i b_n\| \\
 & = \|y_{n+1} - p\| + \left\| \sum_{i=0}^k \alpha_{n,i} T^i p - \alpha_{n,0} y_n - \sum_{i=1}^k \alpha_{n,i} T^i b_n \right\| \\
 & = \|y_{n+1} - p\| + \left\| \sum_{i=1}^k \alpha_{n,i} (T^i p - T^i b_n) + \alpha_{n,0} (p - y_n) \right\| \\
 & \leq \|y_{n+1} - p\| + \sum_{i=1}^k \alpha_{n,i} \|T^i p - T^i b_n\| + \alpha_{n,0} \|y_n - p\| \\
 & \leq \|y_{n+1} - p\| + \sum_{i=1}^k \alpha_{n,i} \left\{ \sum_{j=1}^i \binom{i}{j} a^{i-j} \varphi^j(\|p - T p\|) + a^i \|p - b_n\| \right\} + \alpha_{n,0} \|y_n - p\| \\
 & = \|y_{n+1} - p\| + \sum_{i=1}^k \alpha_{n,i} \left\{ \sum_{j=1}^i \binom{i}{j} a^{i-j} \varphi^j(0) + a^i \|p - b_n\| \right\} + \alpha_{n,0} \|y_n - p\| \\
 & = \|y_{n+1} - p\| + \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \|p - b_n\| + \alpha_{n,0} \|y_n - p\| \\
 & = \|y_{n+1} - p\| + \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \left\| \sum_{r=0}^s \beta_{n,r} T^r p - \sum_{r=0}^s \beta_{n,r} T^r y_n \right\| + \alpha_{n,0} \|y_n - p\|
 \end{aligned}$$

$$\begin{aligned}
&= \|y_{n+1} - p\| + \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \left\| \sum_{r=0}^s \beta_{n,r} (T^r p - T^r y_n) \right\| + \alpha_{n,0} \|y_n - p\| \\
&= \|y_{n+1} - p\| + \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \left\{ \left\| \sum_{r=1}^s \beta_{n,r} (T^r p - T^r y_n) + \beta_{n,0} (p - y_n) \right\| \right\} \\
&\quad + \alpha_{n,0} \|y_n - p\| \\
&\leq \|y_{n+1} - p\| + \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \left\{ \sum_{r=1}^s \beta_{n,r} \|T^r p - T^r y_n\| + \beta_{n,0} \|y_n - p\| \right\} \\
&\quad + \alpha_{n,0} \|y_n - p\| \\
&= \|y_{n+1} - p\| + \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \sum_{r=1}^s \beta_{n,r} \|T^r p - T^r y_n\| \\
&\quad + \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \beta_{n,0} \|y_n - p\| + \alpha_{n,0} \|y_n - p\| \\
&\leq \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \sum_{r=1}^s \beta_{n,r} \left[ \sum_{j=1}^r \binom{r}{j} a^{r-j} \varphi^j(\|p - T p\|) + a^r \|p - y_n\| \right] \\
&\quad + \|y_{n+1} - p\| + \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \beta_{n,0} \|y_n - p\| + \alpha_{n,0} \|y_n - p\| \\
&= \|y_{n+1} - p\| + \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \sum_{r=1}^s \beta_{n,r} \left[ \sum_{j=1}^r \binom{r}{j} a^{r-j} \varphi^j(0) + a^r \|y_n - p\| \right] \\
&\quad + \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \beta_{n,0} \|y_n - p\| + \alpha_{n,0} \|y_n - p\| \\
&= \|y_{n+1} - p\| + \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \left( \sum_{r=1}^s \beta_{n,r} a^r \right) \|y_n - p\| \\
&\quad + \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \beta_{n,0} \|y_n - p\| + \alpha_{n,0} \|y_n - p\| \\
&= \|y_{n+1} - p\| + \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \left\{ \sum_{r=1}^s \beta_{n,r} a^r + \beta_{n,0} \right\} \|y_n - p\| + \alpha_{n,0} \|y_n - p\| \\
&= \|y_{n+1} - p\| + \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \left( \sum_{r=0}^s \beta_{n,r} a^r \right) \|y_n - p\| + \alpha_{n,0} \|y_n - p\| \\
&= \|y_{n+1} - p\| + \left[ \left( \sum_{i=1}^k \alpha_{n,i} a^i \right) \left( \sum_{r=0}^s \beta_{n,r} a^r \right) + \alpha_{n,0} \right] \|y_n - p\| \\
&\leq \|y_{n+1} - p\| + \delta \|y_n - p\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad \blacksquare
\end{aligned}$$

