

ON WEAKER FORMS OF MENGER, ROTHBERGER AND HUREWICZ PROPERTIES

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Abstract. We introduce new star selection principles defined by neighbourhoods and stars which are weaker versions of the of Menger, Rothberger and Hurewicz properties; in particular the properties introduced are between strong star versions and star versions of the corresponding properties defined in [12]. Some properties of these neighbourhood star selection principles are proved and some examples are given.

1. Introduction and definitions

Our notation and terminology are standard as in [6].

Recall the following two classical selection principles, which we consider only for families of open covers of a topological space.

Let \mathcal{A} and \mathcal{B} be families of open covers of a topological space X . Then (see [20], [11]):

$S_1(\mathcal{A}, \mathcal{B})$ (*the Rothberger-type principle*) denotes the selection hypothesis:

For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of \mathcal{A} there is a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \mathbb{N}\} \in \mathcal{B}$.

$S_{fin}(\mathcal{A}, \mathcal{B})$ (*the Menger-type principle*) denotes the selection hypothesis:

For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of \mathcal{A} there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, \mathcal{V}_n is a finite subset of \mathcal{U}_n and $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ is an element of \mathcal{B} .

As usual, for a subset A of a space X and a collection \mathcal{P} of subsets of X , we denote by $\text{St}(A, \mathcal{P}) = \cup\{P \in \mathcal{P} : A \cap P \neq \emptyset\}$ the star of A with respect to \mathcal{P} . In [12], Kočinac introduced star selection principles in the following way:

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DEFINITION 1.1. Let \mathcal{A} and \mathcal{B} be collections of open covers of a space X .

- The symbol $S_1^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of elements of \mathcal{A} one can choose $U_n \in \mathcal{U}_n, n \in \mathbb{N}$, so that $\{\text{St}(U_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$;
- The symbol $S_{fin}^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of elements of \mathcal{A} one can choose finite $\mathcal{V}_n \subset \mathcal{U}_n, n \in \mathbb{N}$, so that $\cup_{n \in \mathbb{N}} \{\text{St}(V, \mathcal{U}_n) : V \in \mathcal{V}_n\} \in \mathcal{B}$;
- The symbol $SS_1^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of elements of \mathcal{A} one can choose $x_n \in X, n \in \mathbb{N}$, so that $\{\text{St}(\{x_n\}, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$;
- The symbol $SS_{fin}^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of elements of \mathcal{A} one can choose finite $A_n \subset X, n \in \mathbb{N}$, $\{\text{St}(A_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

For a space X we use the following notation:

- \mathcal{O} denotes the collection of all open covers of X .
- Ω denotes the collection of all ω -covers of X ; an open cover \mathcal{U} of X is an ω -cover [9] if every finite subset of X is contained in a member of \mathcal{U} .
- Γ denotes the collection of all γ -covers of X ; an open cover \mathcal{U} of X is a γ -cover [9] if it is infinite and each $x \in X$ belongs to all but finitely many elements of \mathcal{U} .

DEFINITION 1.2. A space X is:

- R (*Rothberger*) if the selection hypothesis $S_1(\mathcal{O}, \mathcal{O})$ is true for X ([19], [8], [20]);
- M (*Menger*) if the selection hypothesis $S_{fin}(\mathcal{O}, \mathcal{O})$ is true for X ([17], [10], [8], [11]; Menger property was called Hurewicz in [1] and [15]);
- H (*Hurewicz*) if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite sets such that for each $n \in \mathbb{N}$, $\mathcal{V}_n \subset \mathcal{U}_n$ and for each $x \in X$, $x \in \cup \mathcal{V}_n$ for all but finitely many n ([10]; see the observation on the Menger property);
- SR (*star-Rothberger*) if the selection hypothesis $S_1^*(\mathcal{O}, \mathcal{O})$ is true for X ([12]);
- SSR (*strongly star-Rothberger*) if the selection hypothesis $SS_1^*(\mathcal{O}, \mathcal{O})$ is true for X ([12]);
- SM (*star-Menger*) the selection hypothesis $S_{fin}^*(\mathcal{O}, \mathcal{O})$ is true for X ([12]);
- SSM (*strongly star-Menger*) if the selection hypothesis $SS_{fin}^*(\mathcal{O}, \mathcal{O})$ is true for X ([12]);
- SH (*star-Hurewicz*) if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers one can choose finite $\mathcal{V}_n \subset \mathcal{U}_n, n \in \mathbb{N}$, so that for every $x \in X$, $x \in \text{St}(\cup \mathcal{V}_n, \mathcal{U}_n)$ for all but finitely many n ([3]);
- SSH (*strongly star-Hurewicz*) is the selection hypothesis $SS_{fin}^*(\mathcal{O}, \Gamma)$ is true for X ([3]).

