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ABSTRACT FORMULATION OF AN AGE-PHYSIOLOGY DEPENDENT POPULATION DYNAMICS PROBLEM

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Abstract. We transform a population dynamics problem with an additional structure into an abstract Cauchy problem, and use functional analytic method to solve it. Instead of using the familiar theory of resolvent, we rather replace the condition of well-posedness by its equivalent form in order to prove the existence of a unique weak solution. An *a priori* estimate of the solution is also given as well as the perturbation about the equilibrium point, and by a bounded linear operator.

1. Introduction

A variety of problems in differential equations, (abstract) differential equations, age-dependent population models with or without delay, evolution equations with boundary conditions, can be written as semi-linear Cauchy problems [18]. In this paper, we consider a mathematical model describing the dynamical evolution of an age-structured population with an additional structure, g, say. A brief comment of previous works provides the context for this paper.

Inaba [10] considered a mathematical model of an epidemic spreading in an age-structured population with age-dependent transmission. His analysis is basically on the threshold and stability of the epidemic model. In order to prove the existence and uniqueness of solutions, he transformed the model equations into an abstract Cauchy problem.

Chan and Guo [4] present conditions which guarantee the boundedness and stability of a large time behaviour of the population density distribution for the general logistic model by writing their model equations as an abstract evolution equation.

Magal [13] investigated the existence of compact attractors for time-periodic age-structured models. He considered a population which can be divided into several species, and several patches when there is spatial structure. He noted that it

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is natural to incorporate periodic births and periodic mortalities in fisheries problems, where intra and inter-specific competition as well as migration take place, and considered an abstract formulation of that type of evolution problem.

Over the years, various authors have considered the abstract formulation in handling dynamical systems of real life situations ([9], [5], [18], to name, but a few). Nevertheless very few of these authors give concrete examples of evolution operators (or generators) which satisfy the semigroup properties, except [9], [16], to the best knowledge of this author.

More often than not, it happens that the characteristics lines through a given point in an attempt to solve the system (2.1) below intersect. We cannot therefore expect the existence of classical solution always, for $t \geq 0$. We circumvent this difficulty by making use of the theory of semigroup of linear operators. This technique enables us to ascertain the existence of a unique mild solution under some conditions, and it applies usefully and meaningfully to smooth functions which are C^1.

Liadi and Tchuenche applied the theory of resolvent to show that solutions of abstract delayed differential equations converge asymptotically to zero if starting in the neighbourhood of the origin. Also, using the theory of semigroups of operators, they show that solutions starting in the neighbourhood of the initial distribution will converge to it for suitably chosen constant, which depends on the parameters of the equation.

Motivated by the quasi-linear type equation of an age-structured population with an additional structure [16], we decided to carry out this study in which we define an evolution operator satisfying the semigroup properties.

We now present the plan of the paper. In the next section, we transform the model equation given by the first order quasi-linear partial differential equation (2.1) into a non-homogeneous abstract Cauchy problem. We adapt result of books by Goldstein [8] and Pazy [14]. A classical generator a C_0-semigroup is given, followed by the proof of the existence and uniqueness of solutions. In section 3, we present an *a priori* of weak solutions. Finally in section 4, we consider a perturbation of the non-zero steady state of (2.1).

2. Transformation into Abstract Problem

This section deals with the abstract evolution equations approach to a class of partial differential equations (problem 2.1) motivated by population dynamics. Using an evolution equation approach, sufficient conditions for well-posedness in L^1 of the dynamics and of existence of a weak solution are given. The model equations are

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} + G(a)\frac{\partial u}{\partial g} &= -R(a,g)u(t,a,g) \\ u(0,a,g) &= u_0(a,g) \\ u(t,0,g) &= B(t,g) \end{aligned} \tag{2.1}$$

where $u(t,a,g)$ represents the population density of individuals at time t, age a with physiological variable g, $B(t,g)$ is the renewal equation while $G(\cdot)$ is the velocity of g and $R(\cdot,\cdot)$ is the death modulus.