**THEOREM 3.2.** *Let  $(E, \|\cdot\|)$  be a normed linear space and  $T: E \rightarrow E$  a selfmap of  $E$  satisfying the contractive condition (2.4) and  $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  a subadditive monotone increasing function such that  $\varphi(0) = 0$ . Let  $x_0 \in E$ , and  $\{x_n\}_{n=0}^\infty$  be the Kirk-Mann iterative process defined by (2.2). Suppose that  $T$  has a fixed point  $p$ . Then, the Kirk-Mann iterative process is  $T$ -stable.*

*Proof.* The proof of this theorem is similar to that of Theorem 3.1.  $\blacksquare$



REMARK 3.3. Theorems 3.1 is a generalization and extension of Theorem 2 of Osilike [26], Theorem 2 and Theorem 5 of Osilike and Udomene [27] as well as Theorem 3 of Olatinwo et al. [24]. Theorem 3.2 is a generalization of Theorem 2 of Rhoades [29,31], Theorem 3 of Berinde [2] as well as Theorem 3.2 of Imoru and Olatinwo [12].

## REFERENCES

- [1] S. Banach, *Sur les operations dans les ensembles abstraits et leur applications aux equations integrales*, Fund. Math. **3** (1922), 133–181.
- [2] V. Berinde, *On the stability of some fixed point procedures*, Bul. Stiint. Univ. Baia Mare, Ser. B, Matematica-Informatica, **38**(1) (2002), 7–14.
- [3] V. Berinde, *Iterative Approximation of Fixed Points*, Editura Efemeride, 2002.
- [4] V. Berinde, *On the convergence of the Ishikawa iteration in the class of quasi-contractive operators*, Acta Math. Univ. Comenianae **73** (1) (2004), 119–126.
- [5] V. Berinde, *Iterative Approximation of Fixed Points*, Springer-Verlag, Berlin Heidelberg, 2007.
- [6] S. K. Chatterjea, *Fixed-point theorems*, C. R. Acad. Bulgare Sci. **10** (1972), 727–730.
- [7] C. E. Chidume, *Geometric Properties of Banach Spaces and Nonlinear Iterations*, The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, (October 2000).
- [8] Lj. B. Ćirić, *Generalized contractions and fixed point theorems*, Publ. Inst. Math. (Beograd) (N. S.) **12** (26) (1971), 19–26.
- [9] Lj. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **45** (1974), 267–273.
- [10] C. W. Groetsch, *A note on segmenting Mann iterates*, J. Math. Anal. Appl. **40** (1972), 369–372.
- [11] A. M. Harder and T. L. Hicks, *Stability results for fixed point iteration procedures*, Math. Japonica **33** (5) (1988), 693–706.
- [12] C. O. Imoru and M. O. Olatinwo, *On the stability of Picard and Mann iteration processes*, Carp. J. Math. **19** (2) (2003), 155–160.
- [13] C. O. Imoru, M. O. Olatinwo, and O. O. Owojori, *On the stability results for Picard and Mann iteration procedures*, J. Appl. Func. Diff. Eqns. **1** (1), 71–80.
- [14] S. Ishikawa, *Fixed point by a new iteration method*, Proc. Amer. Math. Soc. **44** (1) (1974), 147–150.
- [15] J. R. Jachymski, *An extension of A. Ostrowski's theorem on the round-off stability of iterations*, Aequ. Math. **53** (1997), 242–253.
- [16] R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc. **10** (1968), 71–76.
- [17] L. Liu, *Fixed points of local strictly pseudo-contractive mappings using Mann and Ishikawa iteration with errors*, Indian J. Pure Appl. Math. **26** (7) (1995), 649–659.
- [18] L. Liu, *Ishikawa and Mann iteration processes with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl. **194** (1995), 114–125.
- [19] G. Liu, D. Lei and S. Li, *Approximating fixed points of nonexpansive mappings*, Internat. J. Math. & Math. Sci. **24** (3) (2000), 173–177.
- [20] M. Maiti and B. Saha., *Approximating fixed points of nonexpansive and generalized nonexpansive mappings*, Internat. J. Math. & Math. Sci. **16** (1) (1993), 81–86.
- [21] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **44** (1953), 506–510.
- [22] M. O. Olatinwo, *Some stability and strong convergence results for the Jungck-Ishikawa iteration process*, Creative Math. & Inf., **17** (2008), 33–42.
- [23] M. O. Olatinwo, O. Owojori and C. O. Imoru, *On Some stability results for fixed point iteration procedure*, J. Math. Stat. **2** (1) (2006), 339–342.