We introduce the following definitions.

DEFINITION 1.3. Let \mathcal{A} and \mathcal{B} be collections of open covers of a space X . A space X satisfies:

NSR(\mathcal{A}, \mathcal{B}) if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of elements of \mathcal{A} one can choose $x_n \in X$, $n \in \mathbb{N}$, so that for every open $O_n \ni x_n$, $n \in \mathbb{N}$, $\{\text{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$;

NSM(\mathcal{A}, \mathcal{B}) if for every sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of elements of \mathcal{A} one can choose finite $A_n \subset X$, $n \in \mathbb{N}$, so that for every open $O_n \supset A_n$, $n \in \mathbb{N}$, $\{\text{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

In particular we give the following definitions:

DEFINITION 1.4. A space X is:

NSR: (*neighbourhood star-Rothberger*) if the selection hypothesis NSR(\mathcal{O}, \mathcal{O}) is true for X ;

NSM: (*neighbourhood star-Menger*) if the selection hypothesis NSM(\mathcal{O}, \mathcal{O}) is true for X ;

NSH: (*neighbourhood star-Hurewicz*) if the selection hypothesis NSM(\mathcal{O}, Γ) is true for X .

NOTE. NSR and NSM spaces were considered in [13] under different names (nearly strongly star-Rothberger and nearly strongly star-Menger spaces).

The following is straightforward:

PROPOSITION 1.5. For a space X the following hold:

1. X is SR iff for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exist $O_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that for every $x \in X$ there exists $n \in \mathbb{N}$ such that $\text{St}(\{x\}, \mathcal{U}_n) \cap O_n \neq \emptyset$.
2. X is SSR iff for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(x_n : n \in \mathbb{N})$ of points of X such that for every $x \in X$ there exists $n \in \mathbb{N}$ such that $\text{St}(\{x\}, \mathcal{U}_n) \ni x_n$.
3. X is NSR iff for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(x_n : n \in \mathbb{N})$ of points of X such that for every $x \in X$ there exists $n \in \mathbb{N}$ such that $\overline{\text{St}(\{x\}, \mathcal{U}_n)} \ni x_n$.
4. X is SM iff for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exist finite $\mathcal{O}_n \subset \mathcal{U}_n$ ($n \in \mathbb{N}$) such that for every $x \in X$ there exists $n \in \mathbb{N}$ such that $\text{St}(\{x\}, \mathcal{U}_n) \cap (\cup \mathcal{O}_n) \neq \emptyset$.
5. X is SSM iff for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that for every $x \in X$ there exists $n \in \mathbb{N}$ such that $\text{St}(\{x\}, \mathcal{U}_n) \cap A_n \neq \emptyset$.
6. X is NSM iff for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that for every $x \in X$ there exists $n \in \mathbb{N}$ such that $\overline{\text{St}(\{x\}, \mathcal{U}_n)} \cap A_n \neq \emptyset$.

7. X is SH iff for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exist finite $\mathcal{O}_n \subset \mathcal{U}_n$ ($n \in \mathbb{N}$) such that for every $x \in X$, $\text{St}(\{x\}, \mathcal{U}_n) \cap (\cup \mathcal{O}_n) \neq \emptyset$, for all but finitely many $n \in \mathbb{N}$.
8. X is SSH iff for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that for every $x \in X$, $\text{St}(\{x\}, \mathcal{U}_n) \cap A_n \neq \emptyset$, for all but finitely many $n \in \mathbb{N}$.
9. X is NSH iff for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that for every $x \in X$, $\overline{\text{St}(\{x\}, \mathcal{U}_n)} \cap A_n \neq \emptyset$, for all but finitely many $n \in \mathbb{N}$.

With the following result we study the NSM property in finite powers of spaces.