In an attempt to solve (2.1), we proceed as follows: pick (t_0,a_0,g); find the characteristic curve through (t_0,a_0,g), and suppose it hits the a-axis at $(0,\bar{a},0)$. Then $u(t_0,a_0,g)=f(\bar{a})$. Unfortunately, it can happen that two characteristics intersect, and so the solution is constrained to take distinct values at the same point. We cannot therefore, expect to have classical solutions of (2.1) always for $t\geq 0$. We therefore employ the theory of semigroup of linear operators which will enable us to ascertain the existence of a unique weak solution of (2.1), which can be posed as a linear abstract Cauchy problem. Our approach relies on the theory of semi-linear evolution equations, and so we consider (2.1) as an ordinary differential equation in an appropriate Banach space. Let

$$\mathcal{A}\phi(a,g) := \frac{-\partial\phi(a,g)}{\partial a}, \qquad a\in[0,A], \quad g\in\Omega$$

where $\mathcal{A}$ is a linear but unbounded operator in $L^1([0,A]\times\Omega;\mathbf{R}^+)$ [15]. In general, $L^1([0,A]\times\Omega;\mathbf{R}^+)$ represents the set of equivalence classes of Lebesgue integrable functions ($u(a,g)$, say) from $[0,A]\times\Omega$ to $\mathbf{R}^+$ [10]. It is assumed that $u(a,g)$ is absolutely continuous on the appropriate interval where it is defined. Also, $u(0,g)=u_0(g):=\phi(g)$. The regularity on the variable g is $u_0(0)=\phi(0)$; $\phi(0)$ is known and satisfies $\phi(0)=\int_0^A\beta(a,0)u(a,0)\,da$. Then, $D(\mathcal{A})=\{u\in L^1([0,A]\times\Omega;\mathbf{R}^+)$, $\frac{du}{da},\frac{du}{dg}\in L^1([0,A]\times\Omega;\mathbf{R}^+)$; $u(0,g)=u_0(g)=\phi(g)\}$, with $\phi(\cdot)$ satisfying the following relation

$$\phi(g)=\int_0^A\beta(a,g)u(a,g)\,da.$$

Now, we define another operator (linear) $F\colon L^1\to L^1$ by

$$F(\phi)(a,g) := -\left(G(a)\frac{\partial}{\partial g}+R(a,g)\right)\phi(a,g),$$

where without any ambiguity, we write L^1 for short where necessary. Hence, problem (2.1) can be rewritten as an evolution equation in the space L^1

$$\begin{aligned}\frac{du(t)}{dt} &= \mathcal{A}u(t)+F(u(t))\\ u(0) &= u_0.\end{aligned} \tag{2.2}$$

Prub [15] gave conditions for stability of equilibrium solutions of (2.2). Since $F(\cdot)\neq 0$, (2.2) is an inhomogeneous abstract Cauchy problem on the Banach space $L^1([0,A]\times\Omega;\mathbf{R}^+)$. Solving it by showing that $\mathcal{A}$ generates a (C_0) contraction semigroup exhibits a number of qualitative properties besides existence, uniqueness, and continuous dependence on initial data or distributions, namely: continuous dependence on $\mathcal{A}$ [8]. Tchuenche [17] give a stability analysis of the extended abstract linear and nonlinear forms of (2.2) with delay. Since $u(t,a,g)$ is non-negative, we seek for solution of (2.2) in a closed convex set K,

$$K := \{\, u(t)\mid u(t)\in L^1, \quad u\geq 0 \quad a.e.\,\},$$

because only non-negative solutions are biological relevant [15]. It is to be noted here that the theory of C_0-semigroup has many applications to problems that are not concerned with the classical solution of differential equations.