- [24] M. O. Olatinwo, O. O. Owajori and C. O. Imoru, *Some stability results on Krasnoselskij and Ishikawa fixed point iteration procedures*, J. Math. Stat. **2** (1) (2006), 360–362.
- [25] M. O. Olatinwo, O. O. Owajori and C. O. Imoru, *Some stability results for fixed point iteration processes*, Aus. J. Math. Anal. Appl., **3** (2) (2006), 1–7.
- [26] M. O. Osilike, *Some stability results for fixed point iteration procedures*, J. Nigerian Math.Soc., **14/15** (1995), 17–29.
- [27] M. O. Osilike and A. Udomene, *Short proofs of stability results for fixed point iteration procedures for a class of contractive-type mappings*, Indian J. Pure Appl. Math. **30** (12) (1999), 1229–1234.
- [28] A. M. Ostrowski, *The round-off stability of iterations*, Z. Angew. Math. Mech. **47** (1967), 77–81.
- [28] B. E. Rhoades, *Fixed point theorems and stability results for fixed point iteration procedures*, Indian J. Pure Appl. Math. **21** (1) (1990), 1–9.
- [30] B. E. Rhoades, *Some fixed point iteration procedures*, Int. J. Math. Math. Sci., **14** (1) (1991), 1–16.
- [31] B. E. Rhoades, *Fixed point theorems and stability results for fixed point iteration procedures, II*, Indian J. Pure Appl. Math. **24** (11) (1993), 691–703.
- [32] B. E. Rhoades, *Fixed point iteration using infinite matrices*, Trans. Amer. Math. Soc. **196** (1974), 161–76.
- [33] B. E. Rhoades, *Comments on two fixed point iteration methods*, J. Math. Anal. Appl. **56** (2) (1976), 741–750.
- [34] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. **226** (1977), 257–290.
- [35] I. A. Rus, *Generalized Contractions and Applications*, Cluj Univ. Press, Cluj Napoca, 2001.
- [36] I. A. Rus, A. Petrusel and G. Petrusel, *Fixed Point Theory, 1950–2000*, Romanian Contributions, House of the Book of Science, Cluj Napoca, 2002.
- [37] S. L. Singh, C. Bhatnagar and S. N. Mishra, *Stability of Jungck-type iterative procedures*, Internatioal J. Math. & Math. Sc. **19** (2005), 3035–3043.
- [38] T. Zamfirescu, *Fixed point theorems in metric spaces*, Arch. Math. **23** (1972), 292–298.
- [39] E. Zeidler, *Nonlinear Functional Analysis and its Applications. Fixed-Point Theorems I*, Springer-Verlag New York, Inc., 1986.

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Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria

E-mail: polatinwo@oauife.edu.ng, molaposi@yahoo.com