PROPOSITION 1.6. *If all finite powers of a space X are NSM, then X satisfies NSM(\mathcal{O}, Ω).*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X and let $\mathbb{N} = N_1 \cup N_2 \cup \dots$ be a partition of \mathbb{N} into infinite (pairwise disjoint) sets. For every $k \in \mathbb{N}$ and every $m \in N_k$ let $\mathcal{W}_m = (\mathcal{U}_m)^k$. Then $(\mathcal{W}_m : m \in N_k)$ is a sequence of open covers of X^k . Applying to this sequence the fact that X^k is NSM we find a sequence $(A_m : m \in N_k)$ of finite subsets of X^k such that for every sequence $(O_m(A_m) : m \in N_k)$ of neighborhoods of A_m , $m \in N_k$, in X^k , the family $\{\text{St}(O_m, \mathcal{W}_m) : m \in N_k\}$ is an open cover of X^k . For every $m \in N_k$, let S_m be a finite subset of X such that $S_m^k \supset A_m$. Consider the sequence of all S_m , $m \in N_k$, $k \in \mathbb{N}$, chosen in this way and denote it $(S_n : n \in \mathbb{N})$. Let $(G_n(S_n) : n \in \mathbb{N})$ be a sequence of neighborhoods of S_n , $n \in \mathbb{N}$. We claim that $\{\text{St}(G_n(S_n), \mathcal{U}_n) : n \in \mathbb{N}\}$ is an ω -cover of X . Let $F = \{x_1, \dots, x_p\}$ be a finite subset of X . Then $\langle x_1, \dots, x_p \rangle \in X^p$. There exists $n \in N_p$ such that $\langle x_1, \dots, x_p \rangle \in \text{St}((G_n(S_n))^p, \mathcal{W}_n)$, so that we have $F \subset \text{St}(G_n(S_n), \mathcal{U}_n)$. ■

2. Basic relations

Recall that (see [5], [7] or [16]) a space X is *strongly star-compact* (resp., *strongly star-Lindelöf*), briefly SSC (resp., SSL), if for every open cover \mathcal{U} of X there exists a finite (resp., countable) subset $A \subset X$ such that $\text{St}(A, \mathcal{U}) = X$; X is *star-compact* (resp., *star-Lindelöf*), briefly SC (resp., SL), if for every open cover \mathcal{U} of X there exists a finite (resp., countable) subset $\mathcal{V} \subset \mathcal{U}$ such that $\text{St}(\cup \mathcal{V}, \mathcal{U}) = X$.

It is natural in this context to introduce the following definition; it also will be useful later.

DEFINITION 2.1. A space X is NSL (*neighbourhood star-Lindelöf*) if for every open cover \mathcal{U} of X there exists a countable subset $A \subset X$ such that for every neighbourhood U of A , $\text{St}(U, \mathcal{U}) = X$.

It is easy to prove the following two propositions:

PROPOSITION 2.2. *X is NSL iff for every open cover \mathcal{U} there is a countable $A \subset X$ such that for every $x \in X$, $\overline{\text{St}(\{x\}, \mathcal{U})} \cap A \neq \emptyset$.*

PROPOSITION 2.3. *A NSL space X having the following property:*

- (*) *for every open cover \mathcal{U} there is an open cover \mathcal{V} such that for every $x \in X$,*

$$\overline{St(\{x\}, \mathcal{V})} \subset St(\{x\}, \mathcal{U})$$

is star Lindelöf.

NSL property is the countable version of weak star-compactness studied in [4]; in [4] it is proved that weak star-compactness is equivalent to SSC in the class of Urysohn spaces (and thus it is equivalent to countable compactness). Such equivalence is not true in the Lindelöf case as the following example shows. Note that in the view to present an example of an NSM not SSM space (see Example 3.1), under the assumption “ $\omega_1 < \mathfrak{d}$ ” we give a space which is NSM (hence NSL) which is not SSL.

EXAMPLE 2.4. A Urysohn NSL space which is not SSL.

Consider $X = \mathbb{P} \times (\omega + 1)$, where \mathbb{P} denotes set of irrational points. Denote \mathcal{T} the standard Tychonoff topology on X . Define a finer topology \mathcal{T}' on X in which a basic neighborhood of a point $\langle x, n \rangle$, where $x \in \mathbb{P}$ and $n \in \omega$, takes the form $(U \setminus A) \times \{n\}$ where U is a neighborhood of x in \mathbb{P} with the standard topology, and A a countable subset of \mathbb{P} not containing x ; a point $\langle x, \omega \rangle$, where $x \in \mathbb{P}$, has basic neighbourhoods of the form $(U \setminus A) \times (n, \omega) \cup \langle x, \omega \rangle$ where U is a neighborhood of x in \mathbb{P} with the standard topology, and A a countable subset of \mathbb{P} .