Let

$$(T(t)u)(a,g) := \begin{cases} u_0(a-t,g), & t < a < A, \\ 0, & \text{elsewhere.} \end{cases}$$

It is an easy matter to show that $T(t)$ satisfies the semigroup properties. This well-defined semigroup $T(t)$ is known as the semigroup of translations in $L^1([0,A] \times \Omega; \mathbf{R}^+)$, and according to a universal principle of conservation of difficulty [18], we have:

$$\|T(t)u\|_{L^1} \le \|u\|_{L^1} \implies \|T(t)\| \le 1.$$

Hence $T(t)$ is a contraction semigroup of class C_0 which is dissipative, i.e. $\mathrm{Re}\langle \mathcal{A}u, u\rangle \le 0$, and

$$\mathrm{Re}\langle T(t)u - u, u\rangle = \mathrm{Re}\langle T(t)u, u\rangle - \|u\|^2 \le \|T(t)u\| \, \|u\| - \|u^2\| \le 0$$

Let $\mathcal{A}$ be the infinitesimal generator of a C_0-semigroup $T(t)$, $t \ge 0$ and $F(\cdot)$ be Fréchet or Gâteaux differentiable and Lipschitz continuous, then by using the variation of constant formula, (2.2) is transformed to the following integral equation:

$$\begin{aligned} u(t) &= T(t)u_0 + \int_0^t T(t-s)F(u(s))\, ds, \quad t \in \mathbf{R}^+ \\ u(0) &= u_0 \end{aligned} \tag{2.3}$$

Continuous solutions of equation (2.3) are called weak or mild solutions of (2.1). If $u_0 \in D(\mathcal{A})$, then $t \mapsto T(t)u_0$ is differentiable, though a solution of (2.3) is not necessarily differentiable, whence the term weak solution.

It is known that the above Cauchy problem is well-posed if the well-posedness condition $\rho(\mathcal{A}) \neq \phi$ is replaced by its equivalent form,

$$\|T(t)u\| \le Me^{wt}\|u_0\|, \qquad t \in \mathbf{R}^+, \quad w \le 0, \quad M > 0,$$

where w, M are constants, and $\rho(\mathcal{A})$ is the resolvent set of $\mathcal{A}$ [14], [7]. The solution of (2.3) is unique if $|\rho(\mathcal{A})| < 1$. This condition which expresses the fact that solutions of system (2.1) depend continuously on the initial distribution, is sometimes difficult to verify [3]. A similar property found in [6] requires the application of the classical Gronwall's inequality. However, with this *proviso* in mind, it may be expedient to show that the operator S defined below (or some power S^n of it) is a contraction.

Lemma 2.1. *Let $u_0 \in L^1([0,A] \times \Omega; \mathbf{R}^+)$, $F\colon L^1 \longrightarrow L^1$. Then, $F(\cdot)$ is Lipschitz continuous.*

Proof. In order to prove this lemma, we shall assume that u_g and v_g can be made arbitrarily small for any $g \in \Omega$. Let $\max(\|G(a)\|, \|R(a,g)\|) = c$, and the first derivative u_g be continuous and bounded in a closed domain $\bar{\Omega}$, then we can now show that F is Lipschitzian.

$$\begin{aligned}\|F(u(s))-F(v(s))\|_{L^1} &= \|G(a)(u_g-v_g)+R(u-v)\| \\ &\le \min(\|G(a)\|,\|R(a,.g)\|)\|u_g+u-(v_g+v)\|_{L^1} \\ &\le c\|u-v\|_{W^{1,1}([0,A]\times\Omega)} \\ &\le c\|u-v\|_{L^1}, \end{aligned} \tag{2.4}$$

by Sobolev imbedding theorem [1], where the constant c depends on the parameters of the equation. ■

LEMMA 2.2. *Let $\tau>0$, $Y:=C([0,\tau],L^1([0,A]\times\Omega;\mathbf{R}^+)$ and N be a closed neighbourhood of u_0 in $W^{1,1}$. Define S by*

$$(Su)(t):=T(t)u_0+\int_0^t T(t-s)F(u(s))\,ds, \qquad 0\le t\le\tau.$$

Then S is a contraction on Y. If

$$u\in\mathcal{M}:=\{\,v\in Y\mid v(0)=u_0,\ v([0,T])\subset N\,\},$$

then $\mathcal{M}$ is a complete metric space and $Su\in Y$.