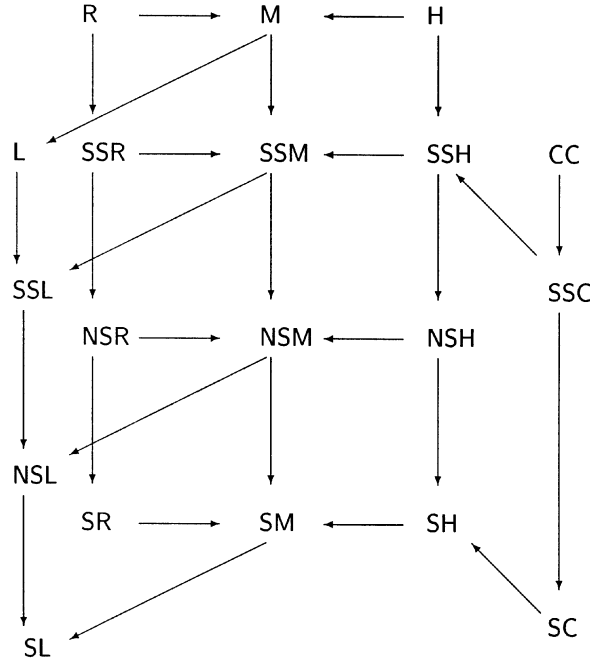
The space (X, \mathcal{T}') is Urysohn since $\mathcal{T}' \supset \mathcal{T}$.

To see that (X, \mathcal{T}') is NSL we notice even more. Since (X, \mathcal{T}) is separable and $Y = \mathbb{P} \times \omega$ is an open subset of it, we have that $(Y, \mathcal{T}|_Y)$ is separable. Let A be countable dense subset of $(Y, \mathcal{T}|_Y)$. It is easy to check that every open neighbourhood of A in topology \mathcal{T}' is dense in (X, \mathcal{T}') .

To see that (X, \mathcal{T}') is not SSL, enumerate all countable subsets of \mathbb{P} as $\{B_\alpha : \alpha < \mathfrak{c}\}$ and represent $\mathbb{P} \times \{\omega\}$ as $\mathbb{P} \times \{\omega\} = \bigcup \{Y_\alpha : \alpha < \mathfrak{c}\}$ where the sets Y_α are pairwise disjoint and all have cardinality \mathfrak{c} . For $z = \langle y, \omega \rangle \in Y_\alpha$, set $U_z = ((\mathbb{P} \setminus B_\alpha) \times \omega) \cup \{z\}$. Then the open cover $\mathcal{U} = \{\mathbb{P} \times \omega\} \cup \{U_z : z \in \mathbb{P} \times \{\omega\}\}$ witness that (X, \mathcal{T}') is not SSL. Indeed let C be a countable subset of X . We have that $C \cap (\mathbb{P} \times \omega) \subset B_\alpha \times \{\omega\}$, for some $\alpha \in \omega$. Since C is countable and Y_α is uncountable, there exists $z = \langle y, \omega \rangle \in Y_\alpha \setminus C$; since the only element of \mathcal{U} containing z is U_z , we have that $z \notin St(C, \mathcal{U})$. \diamond

The implications in the diagram on the next page are obvious (in the diagram CC and L are used to denote countable compactness and the Lindelöf property, respectively).

REMARK 2.5. Since in the class of paracompact Hausdorff we have that $R \Leftrightarrow SR$, $M \Leftrightarrow SM$ (see [12]) and $H \Leftrightarrow SH$ (see [3]), we have that in the class of paracompact Hausdorff spaces all Rothberger-type properties, all Menger-type properties and all Hurewicz-type properties considered are equivalent respectively. Also recall that in the class of paracompact Hausdorff spaces $L \Leftrightarrow SL$ (see [5]). Note that Example 2.4 is not NSM because it contains a copy of \mathbb{P} as a clopen subset and \mathbb{P} (with the standard topology and therefore in the finer topology) is not Menger.



We will present examples showing that the implications $SSM \Rightarrow NSM$, $SSH \Rightarrow NSH$, $SSR \Rightarrow NSR$, $NSR \Rightarrow SR$, $NSM \Rightarrow SM$ and $NSH \Rightarrow SH$ cannot be inverted. Note that the example of an NSM not SSM space also gives an example of an NSM (hence NSL) space which is not SSL ; also the example of a SM not NSM space gives an example of a SL not NSL space.

3. Some examples

Now we show that consistently, NSM , NSH and NSR do not imply SSM , SSH and SSR , respectively. In fact, the examples are not even SSL .

Recall first the definition of \mathfrak{b} , \mathfrak{d} and $cov(\mathcal{M})$. For $f, g \in \mathbb{N}^{\mathbb{N}}$ put

$$f \leq^* g \text{ if } f(n) \leq g(n) \text{ for all but finitely many } n.$$

A subset B of $\mathbb{N}^{\mathbb{N}}$ is *bounded* if there is $g \in \mathbb{N}^{\mathbb{N}}$ such that $f \leq^* g$ for each $f \in B$. $D \subset \mathbb{N}^{\mathbb{N}}$ is *dominating* if for each $g \in \mathbb{N}^{\mathbb{N}}$ there is $f \in D$ such that $g \leq^* f$. The minimal cardinality of an unbounded subset of $\mathbb{N}^{\mathbb{N}}$ is denoted by \mathfrak{b} , and the minimal cardinality of a dominating subset of $\mathbb{N}^{\mathbb{N}}$ is denoted by \mathfrak{d} . A subset X of $\mathbb{N}^{\mathbb{N}}$ can be *guessed* by a function $g \in \mathbb{N}^{\mathbb{N}}$ if for every $f \in X$ the set $\{n \in \mathbb{N} : f(n) = g(n)\}$ is infinite. The minimal cardinality of a subset of $\mathbb{N}^{\mathbb{N}}$ that cannot be guessed is denoted by $cov(\mathcal{M})$ (see [18]).

EXAMPLE 3.1. ($\omega_1 < \mathfrak{d}$) There is a Urysohn NSM space which is not SSL .

EXAMPLE 3.2. ($\omega_1 < \mathfrak{b}$) There is a Urysohn NSH space which is not SSL .

EXAMPLE 3.3. ($\omega_1 < \text{cov}(\mathcal{M})$) There is a Urysohn NSR space which is not SSL.

The space in the three examples is the same. The construction does not depend on cardinality assumptions. Assumptions $\omega_1 < \mathfrak{d}$, $\omega_1 < \mathfrak{b}$ and $\omega_1 < \text{cov}(\mathcal{M})$ are used only in the proof of the properties.

Let S be a subset of \mathbb{R} such that for every non-empty open $U \subset \mathbb{R}$, $|S \cap U| = \omega_1$ (then in particular, $|S| = \omega_1$). Consider $X_S = S \times (\omega + 1)$ topologized as follows: a basic neighbourhood of a point $\langle x, n \rangle$, where $x \in S$ and $n \in \omega$, takes the form $((U \cap S) \setminus A) \times \{n\}$, where U is a neighbourhood of x in the usual topology of \mathbb{R} and A is an arbitrary countable set not containing x ; a point $\langle x, \omega \rangle$, where $x \in S$, has basic neighbourhoods of the form $((U \cap S) \setminus A) \times (n, \omega) \cup \langle x, \omega \rangle$, where U is a neighbourhood of x in the usual topology of \mathbb{R} , A is an arbitrary countable subset of S , and $n \in \omega$.

(1) X_S is not SSL.

(This part of proof does not need assumptions on cardinals.) Enumerate $S = \{s_\alpha : \alpha < \omega_1\}$, and for every $\alpha < \omega_1$ choose an uncountable $A_\alpha \subset S$ so that $A_\alpha \cap A_\beta = \emptyset$ whenever $\alpha \neq \beta$. For $\alpha < \omega_1$ and $a \in A_\alpha$, put $U_a = (\{s_\beta : \beta > \alpha\} \times \omega) \cup (\{a\} \times \{\omega\})$. Then the open cover $\mathcal{U} = \{U_a : a \in A_\alpha, \alpha < \omega_1\} \cup (X_S \setminus (\bigcup\{A_\alpha : \alpha < \omega_1\} \times \{\omega\}))$ witnesses that X_S is not SSL. \triangle

To continue the discussion of the examples, we need an auxiliary definition and a simple lemma.

DEFINITION 3.4. Let $Y \subset X$. Say that Y is *relatively NSM* (*relatively NSH*) *in* X if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X , there is a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X , such that for every open $O_n \supset A_n$, $n \in \mathbb{N}$, $\bigcup\{\text{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\} \supset Y$ (respectively, for every $y \in Y$, $y \in \text{St}(O_n, \mathcal{U}_n)$ for all but finitely many n). Say that Y is *relatively NSR in* X if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X , there are $x_n \in X$, $n \in \mathbb{N}$, such that for every open $O_n \ni x_n$, $n \in \mathbb{N}$, $\bigcup\{\text{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\} \supset Y$.

LEMMA 3.5. *If* $X = \bigcup\{Y_k : k \in \mathbb{N}\}$, *and each* Y_k *is relatively NSM (relatively NSH, relatively NSR) in* X , *then* X *is NSM (respectively, NSH, NSR).*

Proof. Having a sequence of open covers of X , rearrange it as $(\mathcal{U}_{km} : k, m \in \mathbb{N})$, and let $\{\mathcal{U}_{km} : m \in \mathbb{N}\}$ take care of Y_k . \blacksquare

(2) Under $\omega_1 < \mathfrak{d}$, S and the sets $S \times \{n\}$, $n \in \omega$, are NSM.

(Of course, S is with the topology generated by intervals with countably many points removed.)