Proof. Let M, w be such that $\|T(t)\|\le Me^{wt}=O(e^{wt})$, ($T(t)$ is exponentially bounded). Then,

$$\begin{aligned}\|Su-Sv\|_Y &= \sup_{0\le t\le\tau}\|Su(t)-Sv(t)\| \\ &= \sup_{0\le t\le\tau}\left\|\int_0^t T(t-s)\left[F(u(s))-F(v(s))\right]\,ds\right\| \\ &\le Me^{wt}\int_0^\tau \|F(u(s))-F(v(s))\|\,ds \\ &\le Me^{wt}c(\tau)\int_0^\tau \|u(s)-v(s)\|\,ds \\ &< c(\tau)Me^{wt}\,\tau\|u-v\|_Y \longrightarrow 0 \ \text{ as } \ \tau\to 0^+ \end{aligned} \tag{2.5}$$

where $c(\tau)$ is assumed to be bounded by $c(1)$ for $\tau<1$. Now, we can show that $S(\mathcal{M})\subset\mathcal{M}$.

$$\begin{aligned}\|Su-u_0\|_Y &\le \sup_{0\le t\le\tau}\|T(t)u_0-u_0\|+\sup_{0\le t\le\tau}\left\|\int_0^t T(t-s)F(u(s))\,ds\right\| \\ &= J_1(\tau)+J_2(\tau).\end{aligned}$$

$J_1(\tau)\to 0$ as $\tau\to 0$ while

$$J_2(\tau)\le \tau Me^{w\tau}\sup_{0\le t\le\tau}\|F(u(t))\|\to 0 \ \text{ as } \ \tau\to 0^+.$$

Also, $\|J_1(\tau)\|=O(t) \implies \lim\limits_{t\to 0}\dfrac{T(t)u_0-u_0}{t}=\mathcal{A}u_0$. Hence $S(\mathcal{M})\subset\mathcal{M}$ provided τ

is sufficiently small. Applying inequality (2.5) n times, we have

$$\begin{aligned}
\|S^n u(t) - S^n v(t)\|_Y &= \left\| \int_0^t T(t-s)\left\{F(S^{n-1}u(s)) - F(S^{n-1}v(s))\right\} ds \right\|_Y \\
&\le Me^{wt}c \int_0^t (cMe^{ws}\tau s)^{n-1} \sup_{0\le r\le\tau} \|u(r)-v(r)\| \, \frac{dr}{(n-1)!} \\
&\le (Me^{wt}c)^n \sup_{0\le r\le\tau} \|u-v\| \int_0^t \frac{s^{n-1}}{(n-1)!}\, ds \\
&\le \frac{(cMe^{wt})^n}{n!} \sup_{0\le r\le\tau} \|u-v\|_Y s^n \qquad (2.6)
\end{aligned}$$

Let $\alpha = \frac{(cMse^{wt})^n}{n!}$, and choose n large enough so that $\alpha < 1$. Then

$$\|S^n u - S^n v\|_Y \le \alpha \|u-v\|_Y, \qquad u, v \in Y.$$

Hence, S is a contraction and has a unique fixed point in Y. Thus (2.1) possesses a unique continuous (mild) solution on $[0,\tau]$ and since τ is arbitrary, the result holds in the whole of $\mathbf{R}^+$. ∎

3. A Priori Bound of Weak Solutions

THEOREM 3.1. *If $T(t)$ is dissipative and $F\colon L^1([0,A]\times\Omega;\mathbf{R}^+) \longrightarrow L^1([0,A]\times\Omega;\mathbf{R}^+)$ is Lipschitzian, then equation (2.1) possesses a unique bounded weak solution provided there exist constants $M>0$, $w\le 0$ such that $\|T(t)\| = O(e^{\bar\alpha t})$, with $\bar\alpha = cM + w < 0$.*