Recall that a space X is *projectively Menger* if every continuous second countable image of it is Menger (similar definitions for Rothberger and Hurewicz properties) [2], [14]. Since $|S| < \mathfrak{d}$, S and the sets $S \times \{n\}$, $n \in \omega$ are projectively Menger. Since every Lindelöf, projectively Menger space is Menger, we conclude that S and the sets $S \times \{n\}$, $n \in \omega$ are Menger then NSM. \triangle

(3) Under $\omega_1 < \mathfrak{d}$, $S \times \{\omega\}$ is relatively NSM in X_S .

For $a \in S$, $n \in \mathbb{N}$, and a countable subset $A \subset S$, we denote

$$U_{A,n}(a) = \{ \langle a, \omega \rangle \} \cup (((S \cap (a - 1/n, a + 1/n)) \setminus A) \times (n, \omega)).$$

It is clear that these sets form a base at $\langle a, \omega \rangle$.

Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X_S . For every $a \in S$ and every $n \in \mathbb{N}$, pick $f_a(n) \in \mathbb{N}$ and a countable $A_{a,n} \subset S$ so that $U_{A_{a,n}, f_a(n)}(a)$ is a subset of some element of \mathcal{U}_n . It is clear that, given some a , the sets $A_{a,n}$ may be taken the same for all n (just take the union), so $A_{a,n}$ will be denoted just A_a . Further, f_a is a function from \mathbb{N} to \mathbb{N} . Since $|S| < \mathfrak{d}$, there is a function f^* from \mathbb{N} to \mathbb{N} such that for every $a \in S$, $f^*(n) > f_a(n)$ for infinitely many n . For each $n \in \mathbb{N}$, pick a finite $B_n \subset S$ such that for every $x \in [-n, n]$, there is a $b \in B_n$ such that $|x - b| \leq 1/(2f^*(n))$. Put $C_n = B_n \times \{f^*(n)\}$. Let O_n be any neighbourhood of C_n . We have that $\bigcup \{ \text{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N} \} \supset S \times \{ \omega \}$. \triangle

In view of (2), (3) and Lemma 3.5, we have done Example 3.1. \diamond

(4) Under $\omega_1 < \mathfrak{b}$, S is NSH (and thus so are $S \times \{n\}$, $n \in \omega$).

Similar to (2). \triangle

(5) Under $\omega_1 < \mathfrak{b}$, $S \times \{ \omega \}$ is relatively NSH in X_S .

The function f^* and the sets C_n , defined like in (3), now satisfy stronger conditions: for every $a \in S$, $f^*(n) > f_a(n)$ for all but finitely many n , and thus for every open $O_n \supset C_n$, $\langle a, \omega \rangle \in \text{St}(O_n, \mathcal{U}_n)$ for all but finitely many n . \triangle

By (4), (5) and Lemma 3.5, we have finished with Example 3.2. \diamond

(6) Under $\omega_1 < \text{cov}(\mathcal{M})$, S and the sets $S \times \{n\}$, $n \in \omega$, are NSR.

Similar to (2). \triangle

(7) Under $\omega_1 < \text{cov}(\mathcal{M})$, $S \times \{ \omega \}$ is relatively NSR in X_S .

$S \times \omega$ is separable in the usual Tychonoff product topology. Let D be a dense countable subset of this space. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X_S . For every $a \in S$ and every $n \in \mathbb{N}$, pick an element U of the usual topology on S , a countable subset A of S and $m \in \omega$ so that $(U \setminus A) \times (m, \omega) \cup \{ \langle a, \omega \rangle \}$ is contained in some element of \mathcal{U}_n . Further, pick $f_a(n) \in (U \times \{m+1\}) \cap D$. Since $|S| < \text{cov}(\mathcal{M})$, there exists $f \in D^{\mathbb{N}}$ such that for every $a \in S$, the set $\{n \in \mathbb{N} : f(n) = f_a(n)\}$ is infinite. For every $n \in \mathbb{N}$, let O_n be any neighborhood of $f(n)$. We have that $\bigcup \{ \text{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N} \} \supset S \times \{ \omega \}$. \triangle

In view of (6), (7) and Lemma 3.5, we have done Example 3.3. \diamond

PROBLEM 3.6. Do there exist **ZFC** examples of spaces as in Examples 3.1, 3.2 and 3.3?

Now we show that implications $\text{NSM} \Rightarrow \text{SM}$, $\text{NSH} \Rightarrow \text{SH}$ and $\text{NSR} \Rightarrow \text{SR}$ can not be reversed.

EXAMPLE 3.7. A Tychonoff space which is SR and SH (and thus SM), but is not NSL (and thus is neither of NSR, NSH, NSM).

Let $K = D \cup \{ \infty \}$ be the one point compactification of the discrete space D of uncountable cardinality κ . Denote $X_0 = K \times \kappa^+$, $X_1 = D \times \{ \kappa^+ \}$, $X = X_0 \cup X_1$; X has the topology inherited from the product $K \times (\kappa^+ + 1)$.

1) X is not NSL.

Consider the cover $\mathcal{V} = \{K \times \kappa^+\} \cup \{\{d\} \times [0, \kappa^+] : d \in D\}$. Let $B \subset X$ be countable. Pick $p \in D \setminus \pi_K(B)$ (where π_K is the projection of X onto K). Put $U = ((K \setminus \{p\}) \times [0, \kappa^+]) \cap X$. Then U is a neighbourhood of B such that $\text{St}(U, \mathcal{V}) \neq X$ (because $\langle p, \kappa^+ \rangle \notin \text{St}(U, \mathcal{V})$). \triangle

2) X is SR.

(It is worth to note that only here we will use specific properties of the one point compactification. For other parts of the argument, any other compactification bD of D would do.)

We need an auxiliary definition and an easy lemma.

DEFINITION 3.8. Let $Y \subset X$. Say that Y is *relatively SR in X* if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X , there exists $U_n \in \mathcal{U}_n$ such that $\bigcup\{\text{St}(U_n, \mathcal{U}_n) : n \in \mathbb{N}\} \supset Y$.

LEMMA 3.9. *If $X = \bigcup\{Y_k : k \in \mathbb{N}\}$ and each Y_k is relatively SR in X , then X is SR.*

We are going now to prove that: (a) X_1 is relatively SR in X , and (b) X_0 is SSR (hence SR and relatively SR in X). Then by Lemma 3.9 we will have that X is SR.

(a) X_1 is SR in X .

Let $(\mathcal{U}_n : n \in \omega)$ be a sequence of open covers of X . For every $d \in D$ pick an ordinal α_d such that $\{d\} \times [\alpha_d, \kappa^+]$ is a subset of some element of \mathcal{U}_0 . Put $\alpha^* = \sup\{\alpha_d : d \in D\}$. Let $\langle \infty, \alpha^* \rangle \in U \in \mathcal{U}_0$. Then only finitely many points $\langle d, \kappa^+ \rangle$ are not in $\text{St}(U, \mathcal{U}_0)$. Let $\mathcal{U}_1, \mathcal{U}_2$, etc take care about these points. \triangle

(b) X_0 is SSR.

It is well known that a compact space is Rothberger if and only if it is scattered. Since for every ordinal $\gamma, \gamma + 1$ and $K \times (\gamma + 1)$ are compact scattered spaces they are Rothberger.

Claim b.1. If δ is a limit ordinal of countable cofinality, then δ is Rothberger.

Indeed, let $\delta = \lim\{\gamma_n : n \in \omega\}$. Then $\delta = \lim\{\gamma_n + 1 : n \in \omega\}$, each $\gamma_n + 1$ is Rothberger and countable union of Rothberger spaces is Rothberger. \triangle

Claim b.2. Every ordinal δ is SSR.

We only have to prove the case when δ is a limit ordinal of uncountable cofinality. Then it is strongly star-compact. Let $(\mathcal{U}_n : n \in \omega)$ be a sequence of open covers of δ . There is $p \in \delta$ such that $\text{St}(\{p\}, \mathcal{U}_0)$ contains a final tail of δ . Further, $p + 1$ is Rothberger and thus it can be served by $(\mathcal{U}_n : n \geq 1)$. \triangle

Claim b.3. Let \mathcal{U} be an open cover of X_0 . For a finite subset $F \subset K$ denote

$$A_{F, \mathcal{U}} = \{\alpha \in \kappa^+ : \text{there are } \beta < \alpha, U \in \mathcal{U}, \text{ such that } (K \setminus F) \times (\beta, \alpha] \subset U\}.$$