Proof.

$$\begin{aligned}
\|u(t)\|_{L^1} &\le \left\| T(t)u_0\| + \int_0^t \|T(t-s)F(u(s))\, ds \right\| \\
&\le Me^{wt}\|u_0\| + Me^{wt} \int_0^t e^{-ws}\|F(u(s))\|\, ds \\
&\le Me^{wt}\|u_0\| + Me^{wt} \int_0^t ce^{-ws}\|u(s)\|\, ds \\
&\le Me^{wt}\left(\|u_0\| + c\int_0^t e^{-ws}\|u(s)\|\, ds \right) \\
e^{-wt}\|u(t)\| &\le M\|u_0\| + cM\int_0^t e^{-ws}\|u(s)\|\, ds.
\end{aligned}$$

On applying Gronwall's inequality to $e^{-ws}\|u(s)\|$, we have

$$\|u(t)\| \le M\|u_0\|e^{(cM+w)t}. \qquad (3.1)$$

If $cM + w < 0$ then $\|u(t)\| < \infty$ as $t \to \infty$. ∎

LEMMA 3.2. *Let $F(\cdot)$ be Lipschitz continuous in u. If $u \in C([0,T] : L^1)$, then the weak solution of (2.1) is unique.*

Proof. Define $S\colon C([0.T]:L^1) \longrightarrow C([0,T]:L^1)$ and let $u(t)$ and $v(t)$ be two solutions. Then from equation 6.1 [4], we have

$$\begin{aligned}\|(Su)(t)-(Sv)(t)\| &\le \|T(t)u_0 - T(t)u_0\| + \int_0^t \|T(t-s)\|\|F(u(s))-F(v(s))\|\,ds\\ &\le cMe^{wt}\int_0^t e^{-ws}\|u(s)-v(s)\|\,ds\end{aligned}$$

and by the classical Gronwall's lemma, we have $\|u(t)-v(t)\| \le 0$, implying $u(t) = v(t)$. This completes the proof. ■

4. Perturbation about an Equilibrium Point

LEMMA 4.1. *The steady state solution of equation (2.1) is exponentially asymptotically stable if for $M > 0$ and $\alpha < 0$,*

(i) $\|h(t,u(t))\| \le M\|u(t)\|$,

(ii) $M + \alpha < 0$.

Proof. Let z_e be a steady-state solution of (2.1). Then putting $u = z_e + z$ into (2.1), we get

$$\dot z(t) = \mathcal{A}z(t) + h(t,z(t)), \qquad z(0) = z_0$$

where $h(t,z(t)) := F(z_e + z(t))$. By a variation of parameters,

$$z(t) = e^{\mathcal{A}t}z_0 + \int_0^t e^{\mathcal{A}(t-\tau)}h(\tau,z(\tau))\,d\tau. \tag{4.1}$$

Taking the Laplace transform of (4.1) with respect to t, while introducing ξ as the transform variable yields

$$\mathcal{L}\{z(t)\} = \mathcal{L}\{e^{\mathcal{A}t}\}[z_0 + \mathcal{L}\{h(t,z(t))\}] \tag{4.2}$$

and the transform equation takes the form

$$\hat z(\xi) = (\xi I - \mathcal{A})^{-1}[z_0 + \hat h(\zeta, z(\xi))] \tag{4.3}$$

where $(\xi I - \mathcal{A})^{-1}$ is a bounded linear operator provided $\|\mathcal{A}\| < |\xi|$, I being the identity operator. Assuming Laplace invertibility of (4.3) in $\mathrm{Re}(\xi) \le \alpha < 0$, we obtain

$$z(t) = e^{\alpha t}z_0 + \int_0^t e^{\alpha(t-\tau)}h(\tau,z(\tau))\,d\tau. \tag{4.4}$$