We claim that one of the sets $A_{F, \mathcal{U}}$ contains a final tail of κ^+ . Suppose the contrary that is for every finite $F \subset K$, $A_{F, \mathcal{U}}$ does not contain a tail. This means that for

every $\alpha < \kappa^+$ there exists $\beta > \alpha$ such that $\beta \notin A_{F,\mathcal{U}}$. Let $\mathcal{F} = \{F \subset K : F \text{ is finite}\}$; of course $|\mathcal{F}| = \kappa$. For every $F \in \mathcal{F}$ fix $\alpha_{0,F} \notin A_{F,\mathcal{U}}$. Put $\alpha_0 = \sup\{\alpha_{0,F} : F \in \mathcal{F}\}$ (this supremum exists by regularity of κ^+). For every $F \in \mathcal{F}$, fix $\alpha_{1,F} > \alpha_0$ such that $\alpha_{1,F} \notin A_{F,\mathcal{U}}$ (this exists for the hypothesis that $A_{F,\mathcal{U}}$ do not contain tails). Put $\alpha_1 = \sup\{\alpha_{1,F} : F \in \mathcal{F}\}$ and so on (this means that we proceed by induction: for every $n \in \mathbb{N}$, put $\alpha_{n+1} = \sup\{\alpha_{n+1,F} : F \in \mathcal{F}\}$ where for every $F \in \mathcal{F}$, $\alpha_{n+1,F}$ is a fixed element of κ^+ such that $\alpha_{n+1,F} > \alpha_n$ and $\alpha_{n+1,F} \notin A_{F,\mathcal{U}}$). Let $\alpha^* = \sup\{\alpha_n : n \in \mathbb{N}\}$. By regularity of κ^+ we have that $\alpha^* < \kappa^+$ and then the point $\langle \infty, \alpha^* \rangle \in X_0$. Fix $U \in \mathcal{U}$ such that $\langle \infty, \alpha^* \rangle \in U$. Then there exist a finite $F \subset D$ and $\beta < \alpha^*$ such that $(K \setminus F) \times (\beta, \alpha^*] \subset U$. Since $\alpha^* = \sup\{\alpha_n : n \in \mathbb{N}\}$ and $\beta < \alpha^*$, there exists $n \in \mathbb{N}$ such that $\alpha_n > \beta$. Further $\alpha_{n+1,F} \leq \alpha_{n+1} \leq \alpha^*$. Then $(K \setminus F) \times (\alpha_n, \alpha_{n+1,F}] \subset (K \setminus F) \times (\beta, \alpha^*] \subset U$; hence $\alpha_{n+1,F} \in A_{F,\mathcal{U}}$, a contradiction. \triangle

Claim b.4. Let F be such that $A_{F,\mathcal{U}}$ contains a final tail. Consider

$$\mathcal{O} = \{O : O \subset \kappa^+ \text{ open and } (K \setminus F) \times O \text{ is a subset of some element of } \mathcal{U}\}.$$

This is an open cover of the final tail. Then there is $p \in \kappa^+$ such that $\text{St}(\{p\}, \mathcal{O}) \supset [p, \kappa^+)$. (This follows from strong star-compactness.) \triangle

To conclude the proof that X_0 is SSR, consider a double indexed sequence of open covers $(\mathcal{U}_{mn} : m, n \in \omega)$. Pick the set $A_{F,\mathcal{U}}$ (with appropriate F) corresponding to the cover \mathcal{U}_{00} and take $p = p_{00}$ as in Claim b.4. Then $\text{St}(\langle \infty, p_{00} \rangle, \mathcal{U}_{00}) \supset (K \setminus F) \times [p_{00}, \kappa^+)$. Covers $\mathcal{U}_{1,n}$, $n \in \omega$ will be used to serve $K \times (p+1)$. This will leave unserved only some of the points of the form $\langle f, \alpha \rangle$ where $f \in F$. But for each $f \in F$, the set $\{\langle f, \alpha \rangle : \alpha < \kappa^+\}$ is homeomorphic to κ^+ and thus can be served by some of remaining covers by Claim b.1. \triangle

3) X is SH.

In fact, we shall prove that X has the following property (*):

for every open cover \mathcal{U} there is a compact $C \subset X$ such that $\text{St}(C, \mathcal{U}) = X$.

It is easily seen that a space having property (*) is SC and thus SH. So, let \mathcal{U} be an open cover of X . Since $K \times \kappa^+$ is countably compact, there exists a finite subset $E \subset X$ such that $\text{St}(E, \mathcal{U}) \supset K \times \kappa^+$. For each $d \in D$ choose $U_d \in \mathcal{U}$ such that $\langle d, \kappa^+ \rangle \in U_d$ and pick $x_d = \langle d, \gamma_d \rangle \in U_d \setminus \{\langle d, \kappa^+ \rangle\}$. Put $\gamma = \sup\{\gamma_d : d \in D\}$. By regularity of κ^+ , $\gamma < \kappa^+$. Then the set $A = cl_{K \times [0, \gamma]} \{x_d : d \in D\}$ is compact. Further $\text{St}(A, \mathcal{U}) \supset D \times \{\kappa^+\}$. The set $C = E \cup A$ is compact and $\text{St}(C, \mathcal{U}) = X$. \diamond

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