Thus, from hypothesis (i),

$$\|z(t)\| \le e^{\alpha t}\{\|z_0\| + M\int_0^t e^{-\alpha\tau}\|z(\tau)\,d\tau\|\}$$

Hence, by Gronwall's Lemma,

$$\|z(t)\| \le \|z_0\|e^{(M+\alpha)t} \quad \forall t \ge 0 \tag{4.5}$$

and the conclusion follows if $M + \alpha < 0$. ■

Relevant to this result, although not overlapping with it is that of [12], equation (4.5) being a special case of his result. The asymptotic stability of such abstract equations with time delay can be found in Tchuenche [17].

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REFERENCES

[**1**] Adams, R. A., *Sobolev Spaces*, Academic Press, New York San Francisco London, 1975.

[**2**] Bainov, D. and Simeonov, P., *Integral Inequalities and Applications*, Kluver Academic Publishers Group, 1992.

[**3**] Barbu, V., *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Editum Academici Noordhoff International Publishing. Bucaresti Romania Leyden The Netherlands, 1976.

[**4**] Chan, W. L. and Guo, B. Z., *Global Behaviour of Age-dependent Logistic Population Models*, J. Math. Biol. **28**, 225–235.

[**5**] Da Prato, G. and Iannelli, M., *Boundary Value Problem for Age-dependent Equations*, In: P.Clement and Lumer G. (eds), *Evolution Equations, Control Theory and Biomathematics*, Lecture Notes in Pure and Applied Mathematics **155** (1994), 91–115.

[**6**] Di Blasio, G., Iannelli, M. and Sinestrari, E., *Approach to Equilibrium in Age-structured Population with Increasing Recruitment Process*, J. Math. Biol. **13** (1982), 371–382.

[**7**] Engel, K. J., Nagel, R., *One-parameter Semigroups for Linear Evolution Equations*, Springer Verlag, Graduate Texts in Mathematics, 2000.

[**8**] Goldstein, J. A., *Semigroups of Linear Operators and Applications*, Oxford University Press, New York Clarendon Press, Oxford, 1985.

[**9**] Iannelli, M., Kim, M. Y. and Park, E. J., *Asymptotic Behaviour for an SIS Epidemic Model and its Approximation*, Nonlinear Analysis **35** (1999), 797–814.

[**10**] Inaba, H., *Threshold Stability Results for an Age-structured Epidemic Model*, J. Math. Biol. **28** (1990), 411–434.

[**11**] Liadi, M. A. and Tchuenche, J. M., *Asymptotic Behaviour of Solutions of a Delayed Abstract Differential Equation*, submitted.

[**12**] Liadi, M. A., *A Qualitative Study of Andrew Dobson's Problem in Helminth Infection*, PhD Thesis, Fac. Sci., University of Ibadan, 1999.

[**13**] Magal, P., *Compact Attractors for Time-periodic Age-structured population Models*, Elect. J. Diff. Eqns. **65** (2001), 1–35.

[**14**] Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Applied Mathematical Sciences, Springer Verlag, 1983.

[**15**] Prub, J., *Equilibrium Solutions of Age-Specific Population Dynamics of Several Species*, J. Math. Biol. **11** (1981), 65–83.

[**16**] Tchuenche, J. M., *Theoretical Population Dynamics Model of a Genetically Transmitted Disease: Sickle-Cell Anaemia*, Bull. Math. Biol. **69**(2) (2007), 699–730.

[**17**] Tchuenche, J. M., *Asymptotic Stability of an Abstract Delayed Functional Differential Equation, Nonlinear Analysis: Modelling and Control*, **11** (1) (2006), 1–15.

[**18**] Thieme, H., *Semi-flows Generated by Lipschitz Perturbations of Non-densely Defined Operators*, Diff. Int. Eqns. **3**(6) (1990), 1035–1066.

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